MATH. SCAND. 58 (1986), 215–226

# MINIMAL AND DISTAL FUNCTIONS ON SEMIDIRECT PRODUCTS OF GROUPS II

PAUL MILNES[1]

**Abstract.**

Let $G = G_1 \otimes G_2$ be a semidirect product of locally compact groups with multiplication $(s',t')(s,t) = (s'\sigma(t')s, t't)$. The formula $F(s,t) = f(s)$ provides a canonical way to extend a function $f$ on $G_1$ to a function $F$ on $G$. In an earlier paper we studied the case when $G_2$ is compact, and also showed that

(*) if $f$ is point distal, then $F$ is minimal on the discrete version of $G$.

In the present paper it is shown that the point distal hypothesis in (*) cannot be weakened and the main result asserts that, if $G_1$ is compact, then $F$ is a distal function on $G$ for every $f \in C(G_1)$ if and only if the action of $G_2$ on $G_1$ is distal. Many examples are discussed.

## 1. Preliminaries.

Let $G$ be a locally compact group. A bounded complex-valued function $F$ on $G$ is called *right uniformly continuous* if, for any $\varepsilon > 0$, there is a neighbourhood $V$ of the identity $e$ of $G$ such that $|F(s) - F(t)| < \varepsilon$ whenever $st^{-1} \in V$. Let $U(G)$ be the class of such functions. $U(G)$ is a C*-subalgebra of the C*-algebra $C(G)$ of all continuous bounded complex-valued functions on $G$. The *right translate* $R_tF$ of $F \in C(G)$ is defined by

$$R_tF(s) = F(st), \quad s,t \in G,$$

and, if $F \in U(G)$, then the closure $R_GF^-$ of the orbit

$$R_GF = \{R_tF \mid t \in G\}$$

in the topology of pointwise convergence on $G$ is compact in $C(G)$ for that topology. In fact, this compactness property characterizes $U(G)$. See [2] for all this, noting that the space we have called $U(G)$ here is called, for a good reason, LUC$(G)$ in [2]. If $F \in U(G)$, the translation operators $R_t$,

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[1] Research supported in part by NSERC grant A7857.
Received April 4, 1984.

$t \in G$, leave $R_G F^-$ invariant and $(R_G, R_G F^-)$ is a flow. $F$ is called *minimal*, *point distal*, or *distal* if that flow is minimal, point distal with $F$ as special point, or distal, respectively. Specifically, an $F$ in $U(G)$ is:

*minimal* if, whenever $H_1 = \lim_\alpha R_{t_\alpha} F$ (pointwise on $G$), there is a net $\{t_\beta\} \subset G$ such that $F = \lim_\beta R_{t_\beta} H_1$;

*point distal* if, whenever $H_1 = \lim_\alpha R_{t_\alpha} F$ and $\lim_\gamma R_{t_\gamma} H_1 = H' = \lim_\gamma R_{t_\gamma} F$, it follows necessarily that $H_1 = F$; or

*distal* if, whenever $H_1 = \lim_\alpha R_{t_\alpha} F$, $H_2 = \lim_\beta R_{t_\beta} F$, and $\lim_\gamma R_{t_\gamma} H_1 = H' = \lim_\gamma R_{t_\gamma} H_2$, it follows necessarily that $H_1 = H_2$.

Clearly distal functions are point distal, and point distal functions are minimal (see [4], [19]). Also, the limit function $H_1$ of the definition will be minimal or distal if $F$ is minimal or distal, respectively, but can fail to be point distal if $F$ is point distal. Thus, if $F$ is point distal, if $H'$ is some minimal function, and if $\lim_\beta R_{t_\beta} F = \lim_\beta R_{t_\beta} H'$, then $H' \in R_G F^-$ and $H' = F$.

We need to define one more kind of function. For an $f \in U(G)$ and $\varepsilon > 0$, let

$$A_\varepsilon = A_\varepsilon(f) = \{s \in G \mid \|R_s f - f\| < \varepsilon\},$$

a symmetric set. Then $f$ is called *Bohr almost periodic* if, for every $\varepsilon > 0$, there is a compact $K_\varepsilon \subset G$ such that $A_\varepsilon K_\varepsilon = G$. A reference for such functions is [7] (where they are called "uniformly almost periodic" functions).

*Note.* Our terminology would have been more accurate if we had called the functions defined above right minimal, right point distal, etc., since the definitions involved right uniform continuity and right translation. We wish to assert here that, in this paper, a function will be called *left minimal*, *left point distal*, etc., if it satisfies the appropriate analogous condition involving left uniform continuity and left translation.

## 2.

Let $\sigma$ be a homomorphism of a locally compact group $G_2$ into Aut $(G_1)$, the group of automorphisms of another locally compact group $G_1$. The multiplication formula

$$(s', t')(s, t) = (s'\sigma(t')s, t't)$$

gives the product space $G_1 \times G_2$ a group structure. $G_1 \times G_2$ with this multiplication is called a *semidirect product* of $G_1$ and $G_2$ and is designated by $G_1 \otimes G_2$, $G_1 \otimes G_2$ is a topological group if the map

$$(s,t) \to \sigma(t)s, \quad G_1 \times G_2 \to G_1$$

is (jointly) continuous; we will generally assume that this is the case.

In [13] we showed that, for compact $G_2$, the formula $F'(s,t) = f(\sigma(t^{-1})s)$ extends a minimal {point distal} [distal] $f$ on $G_1$ to a minimal {point distal} [distal] $F$ on $G_1 \otimes G_2$. After the first version of [13] was written, T.-S. Wu pointed out to us that the result of the last sentence can be obtained, and generalized, via a theorem of Hahn [8]; see the appendix at the end of this paper. In [10], [13], we also drew the following conclusions.

(i) Let $G = C \otimes T$ be the euclidean group of the plane and let $f$ be a non-trivial continuous character on $C$, e.g., $f(z) = f(x+iy) = e^{ix}$. Then (the corresponding) $F'$ is not left uniformly continuous on $G$ and is not even left point distal on $G_d$ (which is the group $G$ with the discrete topology).

(ii) If $G = G_1 \otimes G_2$ and $f$ is a left point distal function on $G_1$, then $F'$ is a left minimal function on $G_d$.

(iii) In the particular case where $G = \mathrm{R} \otimes \mathrm{R}^+$, the affine group of the line, and $f$ is a non-trivial continuous character on $\mathrm{R}$, $F'$ is left minimal on $G_d$, while a net of right translates of $F'$ converges to the constant function 1: $F'$ is not even minimal on $G_d$. (Of course, $\mathrm{R}^+$ is not compact.)

Professor Wu also pointed out the relevance to (iii) of a construction of Furstenberg [6; II.5.5]. We take this opportunity to thank Professor Wu.

In this paper we wish to emphasize the result (ii) and first restate it in its "right" version.

THEOREM 1. *Let $G = G_1 \otimes G_2$. Then the formula $F(s,t) = f(s)$ extends a point distal function $f$ on $G_1$ to a minimal function $F$ on $G_d$.*

[The proof can go as follows. The function $f$ is right point distal (if and) only if $\hat{f}$, defined by $\hat{f}(s) = f(s^{-1})$ for $s \in G$, is left point distal, which implies $F_1$, $F_1(s,t) = \hat{f}(s)$ is left minimal on $G$ [13; Theorem 3]. But then $F = \hat{F}_1$ is right minimal on $G$.]

The following examples show that the point distal hypothesis can be necessary for the conclusion of Theorem 1.

EXAMPLES AND REMARKS 2. 1. Let $G$ be the discrete group $T \otimes \{\pm 1\}$ (where $T$ is the circle group) with multiplication

$$(w,\varepsilon)(w_1,\varepsilon_1) = (ww_1^{\varepsilon}, \varepsilon\varepsilon_1).$$

On $T$, the function defined by $f_1(w) = \theta$ if $w = e^{i\theta}$, $0 \leqq \theta < 2\pi$, is easily seen to be minimal, but is not point distal, since

$$\lim_n R_{e^{-i/n}} f_1 = f_2,$$

where $f_2(w) = \theta$, if $w = e^{i\theta}$, $0 < \theta \leqq 2\pi$, and

$$\lim_n R_{e^{i/n}} f_2 = \lim_n R_{e^{i/n}} f_1 = f_1,$$

while $f_2 \neq f_1$. Further, the extension $F$ of $f_1$ to $G$ is not minimal. For, if

$$\lim_n R_{(e^{i/n},1)} F(w,\varepsilon) = \lim_n f_1(we^{i\varepsilon/n}) = \begin{cases} f_1(w), & \text{if } \varepsilon = 1 \\ f_2(w), & \text{if } \varepsilon = -1 \end{cases} = H(w,\varepsilon), \text{ say,}$$

then

$$R_{(e^{i\psi},\varepsilon)} H(1,1) \to F(1,1) = 0$$

if and only if $\psi \to 0^+$, while

$$R_{(e^{i\psi},\varepsilon)} H(1,-1) \to F(1,-1) = 0$$

if and only if $\psi \to 0^-$.

2. A similar example can be set up on $G = \mathbf{C} \otimes \{\pm 1\}$, where $\mathbf{C}$ is the complex plane and $(z,\varepsilon)(z_1,\varepsilon_1) = (z + \varepsilon z_1, \varepsilon\varepsilon_1)$. For $j = 1,2$, let $f_j$ be the "piecewise linear" function on $\mathbf{C}$ defined by $f_j(n+iy) = \sin n/|\sin n|$, if $n \neq 0$, and $f_j(iy) = (-1)^{j+1}$, $f$ being linear on each "interval" $\{(x,iy) \mid n \leqq x \leqq n+1\}$. Then one can check directly that $f_1$ is minimal on $\mathbf{C}$ (or one can prove that it is the pointwise limit of translates of the "piecewise linear" almost automorphic function $f_3$ on $\mathbf{C}$ defined by

$$f_3(n+iy) = \cos n/|\cos n|$$

for all integers $n$). If one extends $f_1$ to $F$ on $G$ as above, $F(z,\varepsilon) = f_1(z)$, and, if $\{m_n\}$ is a sequence of integers such that $m_n \pmod{2\pi} \to 0^-$, then, still as above, $\lim_n R_{(m_n,1)} F$ is a function which cannot be right-translated back to $F$. Thus, again, $F$ is not minimal.

3. The functions above provide easy illustrations of the fact (see [1]) that the minimal functions generally do not form an algebra: if $f_1$ and $f_2$ are as in (1) or (2), then neither $f_1 + f_2$ nor $f_1 f_2$ is minimal. We note as well that the almost automorphic function $f_3$ of (2) is point distal [4; Satz 4] and, hence, defining $F(z,\varepsilon) = f_3(z)$ does give a minimal function $F$ on $C \otimes \{\pm 1\}$.

4. It seems highly likely that, if $f_1$ is as in (2), the corresponding $F$ on $(\mathbf{C} \otimes T)_d$, the discrete euclidean group of the plane, is not minimal. However, $F \notin U(\mathbf{C} \otimes T)$ and we are unable to calculate pointwise limits of right translates of $F$.

The next theorem gives a setting, where $F$ (as in Theorem 1) will be in $U(G)$ if $f$ is in $U(G_1)$. The first part of it is due to Wu (see [20]).

THEOREM 3. *Let* $G = G_1 \otimes G_2$ *be a semidirect product of topological groups with* $G_1$ *compact. Let* $f \in C(G_1)$ *and define* $F$ *on* $G$ *by* $F(s,t) = f(s)$.

(i) *$F$ is Bohr almost periodic, hence is in $U(G)$.*
(ii) *The action of $G_2$ on $G_1$ is distal if and only if, for every $f \in C(G_1)$, the corresponding $F$ is a distal function on $G$.*

PROOF. (i) (see [20]).

$$A_\varepsilon = \{(s,t) \mid \| R_{(s,t)}F - F \| < \varepsilon\}$$

contains $\{e\} \times G_2$ for all $\varepsilon > 0$, hence $(G_1 \times \{e\})A_2 = G$ for all $\varepsilon > 0$. That $F \in U(G)$ now follows from [7].

(ii) We note first that, by Theorem A3, the action of $G_2$ on $G_1$ is distal if and only if the flow $(G_1 \otimes G_2, G_1)$,

$$(s,t) \colon s_1 \to s\sigma(t)s_1,$$

is distal. We next assume $(G_1 \otimes G_2, G_1)$ is distal and must show that each $F$, coming from $f \in C(G_1)$, as above, is distal. Suppose

$$H(s,t) = \lim_\alpha R_{(s_\alpha,t_\alpha)}F(s,t) = \lim_\alpha f(s\sigma(t)s_\alpha)$$

for all $(s,t) \in G$. Without loss, we can assume $\lim_\alpha s_\alpha = s_1$ and thus $H(s,t) = f(s\sigma(t)s_1)$ for all $(s,t) \in G$. It follows that the map that takes $s_1 \in G_1$ to the corresponding $H \in U(G)$ is a continuous homomorphism of flow $(G_1 \otimes G_2, G_1)$ onto flow $(G_1 \otimes G_2, X_F)$. Since $(G_1 \otimes G_2, G_1)$ is distal, so is $(G_1 \otimes G_2, X_F)$ [3; Corollary 5.7].

On the other hand, suppose that, for each $f \in C(G_1)$, the corresponding $F$ is distal. For $f \in C(G)$ and $s \in G_1$, let $F_s(s',t') = f(s'\sigma(t')s)$ for all $(s',t') \in G$. Then, if $M \subset C(G_1)$, the map $s \to (F_s)_{f \in M}$ effects a continuous homomorphism of $(G_1 \otimes G_2, G_1)$ into $(G_1 \otimes G_2, \pi\{X_F \mid f \in M\})$. If the map is one-to-one (e.g., if $M = C(G_1)$), then it effects an isomorphism of $(G_1 \otimes G_2, G_1)$ and a subflow of a product of distal flows; thus $(G_1 \otimes G_2, G_1)$ is distal [3; Proposition 5.8], and the proof is complete.

EXAMPLES 4. (i) Let $G'_j$ be abelian groups, $j = 1,2,3$, and let $\psi$ be a homomorphism of $G'_3$ into Hom $(G'_2, G'_1)$, which is an abelian group under addition. Then

$$G = G_1 \otimes G_2 = (G'_1 \times G'_2) \otimes G'_3$$

is a semidirect product of $G_1 = G'_1 \times G'_2$ and $G_2 = G'_3$, the group operation being

$$(s'_1,s'_2,s'_3)(s_1,s_2,s_3) = (s'_1 + s_1 + \psi(s'_3)s_2, s'_2 + s_2, s'_3 + s_3);$$

see [15; Theorem 7]. The action of $G'_3$ on $G'_1 \times G'_2$ is given by

$$s_3 : (s_1,s_2) \to (s_1 + \psi(s_3)s_2, s_2)$$

and is easily seen to be distal. Thus if $G'_1 \times G'_2$ is compact, $G$ satisfies all the hypotheses of Theorem 3. Further, if $f \in C(G'_1 \times G'_2)$, then $F$ (defined on $G$ by $F(s_1,s_2,s_3) = f(s_1,s_2)$ as in Theorem 3) is also left minimal on $G_d$. [For, if

$$\lim_\alpha L_{(s_{1\alpha},s_{2\alpha},s_{3\alpha})}F(s_1,s_2,s_3) = \lim_\alpha f(s_{1\alpha} + s_1 + \psi(s_{3\alpha})s_2, s_{2\alpha} + s_2)$$

$$= f(s'_1 + s_1 + \varphi(s_2), s'_2 + s_2) = H(s_1,s_2,s_3), \text{ say}$$

for some $s'_1 \in G'_1, s'_2 \in G'_2$ and $\varphi \in \mathrm{Hom}(G'_2, G'_1)$, and for all $(s_1,s_2,s_3) \in G$, then

$$\lim_\alpha L_{(-s'_1 + \varphi(s'_2), -s'_2, -s_{3\alpha})}H = F.]$$

We doubt that such $F$'s (on $G = G_1 \otimes G_2$ as in Theorem 3) will always be left minimal, even with the added assumption that the action of $G_2$ on $G_1$ is distal; certainly when the action of $G_2$ is allowed not to be distal or when $G_1$ is allowed not to be compact, such $F$'s need not be left minimal; see Example 4 (vi), below (details in [14]), and (iii) at the beginning of this section.

(ii) Let $K$ be a locally compact abelian group with dual group $\hat{K}$. Then the Heisenberg group $G = (T \times K) \otimes \hat{K}$, where $T$ is the circle group and

$$(w',s',\hat{s}')(w,s,\hat{s}) = (w'w\hat{s}'(s), s^1 + s, \hat{s}' + \hat{s}),$$

is of the form discussed in (i).

(iii) When $K = T$, $\hat{K} = \mathbf{Z}$ and Theorem 3 implies that, if $f$ is defined on $T$ by $f(w_1) = w_1$, then $F(w_1,w_2,n) = w_1$ defines a distal function $F$ on $G = (T \times T) \otimes \mathbf{Z}$. We wish to point out now that $F$, although left minimal on $G_d$, is not left point distal on $G_d$. For, suppose $h_1$ is the character on $T_d$ such that $h_1(w) = w$, if $w$ is a root of unity, and $h_1(w) = 1$ otherwise; since, for any finite $A \subset T$, there is a $w_A \in T$ such that $w_A w$ is not a root of unity for any $w \in A$, it follows that $\lim_A L_{w_A} H_1 = 1$ (pointwise on $T$). So, if $\{n_\alpha\}$ is a net in $\mathbf{Z}$ such that

$$\lim_\alpha L_{(1,1,n_\alpha)}F(w_1,w_2,n) = \lim_\alpha w_1 w_2^{n_\alpha} = w_1 h_1(w_2) = H(w_1,w_2,n), \text{ say,}$$

for all $(w_1,w_2,n) \in G$, then

$$\lim_A L_{(1,w_A,0)} H = F = \lim_A L_{(1,w_A,0)} F,$$

as required.

(iv) Another way to view Example (iii) is as follows. Let $\psi$ be the automorphism of $T^2 \simeq [0,1)^2$ corresponding to matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$,

$$\psi(a,b) = \big((a+b)(\mathrm{mod}\,1), b\big).$$

Iteration of $\psi$ gives an action of Z on $T^2$ and the resulting semidirect product $T^2 \otimes \mathrm{Z}$ is isomorphic to that given above.

Analogously, if $A$ is any upper triangular $n \times n$ matrix with integral entries and 1's on the diagonal, then iteration of the corresponding automorphism of $T^n$ gives a distal action of Z on $T^n$ (see [5], [16]).

(v) To get a non-distal action of Z on $T^n$, one starts with an invertible $n \times n$ matrix $A$ with integral entries, determinant 1, and an eigenvalue not equal to 1; e.g., $A = \left(\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right)$ yields an expansive automorphism of $T^2$ (see [17]) and a non-distal action of Z on $T^2$ (see [7]).

(vi) Another example of non-distal action appears in the semidirect product $G = T^T \otimes T_d$ with multiplication

$$(h',w')(h,w) = (h' R_{w'} h,\ w'w),$$

where $T$ is the circle group and $T_d$ is the same group with the discrete topology. (The non-normal subgroup, i.e., the acting subgroup, must have the discrete topology to make $G$ a topological group.) Here, let $h \in T^T$ be defined by $h(\zeta) = -1$ if $\zeta = -1$, $h(\zeta) = 1$ otherwise, and let $\{w_n\} = \{e^{i/n}\}$. Then

$$\lim_n R_{w_n} h = 1 = \lim_n R_{w_n} 1.$$

This example is used in [14] to illustrate some pathology of the Bohr almost periodic functions and appears in [15] in connection with the Ellis group of a distal flow $(\mathrm{R} \otimes \mathrm{R}^+, X)$, where $\mathrm{R} \otimes \mathrm{R}^+$ is the affine group of the line.

(vii) The construction of Furstenberg [6; 5.5, p. 27] also gives non-distal actions. As a special case, consider $\mathrm{R} \otimes \mathrm{R}^+$, the affine group of the line with multiplication

$$(x',y')(x,y) = (x' + y'x, y'y).$$

Of course, R is not compact, but one can extend the action of $\mathrm{R}^+$ on R to the almost periodic compactification AR of R, which we will view as $(\mathrm{R}_d)\hat{}$, the set of *all* characters on R. One then gets a semidirect product $G = \mathrm{AR} \otimes \mathrm{R}_d^+$ with multiplication

$$(h',y')(h,y) = (h' R_{y'} h, y' y),$$

where $R_{y'} h(x) = h(xy')$ for $h \in \mathsf{R}_a$, $x \in \mathsf{R}$, $y' \in \mathsf{R}^+$, and, again, the acting subgroup must have the discrete topology to make $G$ a topological group. Let $h \in \mathsf{AR}$ be the character such that $h(x) = e^{ix}$ if $x \in \mathsf{Q}$, $h(x) = 1$ otherwise. It follows readily that there exists a net of $y$'s such that

$$\lim R_y h = 1 = \lim R_y 1.$$

REMARK 5. In [20], Wu uses the non-distality of the action of $G_2$ on $G_1$ to conclude the existence of a function Bohr almost periodic, but not left Bohr almost periodic, on $G = G_1 \otimes G_2$. Example 4 (iii) shows that the weaker assumption of non-equicontinuity of the action of $G_2$ on $G_1$ can be enough to ensure the existence of such functions, and we point out that, in this example, $G_2 = \mathsf{Z}$ does act equicontinuously on each minimal subset of $G_1 = T \times T$ (see [2; Example V. 1.6]). It follows from ideas as in [9; Theorem 3.2] that, if the action of $G_2$ on compact $G_1$ is equicontinous, then all $F$'s on $G$ coming from $f$'s on $G_1$ as above will, in fact, be almost periodic.

In the corollary which follows, $G_1$ is not assumed to be compact, but the other hypotheses ensure that the necessary work can be done on $\mathrm{A}G_1 \otimes (G_2)_d$, where $\mathrm{A}G_1$ is the almost periodic compactification of $G_1$ and $(G_2)_d$ is the discrete version of $G_2$.

COROLLARY 6. *Let $G = G_1 \otimes G_2$ be a semidirect product with $G_2$ acting distally on $G_1$, and suppose the action of $G_2$ on $G_1$ extends to a distal action of $G_2$ on $\mathrm{A}G_1$. Then the formula*

(*) $F(s,t) = f(s), \quad (s,t) \in G,$

*extends each almost periodic function $f$ on $G_1$ to a distal function $F$ on $G_1 \otimes (G_2)_d$.*

EXAMPLES 7. (i) In Example 4 (vii), the distal action of $\mathsf{R}^+$ on $\mathsf{R}$ is seen not to extend to a distal action of $\mathsf{R}^+$ on $\mathsf{AR}$. Somewhat similarly, the action of $T$ on $\mathsf{C}$ in the euclidean group of the plane $\mathsf{C} \otimes T$ can be seen not to extend to a distal action of $T$ on $\mathsf{AC}$.

(ii) One can use the action of a group $G'$ on itself by inner automorphisms to form a semidirect product $G = G' \otimes G'$, and this action is often distal (see [18]). There is a problem with this setting: on the one hand, one wants $G'$ to have many almost periodic functions, i.e., $G'$ should be "fairly abelian", while, on the other hand, one wants the action of $G'$ on itself by inner automorphisms to be significant, i.e., $G'$ must be "somewhat non-abelian". The euclidean group of the plane $G' = \mathsf{C} \otimes T$ presents an example of what can happen. The action of $G'$ on itself by inner automorphisms is distal, but $AG' \simeq T$ (see [11]) and the resulting action of $G'$ on $AG'$ is trivial; hence $AG' \otimes G'$ is just the direct product $T \times (\mathsf{C} \otimes T)$. The same sort of collapsing happens for $G' = \mathsf{R} \otimes \mathsf{R}^+$, where $AG' \otimes G' \simeq \mathsf{R}^+ \times (\mathsf{R} \otimes \mathsf{R}^+)$ (the action of $G'$ on $AG'$ being distal, trivially, although the action of $G'$ on $G'$ by inner automorphisms is not distal), and also for $G' = (T \times T) \otimes \mathsf{Z}$ of Example 4 (iii), where

$$AG' \otimes G' \simeq (T \times \mathsf{Z}) \times ((T \times T) \otimes \mathsf{Z}).$$

The action of this last group on itself even extends to a distal action on the distal compactification $DG' \simeq (T \times T) \otimes D\mathsf{Z}$ (see [12]) of $G'$ (where $D\mathsf{Z}$ is the distal compactification of $\mathsf{Z}$), the extended action being given by

$$(w_1, w_2, n): (w_1', w_2', \mu) \to (w_1'(w_2')^n \mu(w_2)^{-1}, w_2', \mu).$$

(We must define $\mu(w_2)$. There is a canonical continuous homomorphism $\gamma$ of $D\mathsf{Z}$ onto $A\mathsf{Z} \simeq \hat{T}_d$ (see [1], [13]). $\mu(w_2)$ is just $\gamma\mu(w_2)$.) This example has been set up to answer a question concerning a possible generalization of Corollary 6: does the formula (*) always extend a distal $f$ on $G_1$ to a distal $F$ on $G_1 \otimes (G_2)_d$ if the action of $G_2$ on $G_1$ extends to a distal action on $DG_1$? That the answer is "no" is shown by the distal function $f$ defined by $f(w_1, w_2, n) = w_1$ on $(T \times T) \otimes \mathsf{Z} = G'$. The corresponding $F$ is readily seen to be minimal, but not point distal on $G' \otimes G_d'$.

A final example we want to mention is the "symmetrical" Heisenberg group $G = (T \times \mathsf{R}) \otimes \mathsf{R}$ with operation

$$(w', x', y')(w, x, y) = (w' w e^{ixy'}, x' + x, y' + y).$$

The formula (*) extends the almost periodic function $f$ on $T \times \mathsf{R}$, $f(w,x) = w$, to a function $F$ that is neither left nor right point distal on $G_d$, but is both left and right minimal on $G_d$.

**Appendix.**

Here we state and discuss briefly some theorems about embedding flows. The first two are due to Hahn [8] and were pointed out to the author by T.-S. Wu.

THEOREM A1. *Let $G = G_1 \otimes G_2$ be a semidirect product of locally compact groups with $G_2$ compact and let $(G_1, X)$ be a flow. Then*

$$(s,t): (x, t_0) \to (\sigma(tt_0)^{-1} sx, tt)$$

*makes $(G_1 \otimes G_2, X \times G_2)$ a flow that is minimal {point distal} [distal] if and only if $(G_1, X)$ is minimal {point distal} [distal].*

The point here is that $(G_1, X)$ is embedded in $(G_1 \otimes G_2, X \times G_2)$,

$$(G_1, X) \simeq (G_1 \times \{e\}, X \times \{e\}).$$

We indicate how this theorem shows the formula $F'(s,t) = f(\sigma(t^{-1})s)$ extends a distal $f$ on $G_1$ to a distal $F$ on $G_1 \otimes G_2$ (as asserted in the second paragraph of Section II). We assume $f$ is a distal function, that is, $(G_1, X_f)$ is a distal flow, and then form the extended flow $(G_1 \otimes G_2, X_f \times G_2)$, which is distal by the theorem. Hence, if $F_1$ is the continuous function,

$$F_1 : (X_f \times G_2) \to \mathbf{C}, \quad F_1(h,t') = h(e),$$

(where $e$ is the identity of $G_1$) then the function

$$(s,t) \to F\big((s,t)(f,e)\big) = F_1(R_{\sigma(t^{-1})s}f, t) = f\big(\sigma(t^{-1})s\big) = F'(s,t)$$

is a distal function on $G_1 \otimes G_2$ [1; section 4].

The setting of Theorem A1 can be broadened a little.

THEOREM A2. *Let $G$ be a locally compact group with closed normal subgroup $G_1$ such that $G_2 = G/G_1$ is compact. Then every minimal {point distal} [distal] flow $(G_1, X)$ can be extended to a minimal {point distal} [distal] flow $(G, Y)$.*

HINT AT PROOF. On $X \times G$, let $\rho$ be the equivalence relation with equivalence classes

$$\{(sx, ts^{-1}) \mid s \in G\}, \quad (x,t) \in X \times G.$$

Then $Y = (X \times G)/\rho$ and $(G, Y)$ is determined by the action,

$$t_1 : (x,t) \to (x, t_1 t), \quad x \in X, \ t_1, t \in G.$$

The verification that $(G, Y)$ is as claimed is straightforward; we mention only that the homeomorphism of $X$ into $Y$ is given by the map $x \to (x,e)$. (To be precise, the image of $x$ under this map is the $\rho$-class of $(x,e)$.)

We state one more theorem.

THEOREM A3. *Let $G_1 \otimes G_2$ be a semidirect product of locally compact groups with $G_1$ compact. Then*

$$(s,t) : s_1 \to s\sigma(t)s_1, \ G_1 \to G_1$$

*makes $(G_1 \otimes G_2, G_1)$ a flow. This flow is distal if and only if the flow $(G_2, G_1)$ is distal.*

NOTE ADDED IN PROOF. In Example 4 (i) we give a semidirect product formulation of the group $(G_1' \times G_2') \otimes G_3'$. Since this paper was submitted, we have noted that such groups are just the groups of Heisenberg type of H. Reiter, Comment. Math. Helv. 49 (1974), 333–364.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO
N6A 5B7 CANADA