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THE BONNET–MYERS THEOREM IS TRUE FOR RIEMANNIAN HILBERT MANIFOLDS

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Abstract.
The Bonnet–Myers Theorem is shown to be true for Riemannian Hilbert manifolds.

The result known as the Bonnet–Myers Theorem gives an upper bound for the diameter of Riemannian manifolds with sectional curvature bounded from above. See [4] for a discussion of the history behind this result.

In this note we show that this result remains true also in the category of Riemannian Hilbert manifolds.

Let $d(M)$ denote the diameter of M defined by

$$d(M) = \sup_{p,q \in M \times M} d(p,q).$$

Here, $d(\cdot,\cdot)$ denotes geodesic distance in M. The Bonnet–Myers Theorem can be stated as follows:

THEOREM 1. *Let M be a complete connected C^∞ Riemannian Hilbert manifold. Assume that K, the sectional curvature of M, satisfies $K \geqq K_1$ for some constant $K_1 > 0$. Then the diameter of M satisfies*

$$d(M) \leqq \pi/\sqrt{K_1}.$$

REMARK. If M is finite dimensional then, as is well known, under the above conditions both M and its universal covering space are compact which implies that the first fundamental group $\pi_1(M)$ is finite. It is unclear whether there is any reasonable generalization of this to the infinite dimensional case.

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PROOF OF THEOREM 1. We will prove the Bonnet–Myers Theorem by noting that the Morse–Schoenberg Theorem makes sense for Riemannian Hilbert manifolds. To see this, it is practical to work in the setting of Sobolev manifolds of curves, see [3, Chapter 2] for a description of the classical and [2, §6] for the Hilbert manifold case. In particular, there is a naturally defined Riemannian metric, which we will denote $\langle \cdot, \cdot \rangle_1$, on the H^1-loop space $\Omega_{pq}M$ of H^1-curves connecting p and q.

Let $E(c)$ denote the energy function with respect to the Riemannian metric of M, calculated at $c \in \Omega_{pq}M$. Then the Hessian, $D^2E(c)$ is a bounded symmetric bilinear form and we can define a selfadjoint operator

$$A_c\colon T_c \to T_c \text{ by } \langle A_c X, Y \rangle_1 = D^2E(c)(X,Y),$$

cf. [5, Theorem 5.35]. Here we have used T_c to denote $T_c\Omega_{pq}M$, the tangent space at c of $\Omega_{pq}M$.

Since A_c is selfadjoint, it follows from the general results of spectral theory for selfadjoint operators (see e.g. [5, Chapter 7]) that there is an orthogonal decomposition of T_c into negative, zero and positive eigenspaces of A_c:

$$(*) \qquad T_c = T_c^- \oplus T_c^0 \oplus T_c^+.$$

Furthermore, these are closed.

In the case where M is finite dimensional, it is known that $A_c = I + k_c$, where k_c is a compact operator (cf. [3, Lemma 2.5.2]). From this it follows that T_c^- and T_c^+ are finite dimensional in this case. This is not true in general, however.

For our present purposes it is sufficient that the decomposition $(*)$ exists. We now define the *index* of c to be the dimension of T_c^- and the *nullity* of c to be the dimension of T_c^0. We allow both the index and the nullity to be equal to infinity.

With this definition of index and nullity, the Morse–Schoenberg Comparison Theorem makes sense even for Hilbert manifolds.

We are now able to prove the Bonnet–Myers Theorem along the standard lines. With the assumptions of Theorem 1, assume that there exist p,q so that $d(p,q) > \pi/\sqrt{K_1}$. Then Ekelands version of the Hopf–Rhinow Theorem for Riemannian Hilbert manifolds [1, Theorem B] implies that there exists a $q' \in M$ such that $d(p,q') > \pi/\sqrt{K_1}$ and p and q' are connected by a minimizing geodesic. But this is impossible, by the Morse–Schoenberg Theorem.

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