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MULTI-VALUED NAMIOKA THEOREMS

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Abstract.

The paper has partially a survey character. The existing "point-valued" and "multi-valued Namioka theorems" on separate and joint continuity are reviewed and it is shown that, in most cases, the multi-valued ones can be deduced from their point-valued counterparts. This gives a unification and also some mild improvements of previous results of Christensen and Christensen–Kenderov.

1. Minimal usco maps.

In this paper the word "map" will be reserved for "set-valued map".

Let X, Y be the Hausdorff spaces and $F: X \to Y$ a map (i.e., F is a function from X into the power set of Y). Such an F is said to be upper semi-continuous at x (usc at x), if, for every open set V containing $F(x)$, there exists a neighbourhood U of x such that

$$F(U) := \bigcup \left\{ F(x) : x \in U \right\} \subset V;$$

F is upper semi-continuous (usc), if it is usc at x for every x in X. We will say, shortly, that F is usco, if it is usc and, for every x in X, $F(x)$ is *compact and non-empty*. We denote $\mathrm{Gr}(F)$ the graph of F:

$$\mathrm{Gr}(F) = \bigcup \left\{ \{x\} \times F(x) : x \in X \right\}.$$

Let $G: X \to Y$ be another map. Then G is said to be contained in F ($G \subset F$), if, for every x in X, $G(x) \subset F(x)$. Of course, the relation $\subset$ is a partial order relation in the family of maps, and we may consider minimal elements with respect to this relation. Thus a map F is *minimal* usco, if it is usco and does not contain properly any other usco map.

The proof of the following lemma is easy and may be found e.g. in [4].

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LEMMA 1. *Let F be a* usco *map, and suppose that G is a map such that $G \subset F$ and $G(x) \neq \emptyset$ for every x in X. Then G is* usco *iff* $\mathrm{Gr}(G)$ *is a closed set in $X \times Y$.*

2. Topological games.

Around 1930, S. Mazur invented what is now called the *Banach–Mazur game* (see Oxtoby [14, p. 26]) between players α and β on the real line. This game, call it $\boldsymbol{G}$, when transferred onto a general Hausdorff space X (see Choquet [1]), can be described as follows. Player β, who begins the game, plays at his nth move an open non-empty set U_n. Player α answers by choosing an open non-empty set V_n contained in U_n, then β chooses U_{n+1} in V_n etc.... We say that α *wins* whenever

$$\bigcap_{n=1}^{\infty} V_n \neq \emptyset \tag{1}$$

The following remarkable result, due to Krom ([12, Theorem 1], see also [9]), characterises Baire spaces in terms of the game $\boldsymbol{G}$.

LEMMA 2. *A Hausdorff space x is Baire iff it is β-unfavourable in the game $\boldsymbol{G}$.*

Thus, X is Baire iff whatever a strategy (i.e., a function s from finite sequences of open non-empty sets of X such that $U_n = s(U_1, V_1, \ldots U_{n-1}, V_{n-1})$, an open non-empty set, is contained in V_{n-1}) of β, the player α can choose his moves in such a way that he wins.

Let now $\mathscr{A}$ be a class of subsets of X and define a more restrictive game $\boldsymbol{G}_{\mathscr{A}}$ as follows. Player β, who begins the game, plays at his nth move an open non-empty set U_n. The player α answers by choosing an open set V_n *and* a set $A_n \in \mathscr{A}$. The choice, as in $\boldsymbol{G}$, must be such that $\ldots U_n \supset V_n \supset U_{n+1} \ldots$ but now α wins whenever

$$\bigcap_{n=1}^{\infty} V_n \cap \overline{\bigcup_{n=1}^{\infty} A_n} \neq \emptyset. \tag{2}$$

It is clear that a space X which is β-unfavourable for $\boldsymbol{G}_{\mathscr{A}}$, is so for $\boldsymbol{G}$ as well, i.e., is a Baire space by Lemma 2. Furthermore, *if X is Baire and contains a dense set A from $\mathscr{A}$, then X is β-unfavourable for the game $\boldsymbol{G}_{\mathscr{A}}$*, since then the condition (2) reduces to (1).

The following classes $\mathscr{A}$ have been considered so far:

$\mathscr{A}$ = the class of singletons in X; then $G_{\mathscr{A}}$ is denoted G_σ (Christensen [2], Saint-Raymond [16]).

$\mathscr{A}$ = the class of compact subsets in X; then $G_{\mathscr{A}}$ is denoted G_K (Deville [7], Talagrand [18]).

$\mathscr{A}$ = the class of K-analytic [see 15] subsets in X; then $G_{\mathscr{A}}$ is denoted G_A (Deville, loc. cit., Debs [6]).

3. Namioka theorems.

Let X, Z be Hausdorff spaces, and $f: X \times Z \to [0,1]$ a separately continuous function, i.e., f is continuous when one variable is fixed. The problem of finding conditions on X, Z ensuring that $X \times Z$ has a "large" subset of points (x,z) at which f is jointly continuous (i.e., continuous at (x,z) in the usual sense as a function of two variables) is classic. A breakthrough in this area of research is the 1974 paper of Namioka [13] (in which the information on the "pre-Namioka era" can be found).

Let $C(Z)$ be the familiar Banach space of bounded continuous functions from Z into R with the sup norm, and denote $C_p(Z)$ the space $C(Z)$ when equipped with the topology of pointwise convergence. Before we proceed further, note that the original problem may be given the following form:

Let a continuous function $f: X \to C_p(Z)$ be given. Find conditions on X and Z ensuring that X has a "large" subset of points at which $f: X \to C(Z)$ is continuous (i.e., at which f is norm continuous).

Then the main result of Namioka (loc. cit., cf. also Christensen [2, Proof of Theorem 1, Step 1]) asserts that, *if X is countably Čech complete and Z is a compact Hausdorff space, then* ("large" means "a dense G_δ", i.e.), *X has a dense G_δ of points at which f is norm continuous.*

The work by Namioka has triggered an extensive further research, and by now the above mentioned result has been generalised in many ways (see [2–7], [11], [16–19]). In order to discuss briefly the relevant results, it will be convenient to place ourselves in a still more general framework. Thinking of the space $C(Z)$ with the topology of pointwise convergence and the norm metric, of a Banach space Y with the weak topology and the norm metric, or of a dual Banach space Y^* with its weak* topology and the norm metric, we shall consider the triple (Y,ω,ρ) where $Y = (Y,\omega)$ is a Hausdorff space and ρ is a stronger metric on Y (i.e., ρ is such that the identity $i: (Y,\rho) \to (Y,\omega)$ is continuous).

We shall say that (Y,ω,ρ) is *appropriate* (for Namioka Theorem), if, for every Baire space X and any continuous function $f: X \to (Y,\omega)$, the set of all points at which f is ρ-continuous is *dense*.

We shall say that X is a *Namioka space*, if, for every compact Hausdorff space Z, and any continuous function $f: X \to C_p(Z)$, the set of all points at which f is norm continuous is *dense*.

Let $f: X \to Y$ be given. The *ρ-oscillation* $o(f, x_0)$ of f at x_0 is defined as follows.

$$o(f,x_0) = \inf\bigl(\rho\text{-diam}\, f(U) : U \in \mathcal{U}(x_0)\bigr)$$

where $\mathcal{U}(x_0)$ is a base for the neighbourhoods of x_0 and

$$\rho\text{-diam}\, f(U) = \sup\bigl(\rho(y,y') : y,y' \in f(U)\bigr).$$

It can directly be verified that the oscillation is an upper semi-continuous function, and therefore the set

$$\Xi_0(f) = \{x \in X : o(f,x) = 0\}$$

is always a G_δ set (possibly empty). Furthermore, it can easily be observed that $\Xi_0(f)$ is precisely the set of all points in X at which f is ρ-continuous. Thus in the definitions above the denseness condition means *automatically* that the set in question is a dense G_δ set. Another important observation is that the definition of ρ-oscillation can be applied to any (set-valued) map $F: X \to Y$ and then *$\Xi_0(F)$ is precisely the (G_δ) set of all those x in X at which F admits a singleton in Y as its value and its ρ-upper semi-continuous* (i.e., $F: X \to (Y, \rho)$ is usc at x).

THEOREM 1. *Let X be a Baire space, (Y, ω, ρ) an appropriate space, and $F: X \to (Y, \omega)$ a minimal* usco *map. Then $\Xi_0(F)$ is dense.*

In other words, appropriate spaces are also appropriate for minimal usco maps. In view of this result, one would perhaps like, at first, to show that Namioka spaces are "Namioka for minimal usco maps". After a closer look such a result seems highly improbable (although I have no counter-example) since Namioka spaces are defined precisely as those for which "point-valued" Namioka Theorem holds. However, we will see that *all Namioka spaces that we are able to recognize in topological terms* are indeed "Namioka for minimal usco maps".

J. P. R. Christensen was the first to realize [2], [3] that by using the language of topological games more general statements than the original Namioka Theorem (with range $C(Z)$) can be obtained. Then, it has been shown that spaces which are β-unfavourable for $\boldsymbol{G}$ (Saint-Raymond [16]), for $\boldsymbol{G}_K$ (Talagrand [18]), for $\boldsymbol{G}_A$ (Debs [6]), are all Namioka. The result of Debs seems to give the most general class of "recognizable" Namioka spaces known so far. On the other hand, Baire spaces cannot be reached, in general, since Talagrand [18] gave an example of an α-favourable for $\boldsymbol{G}$ (whence Baire) space X that is not a Namioka space.

THEOREM 2. *Let X be a Hausdorff space that is β-unfavourable for the game $\mathbf{G}_A$. Let Z be a compact Hausdorff space and $F: X \to C_p(Z)$ a minimal* usco *map. Then $\Xi_0(F)$ is dense in X.*

4. The proofs.

Both Theorems rely, essentially, on the same argument, and therefore both proofs will be carried out together.

Recall that a function p from a Hausdorff space X_1 onto a Hausdorff space X_2 is *perfect*, if it is continuous, closed, with compact fibers. Then its inverse $P = p^{-1}: X_2 \to X_1$ is a usco map, and p is *irreducible* whenever P is minimal usco.

Let U be a subset of X_1. Then we will denote

$$P^{-1}(U) = \{x \in X_2 : P(x) \subset U\}.$$

LEMMA 3. *Let X_1, X_2 be Hausdorff spaces and let $p: X_2 \to X_1$ be perfect and irreducible. Suppose that X_1 is β-unfavourable for the game $\mathbf{G}$ (respectively $\mathbf{G}_\sigma$, $\mathbf{G}_K$, $\mathbf{G}_A$). Then X_2 is so too.*

PROOF. We shall first indicate the idea of the proof, which is the same for all games considered. Let s be a strategy for the player β in X_2. Then

$$t = P^{-1} \circ s$$

is the strategy for β in X_1. The player α can win in X_1, thus transferring back his moves to X_2, he wins against s therein. Hence X_2 is β-unfavourable.

(1) We suppose that X_1 is β-unfavourable for $\mathbf{G}$.

Let $U_1^{(2)}$ be the first move of β in X_2 according to his strategy s. Define

$$U_1^{(1)} = \{x \in X_1 : P(x) \subset U_1^{(2)}\}$$

as the first move of β in X_1. $U_1^{(1)}$ is open by the upper semi-continuity of P and we have to check that it is non-empty. However, if it is empty, then the map

$$x \mapsto P(x) \cap (X_2 \setminus U_1^{(2)})$$

is usco and is properly contained in P. This is impossible by the minimality of P.

Let $V_1^{(1)}$ be the first move of α in X_1. Then define

$$V_1^{(2)} = P(V_1^{(1)}) = p^{-1}(V_1^{(1)})$$

as the first move of α in X_2. Now,

$$U_2^{(2)} = s(U_1^{(2)}, V_1^{(2)})$$

is the 2nd move of β in X_2, and we set $U_2^{(1)} = P^{-1}(U_2^{(2)})$, which is as needed by the argument used at the first step. Continuing this process, the strategy $t = P^{-1} \circ s$ is defined for β in X_1.

As X_1 is β-unfavourable, α can choose his moves so that $\bigcap_{n=1}^{\infty} V_n^{(1)} \neq \emptyset$. Then

$$\bigcap_{n=1}^{\infty} V_n^{(2)} = \bigcap_{n=1}^{\infty} p^{-1}(V_n^{(1)}) = p^{-1}\left(\bigcap_{n=1}^{\infty} V_n^{(1)}\right)$$

is non-empty, i.e., X_2 is β-unfavourable for $\boldsymbol{G}$.

(2) In other cases the proof is similar. We only consider the game $\boldsymbol{G}_A$ of Deville. According to its rules, the player α chooses now $V_1^{(1)}$ *and* a K-analytic set $A_n^{(1)}$ at his nth move in X_1. Since one of possible definitions of K-analytic set is that it is the image of irrationals under a usc compact-valued map (see [15]), $A_n^{(2)} = P(A_n^{(1)})$ is K-analytic in X_2. It remains to show that

$$\bigcap_{n=1}^{\infty} V_n^{(2)} \cap \overline{\bigcup_{n=1}^{\infty} A_n^{(2)}} \neq \emptyset.$$

Suppose the contrary. Then, since $V_n^{(2)} = p^{-1}(V_n^{(1)})$,

$$p\left(\bigcap_{n=1}^{\infty} V_n^{(2)}\right) = \bigcap_{n=1}^{\infty} V_n^{(1)}$$

is disjoint from the closed set $p\left(\overline{\bigcup_{n=1}^{\infty} A_n^{(2)}}\right)$. Hence $\overline{\bigcup_{n=1}^{\infty} A_n^{(1)}}$ is disjoint with $\bigcap_{n=1}^{\infty} V_n^{(1)}$: a contradiction with the winning choice of α in X_1.

We proceed to the proofs of Theorems 1 and 2. Let X be a Baire space (respectively β-unfavourable for $\boldsymbol{G}_A$), and F a minimal usco map into (Y, ω) (respectively $C_p(Z)$). Consider the first projection

$$p_1 : \mathrm{Gr}(F) \to X.$$

Then p_1 is perfect and irreducible. Indeed, the inverse to p_1:

$$x \longmapsto \{x\} \times F(x)$$

is minimal usco since, by Lemma 1, $\mathrm{Gr}(F)$ is the minimal graph of a usco map. Thus, by Lemma 3, $\mathrm{Gr}(F)$ is Baire (applying this Lemma twice, but see also Remark 4 next section) or β-unfavourable for $\boldsymbol{G}_A$ according to the case considered. Let

$$p_2 : \mathrm{Gr}(F) \to Y \text{ (respectively } C(Z))$$

be the 2nd projection. In both cases, by "point-valued" Namioka Theorem, $\Xi_0(p_2)$ is dense in $\mathrm{Gr}(F)$. Since X is (at least) Baire, it suffices to show that

$$\Xi_\varepsilon(F) = \{x \in X : o(F, X) < \varepsilon\}$$

is dense in X. Suppose the contrary. Then there exists an open set W_0 in X such that, for every open non-empty set $V \subset W_0$, ρ-diam $F(V) \geqq \varepsilon$. On the other hand, take $w \in \Xi_0(p_2) \cap p_1^{-1}(V)$ and an open ball U with diameter less than ε around $p_2(w)$. Then, if

$$W = F^{-1}(U),$$

we have

$$F(W) = p_2(p_1^{-1}(W)) = U.$$

Thus W is open non-empty subset of W_0 such that ρ-diam $F(W) < \varepsilon$. We have obtained a contradiction. Theorems 1 and 2 are proved.

5. Concluding remarks.

1) The first multi-valued Namioka Theorem (corresponding to our Theorem 2 above) has been obtained by J. P. R. Christensen ([4, Theorem 1]) who noticed that his "point-valued" argument in [2] can be suitably modified. A predecessor to our Theorem 1 also exists (Christensen–Kenderov [5]). Again, it is obtained, for (Y^*, weak*) with the Radon–Nikodym property, by a suitable adaptation of the "point-valued" argument. Another appropriate spaces are given in [6], [7], and it may be noted that, generally, a weak or weak* Banach space whose unit ball can be "fragmented by the norm metric" (cf. [8], [10]) is appropriate.

2) Minimal usco maps are by no means rare creatures. In fact, by applying Kuratowski–Zorn Lemma and by taking into account e.g. Lemma 1 above, it follows easily that *any* usco *map contains a minimal one*. Multi-valued Namioka Theorems give the first important insight into their structure. Further results in this direction (i.e., giving rather more precise descriptions of minimal usco maps) have been obtained, applying Namioka Theorems, by the author and will appear elsewhere.

3) Christensen and Christensen with Kenderov give also multi-valued versions of B. E. Johnson's theorem ([4, Theorem 3], [5, Theorem 1.8]). It may be of interest to note that also these results can be obtained as corollaries to Namioka Theorem rather than the independent ones. The following statement "contains" both of them.

THEOREM 3. *Let* $F: X \to (Y, \omega, \rho)$ *be a minimal* ω-usco *map such that* $\Xi_0(F)$ *is dense in* X. *If every family of disjoint open sets in* X *is at most countable, then the range of* F *is* ω-*separable. Moreover, if* (Y, ω) *is a Banach space with its weak topology and* ρ *is the norm metric, then the range of* F *is norm separable.*

PROOF. Observe first that $F(\Xi_0(F))$ is ρ-separable. Indeed, otherwise there exists $\varepsilon > 0$ and an uncountable subset H in $F(\Xi_0(F))$ such that the open balls

$$B(z,\varepsilon) = \{y \in Y : \rho(z,y) < \varepsilon\}, \quad z \in H$$

are disjoint. As F is ρ-usc at any point of $\Xi_0(F)$, $F^{-1}(B(x,\varepsilon))$ are disjoint open sets in X which is impossible by the condition on x. Now, the closure E of $F(\Xi_0(F))$ in (Y, ω) contains the range of F by its minimality, which shows the first part of the theorem. The second follows by the usual argument.

4) Lech Drewnowski has shown to me a simple game-free proof of the Baire space part of Lemma 3. This is based on the fact that, if V is open and dense in X_2, then $P^{-1}(V)$ is so too in X_1. I decided to keep the unified game-theoretic approach.

5) I would like to thank Professor Isaac Namioka for the discussions, concerning Namioka's Theorem for usco maps, during his stay in the Banach Center, Warsaw, in the Spring 1984. The present paper is a development of his idea of looking onto the graph and the second projection.

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