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SPECTRAL PROPERTIES OF VAGUELY ELLIPTIC PSEUDO DIFFERENTIAL OPERATORS WITH MOMENTUM DEPENDENT LONG RANGE POTENTIALS USING TIME DEPENDENT SCATTERING THEORY – II

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Abstract.

Enss' method is developed for $h_0(P)+W_S+W_L(Q,P)$ on $L^2(\mathsf{R}^n)$, where (i) h_0 is a smooth real valued function (ii) W_S is a short range perturbation and (iii) $W_L(Q,P)$ is a smooth long range perturbation. Asymptotic completeness is proved, when $h_0(\infty)=\infty$ and the closure of the set of critical values is countable.

1. Introduction.

The method of Enss [4] was extended to include vaguely elliptic operators with short range potentials in [19]. In [10] we tried to extend the results of [19] to include long range perturbations using the techniques of [5]. We could prove in [10] asymptotic completeness for the pair (H_0, H) with $H_0 = h_0(P)$, $H = H_0 + W_S(Q,P) + W_L(Q,P)$, if

(i) W_S, W_L are short and long range perturbations,
(ii) h_0 is a C^∞ function of at most polynomial growth such that $h_0(\infty) = \infty$,
(iii) $C = \{\xi : h_0'(\xi) = 0 \text{ or } \det h_0''(\xi) = 0\}$ is a set of measure 0 and,
(iv) the closure of $\{h_0(\xi) : \xi \in C\}$ is countable.

While condition (iv) is satisfied when $h_0(\xi_1,\xi_2) = (\xi_1^2+\xi_2^2)^r$, $r>0$ or $h_0(\xi_1,\xi_2) = \xi_1^4+\xi_2^4+a(\xi_1^2+\xi_2^2)$, $a>0$ or $h_0(\xi_1,\xi_2) = \xi_1^4+\xi_2^6+a(\xi_1^2+\xi_2^2)$, $a>0$ it is not satisfied for the elliptic case $h_0(\xi_1,\xi_2)=\xi_1^4+\xi_2^4$. The main aim of this article is to overcome this highly unsatisfactory state of affairs. For a subsidiary aim see next section.

Recently in [6] a stationary theory has been developed for simply characteristic operators with short range potentials. In [14], [11] Enss'

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method is developed for simply characteristic operators with short range potentials. In [13] Enss' theory is developed for parabolic and hyperbolic operators of order 2 with long range potentials. While for the pair $(P_1 P_2, P_1 P_2 + W_S)$ [on $L^2(\mathbb{R}^2)$], [6], [11] could prove asymptotic completeness, for the pair $(P_1^2 P_2^2, (P_1 P_2 + W_S)^2)$ both [6] and [11] remain silent. But the "philosophy" [2] of invariance principle [18], [1], [7] in scattering theory demands some result for the pair $((P_1^2 P_2^2, (P_1 P_2 + W_S)^2)$. The subsidiary aim of this article is to prove some results for the pair $(h_0(P), h_0(P) + W_S + W_L))$ with h_0 as general as possible. In particular we have

(i) the range of $\Omega_{\pm}$ is contained in the space of scattering states of H and,
(ii) a characterization of the orthogonal complement of Range $\Omega_{\pm}$ in the space of scattering states.

For details cf. Theorem 2.3 (v), (vi), (vii). For related works on Enss' method see [16], [9] and references therein.

Finally we sketch the contents. In section 2 we state the assumptions on the Hamiltonians and the main results. In section 3 we prove the existence of the wave operator. In section 4 we construct, by the method of iterations, a position – momentum dependent evolution to be used in section 5. In section 5, the evolution of section 4 approximates the total evolution. In section 6, Range $\Omega_{\pm}$ is characterized and, in the vaguely elliptic case, asymptotic completeness proved.

The ideas of [13] are freely used in this article.

2. Statement of the result.

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}^n$ and $C_0^\infty(\mathbb{R}^n)$ denote the space of all smooth functions with compact support. On $\mathscr{S}$ we define the Fourier transform $\mathscr{F}$ by

$$(\mathscr{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{1}{2}n} \int dx \; \exp[-ix\xi] f(x).$$

$\mathscr{F}$ can be extended, by the Plancheral Theorem, to a unitary map on $L^2(\mathbb{R}^n)$ and this extension shall also be denoted by the same letter $\mathscr{F}$.

Let $Q = (Q_1, \ldots, Q_n)$, $P = (P_1, \ldots, P_n)$ be the position and momentum operators on $L^2(\mathbb{R}^n)$ given by $(Q_j f)(x) = x_j f(x)$, $P_j = -iD_j$, $D_j = \partial/\partial x_j$. We denote by $F(M)$ the indicator function of the set M.

For any self adjoint operator A on $L^2(\mathbb{R}^n)$ we define the spaces of scattering states $M(1, \pm\infty, A)$, $M(2, \pm\infty, A)$ cf. [1], [16], [9]. Let $A_t = \exp[-itA]$. Put

$$M(1,\pm\infty,A)=\{f\in L^2(\mathrm{R}^n):\ \lim_{t\to\pm\infty}\|F(|Q|\leqq r)A_t f\|=0 \quad \text{for each } r>0\}$$

$$M(2,\pm\infty,A)=\{f\in L^2(\mathrm{R}^n):\ \lim_{T\to\infty}\frac{1}{T}\int_0^{\pm T}dt\,\|F(|Q|\leqq r)A_t f\|=0 \quad \text{for each } r>0\}.$$

The linear spaces $M(1,\pm\infty,A)$, $M(2,\pm\infty,A)$ are easily seen to be closed.

For A as above we denote by $H_p(A)$, $H_c(A)$, $H_{ac}(A)$ the closed linear span of eigenvectors of A, the continuous space of A and the absolutely continuous space of A, respectively; $E_p(A)$, $E_c(A)$, $E_{ac}(A)$ stand for the orthogonal projections onto $H_p(A)$, $H_c(A)$, and $H_{ac}(A)$, respectively.

We put $\langle Q\rangle=(1+Q^2)^{\frac{1}{2}}$ and $\langle t\rangle=(1+t^2)^{\frac{1}{2}}$ for real t. All generic constants will be denoted by the same letter K.

Now we state our assumptions A1, ..., A8. Theorem 2.3 is valid under A1, ..., A4. Theorem 2.3 is about

(i) the existence of the wave operators and
(ii) the relation between the ranges of the wave operators and the scattering states.

For Theorem 2.4 on completeness we need, in addition the assumptions A5, A7, A8.

A1 (Condition on the free Hamiltonian). $h_0\colon \mathrm{R}^n\to\mathrm{R}$ is a C^∞ function such that h_0 and all its derivatives are of at most polynomial growth. The free Hamiltonian H_0 is $H_0=h_0(P)$ with its maximal domain.

A2 (Condition on the long range perturbation). $W_L\colon \mathrm{R}^n\times\mathrm{R}^n\to\mathrm{R}$ is a C^∞ function such that there is some δ in $(0,1]$ so that for any bounded subset B of R^n we have

$$\sup_{\xi\in B}|D_\xi^\alpha D_x^\beta W_L(x,\xi)|\leqq K(B,\alpha,\beta)(1+|x|)^{-|\beta|-\delta}$$

for all multi indices α,β. Here $K(B,\alpha,\beta)$ is a constant. The pseudo differential operator $W_L(Q,P)$ is given by

$$[W_L(Q,P)f](q)=(2\pi)^{-\frac{1}{2}n}\int d\xi\ W_L(q,\xi)\hat{f}(\xi)\exp[iq\xi].$$

We assume that W_L maps $\mathscr{S}$ into $L^2(\mathrm{R}^n)$. (This may impose some condition on the growth of ξ in $W_L(x,\xi)$.)

A3 (Condition on the short range perturbation). $\operatorname{Dom} W_S\supseteqq\mathscr{S}$ and there exists some $\varepsilon_0>0$ such that $W_S\varphi(P)\langle Q\rangle^{1+\varepsilon_0}$ is a bounded operator for each φ in $C_0^\infty(\mathrm{R}^n)$.

A4 (Condition on the total Hamiltonian). The operator $H = H_0 + W_S + W_L(Q,P)$ defined on $\mathscr{S}$ is symmetric and has a self adjoint extension denoted by the same letter H.

A5 (Compact perturbation). The operator $(H+i)^{-1} - (H_0+i)^{-1}$ is compact.

A6 (Quasi divergent condition [3]). There exists an integer $M \geqq 0$ such that

$$\lim_{|\xi|\to\infty} \sum_{|\alpha|\leqq M} |D^\alpha h_0(\xi)| = \infty \quad \text{and}$$

$$\sum_{|\beta|=M+1} |D^\beta h_0(\xi)| \leqq K\{1 + \sum_{|\alpha|\leqq M} |D^\alpha h_0(\xi)|\}$$

for a suitable constant K.

A7 (Vaguely elliptic condition). $\lim_{|\xi|\to\infty} |h_0(\xi)| = \infty$.

A8 (Condition on the critical values). If $C_v = \{h_0(\xi) : h_0'(\xi) = 0\}$ is the set of critical values for h_0, then $\bar{C}_v$, the closure of C_v, is a countable set.

With the above assumptions we develop the theory now. For example where A1, ..., A5, A7, A8 are satisfied cf. [19], [10], [20], [21].

With h_0 as in the assumption A1 let us put

$$G = \{\xi : h_0'(\xi) \neq 0\}. \tag{2.1}$$

THEOREM 2.1. *Let* A1 *hold. Then*

(i) $H_{ac}(H_0) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset G\} = \mathscr{F}^{-1} L^2(G)$,

(ii) $H_{ac}(H_0) \subseteqq M(1, \pm\infty, H_0) \subseteqq M(2, \pm\infty, H_0)$.

THEOREM 2.2. *Let* A1, ..., A5 *and* A6 *or* A7 *hold. Then*

(i) $F(|Q| \leqq r)(H+i)^{-1}$ *is compact for each* $r > 0$.

(ii) $H_{ac}(H) \subseteqq M(1, \pm\infty, H) \subseteqq H_c(H) \subseteqq M(2, \pm\infty, H)$.

THEOREM 2.3. *Let* A1, ..., A4 *hold. Define the free and total evolutions* U_t, V_t *by* $U_t = \exp[-itH_0]$, $V_t = \exp[-itH]$. *Then we can find a* C^∞ *function* $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that the following hold:*

(i) $\Omega_\pm = \text{s-lim}_{t\to\pm\infty} V_t^* \exp[-iX(t,P)] E_{ac}(H_0)$ *exists.*

(ii) $\Omega_\pm^* \Omega_\pm = E_{ac}(H_0)$,

(iii) (*intertwining relations*) $V_t \Omega_\pm = \Omega_\pm U_t$ *for all* t,

(iv) $\operatorname{Range} \Omega_\pm \subseteqq H_{ac}(H) \subseteqq H_c(H)$,

(v) $\operatorname{Range} \Omega_\pm \subseteqq M(1, \pm\infty, H) \subseteqq M(2, \pm\infty, H)$,

(vi) $M(1,\pm\infty,H)\ominus \text{Range}\,\Omega_\pm$
$= \{f \in M(1,\pm\infty,H): \lim_{t\to\pm\infty} \|\gamma(P)V_t f\| = 0$
for each γ *in* $C_0^\infty(G)\}$,

(vii) $M(2,\pm\infty,H)\ominus \text{Range}\,\Omega_\pm$
$= \{f \in M(2,\pm\infty,H): \lim_{T\to\infty} (1/T)\int_0^{\pm T} dt\, \|\gamma(P)V_t f\| = 0$
for each γ *in* $C_0^\infty(G)\}$.

THEOREM 2.4. *Let* A1,... A5, A7, A8 *hold. Then*

(i) $\text{Range}\,\Omega_\pm = H_{ac}(H) = M(1,\pm\infty,H) = H_c(H) = M(2,\pm\infty,H)$.
(ii) *Any eigenvalue of* H *not in* $\overline{C}_v$ *is of finite multiplicity. All such eigenvalues can accumulate only at the points of* $\overline{C}_v$.

REMARK 2.5. Using the assumption A5 and Stone–Weierstrass Theorem one can easily show that the operator $\varphi(H)-\varphi(H_0)$ is compact for each continuous function φ on R with $\varphi(\pm\infty)=0$. For details we refer to [16].

3. Proof of Theorem 2.1., 2.2., 2.3. (i)...(v).

PROOF OF THEOREM 2.1. For (i) we refer to Theorem 1 of [20]. For (ii) note that we have clearly $M(1,\pm\infty,A) \subseteqq M(2,\pm\infty,A)$ for any self adjoint A. Since $M(1,\pm\infty,H_0)$ is closed it suffices to show that the dense subspace D of $H_{ac}(H_0)$ given by

$$D = \{f \in \mathscr{S} : \hat{f} \in C_0^\infty(G)\}$$

is $\subseteqq M(1,\pm\infty,H_0)$. For any f in D choose $\varphi \in C_0^\infty(G)$ such that $\varphi(P)f = f$. The result follows by noting that $F(|Q| \leqq r)\varphi(P)$ is compact for each $r > 0$.

PROOF OF THEOREM 2.2 (i). By the compactness condition A5 it suffices to show the compactness of $F(|Q| \leqq r)(H_0+i)^{-1}$. If case A7 holds, this is clear. If A6 holds then apply Theorems A1 and 9 of [3] to get the result.

(ii) By (i) it is clear that $H_{ac}(H) \subseteqq M(1,\pm\infty,H)$. By RAGE Theorem [see 18] we get $H_c(H) \subseteqq M(2,\pm\infty,H)$. It remains to show $M(1,\pm\infty,H) \subseteqq H_c(H)$. For this define the space of bound states by

$$M(0,\pm,H) = \{f: \lim_{r\to\infty} \sup_{t\gtrless 0} \|F(|Q| \geqq r)V_t f\| = 0\}.$$

Then using the techniques of the proof of Propositions 7.1, 7.2, of [1], we see that $H_p(H) \subseteqq M(0,\pm,H)$, $M(0,\pm,H) \perp M(1,\pm\infty,H)$, and finally $M(1,\pm\infty,H) \subseteqq H_c(H)$.

Following [8], [12] we cut down the long range potential W_L in a time dependent manner in the position variable. In section 4 we shall further restrict the momentum variable.

Take χ_0 in $C^\infty(\mathrm{R}^n)$ so that

(3.1) $$0 \leqq \chi_0 \leqq 1, \quad \chi_0 = 1 \text{ for } |x| \geqq 2, \quad 0 \text{ for } |x| \leqq 1.$$

Set for ρ in $(0,1]$ and real t

(3.2) $$W(\rho,t,x,\xi) = \chi_0(\rho x)\chi_0(x \log\langle t\rangle/\langle t\rangle)W_L(x,\xi)$$

where, we recall, $\langle t\rangle = (1+t^2)^{\frac{1}{2}}$ and $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$. Then it is easy to see, by the long range assumption A2, as in [8] that, with $\delta_0 = \delta/3$,

(3.3) $$\sup_{\xi\in B} |D_\xi^\alpha D_x^\beta W(\rho,t,x,\xi)| \leqq K(B,\alpha,\beta)\rho^{\delta_0}\langle t\rangle^{-|\beta|-\delta_0}$$

for each bounded subset B of R^n. Here $K(B,\alpha,\beta)$ is independent of ρ, t. For future use in section 5 we note that

(3.4) $$\sup_{\xi\in B} |D_\xi^\alpha D_x^\beta W(\rho,t,x,\xi)| \leqq K(B,\alpha,\beta)\rho^{\delta_0}\langle t\rangle^{-|\beta|-\delta_0}\langle x\rangle^{-\delta_0/2}$$

for any bounded set B of R^n.

Without loss of generality we can assume, decreasing δ_0 if necessary, that $\delta_0 \notin \{1,\frac{1}{2},\frac{1}{3},\dots\}$. Choose the positive integer m_0 such that

(3.5) $$m_0\delta_0 < 1 < (m_0+1)\delta_0.$$

Now define $Y(m,\rho,t,\xi)$ for $m = 1,\dots m_0$ by

(3.6) $$\begin{aligned} Y(0,\rho,t,\xi) &= 0 \\ Y(m,\rho,t,\xi) &= \int_0^t ds\, W(\rho,s,sh_0'(\xi)+Y_\xi'(m-1,\rho,s,\xi),\xi) \end{aligned}$$

Put

(3.7) $$X(\rho,t,\xi) = th_0(\xi) + Y(m_0,\rho,t,\xi).$$

Now we have

LEMMA 3.1. *Let* A1, A2 *hold and* X, Y *be as in* (3.6), (3.7). *Let* $f \in \mathscr{S}$ *be such that* $\hat f \in C_0^\infty(G)$ [cf. (2.1)]. *Then*

(i) $\|\langle Q\rangle^{-\sigma}\exp[-iX(\rho,t,P)]f\| \leqq K(\sigma,\rho,f)\langle t\rangle^{-\sigma}$ *for each* $\sigma \geqq 0$,

(ii) $\int_{-\infty}^{\infty} dt\,\|[W_L(Q,P) - W(\rho,t,Q,P)]\exp[-iX(\rho,t,P)]f\| < \infty$,

(iii) $\int_{-\infty}^{\infty} dt\,\|[W(\rho,t,Q,P) - \partial Y(m_0,\rho,t,P)/\partial t]\exp[-iX(\rho,t,P)]f\| < \infty$,

(iv) $\lim_{s\to\pm\infty} Y(m_0,\rho,t+s,\xi) - Y(m_0,\rho,s,\xi) = 0$ *for each* ξ *and real* t.

PROOF. (i) and (ii) are easy consequences of the method of stationary phase [5]. For (iii), cf. [12]. For (iv) also one can refer to [12].

We are now ready to prove Theorem 2.3 (i)... (v).

PROOF OF THEOREM 2.3. (i)... (v). (i) For f as in Lemma 3.1. it is easy to see that

$$\int_{-\infty}^{\infty} dt \left\| \frac{d}{dt} V_t^* \exp[-iX(t,P)]f \right\| < \infty,$$

where $X(t,P) = X(1,t,P)$. Now the result follows by density arguments using Theorem 2.1. (i).

(ii) is clear. (iii) follows from Lemma 3.1 (iv) for $\rho = 1$. (iv) is a consequence of (iii).

(v) With f as in Lemma 3.1. we get

$$\lim_{t\to\pm\infty} \| V_t \Omega_\pm f - \exp[-iX(t,P)]f \| = 0 \tag{3.8}$$

and, by Lemma 3.1 (i),

$$\lim_{t\to\pm\infty} \| F(|Q| \leqq r) \exp[-iX(t,P)]f \| = 0 \quad \text{for each } r > 0. \tag{3.9}$$

Now by (3.8) and (3.9), $\Omega_\pm f \in M(1, \pm\infty, H)$, Now the result follows by density arguments.

4. Iterations for the solution of a Hamilton Jacobi equation.

For each real valued φ in $C_0^\infty(G)$ define $W(\varphi,\rho,t,x,\xi)$, $Y(m,\varphi,\rho,t,x,\xi)$, $X(\varphi,\rho,t,x,\xi)$ by

$$\begin{aligned} W(\varphi,\rho,t,x,\xi) &= W(\rho,t,x,\xi)\varphi(\xi) \\ Y(0,\varphi,\rho,t,x,\xi) &= 0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} Y(m,\varphi,\rho,t,x,\xi) = \int_0^t ds\, & W(\varphi,\rho,s,x+s(\varphi h_0)'(\xi) + \\ & + Y_\xi'(m-1,\varphi,\rho,s,x,\xi),\xi) \quad \text{for } m = 1,\dots m_0, \end{aligned} \tag{4.2}$$

$$X(\varphi,\rho,t,x,\xi) = x\cdot\xi + t(\varphi h_0)(\xi) + Y(m_0,\varphi,\rho,t,x,\xi). \tag{4.3}$$

Note that by (3.3) and (3.4), we have

$$\sup_{x,\xi} |D_\xi^\alpha D_x^\beta W(\varphi,\rho,t,x,\xi)| \leqq K(\varphi,\alpha,\beta)\rho^{\delta_0}\langle t\rangle^{-|\beta|-\delta_0} \tag{4.4}$$

and

$$\sup_{\xi} |D_\xi^\alpha D_x^\beta W(\varphi,\rho,t,x,\xi)| \leqq K(\varphi,\alpha,\beta)\rho^{\delta_0}\langle t\rangle^{-|\beta|-\delta_0}\langle x\rangle^{-\delta_0/2}. \tag{4.5}$$

The aim of this section is to prove decay estimates for X. These estimates will be used in the next section.

LEMMA 4.1.
(i) $\sup_{x,\xi} |D_\xi^\alpha Y(m,\varphi,\rho,t,x,\xi)| \leqq K(m,\varphi,\alpha)\rho^{\delta_0}\langle t\rangle^{1-\delta_0}$,
(ii) $\sup_{t,x,\xi} |D_\xi^\alpha D_x^\beta Y(m,\varphi,\rho,t,x,\xi)| \leqq K(m,\varphi,\alpha,\beta)\rho^{\delta_0}$ *if* $\beta \neq 0$,
(iii) $\sup_{x,\xi} |D_\xi^\alpha D_x^\beta [Y(m,\varphi,\rho,t,x,\xi) - Y(m-1,\varphi,\rho,t,x,\xi)]|$
$\leqq K(m,\varphi,\alpha,\beta)\rho^{\delta_0}\langle t\rangle^{1-m\delta_0}$ *for* $1 \leqq m \leqq m_0$,
(iv) $\sup_{x,\xi} |D_\xi^\alpha D_x^\beta \{W(\varphi,\rho,t,X'_\xi(\varphi,\rho,t,x,\xi),\xi) - W(\varphi,\rho,t,x+t(h_0\varphi)'(\xi)+$
$+Y'_\xi(m_0-1,\varphi,\rho,t,x,\xi),\xi)\}| \leqq K(\varphi,\alpha,\beta)\rho^{\delta_0}\langle t\rangle^{-(m_0+1)\delta_0}$.

PROOF. (i) By induction on m. For $m = 0$ it is trivial. Assume the result to be true for $m-1$. For m the result is obvious by (4.4), when $\alpha = 0$. So we can assume $|\alpha| \geqq 1$. Put

$$\begin{aligned} Z &= Z(m-1,\varphi,\rho,s,x,\xi) \\ &= x + s(h_0\varphi)'(\xi) + Y'_\xi(m-1,\varphi,\rho,s,x,\xi). \end{aligned} \tag{4.6}$$

By the induction hypothesis we see that

$$\sup_{x,\xi} |D_\xi^\alpha Z| \leqq K(\alpha)\langle s\rangle \quad \text{for } \alpha \neq 0. \tag{4.7}$$

Now

$$Y(m,\varphi,\rho,t,x,\xi) = \int_0^t ds\, W(\varphi,\rho,s,Z,\xi). \tag{4.8}$$

By induction on $|\alpha|$ ($\geqq 1$) it is easy to see that $D_\xi^\alpha\{W(\varphi,\rho,s,Z,\xi)\}$ is a finite linear combination of terms of the form $W_{i,j}(\varphi,\rho,s,Z,\xi)D_\xi^{\alpha_1}Z \ldots D_\xi^{\alpha_k}Z$, where

(a) $W_{i,j}(\varphi,\rho,s,x,\xi) = D_\xi^j D_x^i W(\varphi,\rho,s,x,\xi)$,
(b) $1 \leqq |i+j| \leqq |\alpha|$,
(c) the product $D_\xi^{a_1} Z \ldots D_\xi^{a_k} Z$ may or may not appear. If it appears, then,
(d) $1 \leqq k \leqq |i|$,
(e) $1 \leqq |a_1|,\ldots,|a_k| \leqq |\alpha|$.

By (4.7) and (4.4) we see, for any typical term

$$W_{i,j}(\varphi,\rho,s,Z,\xi) D_\xi^{a_1} Z \ldots D_\xi^{a_k} Z,$$

the estimate

$$(4.9)\quad |W_{i,j}(\varphi,\rho,s,Z,\xi) D_\xi^{a_1} Z \ldots D_\xi^{a_k} Z| \leqq K(\varphi,a_1,\ldots,a_k,i,j)\rho^{\delta_0}\langle s\rangle^{-\delta_Q}.$$

Now the result follows from (4.8) and (4.9).

(ii) The proof is similar to that of (i) viz by induction on m. For $m=0$ it is clear. Assume the result to be true for $m-1$. With Z as in (4.6) it is clear by the induction hypothesis that

$$(4.10)\qquad |D_\xi^\alpha D_x^\beta Z| \leqq K(\alpha,\beta) \quad \text{for } \beta \neq 0.$$

By induction on $|\alpha+\beta|\,(\geqq 1)$ with $\beta \neq 0$ it is easy to see that $D_\xi^\alpha D_x^\beta W(\varphi,\rho,s,Z,\xi)$ is a finite linear combination of terms of the form $W_{i,j}(\varphi,\rho,s,Z,\xi) D_\xi^{a_1} D_x^{b_1} Z \ldots D_\xi^{a_k} D_x^{b_k} Z$, where

(a) $W_{i,j}(\varphi,\rho,s,x,\xi) = D_\xi^j D_x^i W(\varphi,\rho,s,x,\xi)$,
(b) $1 \leqq |i+j| \leqq |\alpha+\beta|$,
(c) the product $D_\xi^{a_1} D_x^{b_1} Z \ldots D_\xi^{a_k} D_x^{b_k} Z$ always appears,
(d) $1 \leqq k \leqq |i|$,
(e) $1 \leqq |a_1+b_1|,\ldots |a_k+b_k| \leqq |\alpha+\beta|$,
(f) $|b_1+\ldots+b_k| \geqq 1$.

Now the result follows as in (i) using (4.7), (4.8), and (4.10).

(iii) Again we use induction on m. For $m=1$ we refer to (i) and (ii). Assume the result to be true for $m-1$. Write

$$\begin{aligned} &Y(m,\varphi,\rho,t,x,\xi) - Y(m-1,\varphi,\rho,t,x,\xi) \\ &\quad = \textstyle\int_0^1 d\theta \int_0^t ds [(\nabla_x W)(\varphi,\rho,s,x+s(h_0\varphi)'(\xi) + \theta Y'_\xi(m-1,\varphi,\rho,s,x,\xi) + \\ &\qquad + (1-\theta) Y'_\xi(m-2,\varphi,\rho,s,x,\xi),\xi)] \cdot \\ &\qquad \cdot [Y'_\xi(m-1,\varphi,\rho,s,x,\xi) - Y'_\xi(m-2,\varphi,\rho,s,x,\xi)]. \end{aligned}$$

Now the result follows as in (i) and (ii).

(iv) Follows from (iii) by observing

$$\begin{aligned} &W(\varphi,\rho,t,X'_\xi(\varphi,\rho,t,x,\xi),\xi) - W(\varphi,\rho,t,x+t(h_0\varphi)'(\xi) + \\ &\qquad + Y'_\xi(m_0-1,\varphi,\rho,t,x,\xi),\xi) \end{aligned}$$

$$= \int_0^1 d\theta [(\nabla_x W)(\varphi, \rho, t, x + t(h_0\varphi)'(\xi) + \\ + \theta Y'_\xi(m_0, \varphi, \rho, t, x, \xi) + (1-\theta) Y'_\xi(m_0 - 1, \varphi, \rho, t, x, \xi), \xi)] \cdot \\ \cdot [Y'_\xi(m_0, \varphi, \rho, t, x, \xi) - Y'_\xi(m_0 - 1, \varphi, \rho, t, x, \xi)].$$

LEMMA 4.2.

(i) *Let $Y(m, \rho, t, \xi)$ be as in (3.6.). If the real valued φ in $C_0^\infty(G)$ is such that $\varphi = 1$ on an open set B, then*

$$D_\xi^\alpha Y(m, \rho, t, \xi) = D_\xi^\alpha Y(m, \varphi, \rho, t, 0, \xi)$$

for ξ in B, all α and m.

(ii) *Let $X(\rho, t, \xi)$ be as in (3.7). Then* $\lim_{t\to\pm\infty}[X(1,t,\xi) - X(\rho,t,\xi)]$ *exists for ξ in G [of (2.1)].*

(iii) $\sup_{t,x,\xi} \langle x \rangle^{-1} | Y'_\xi(m_0 - 1, \varphi, \rho, t, x, \xi) - Y'_\xi(m_0 - 1, \varphi, \rho, t, 0, \xi)| < \infty.$

PROOF. (i) By induction on m. For $m = 0$ it is clear. Assume the result to be true for $m-1$. Then

$$\begin{aligned} &Y(m, \varphi, \rho, t, 0, \xi) \\ &\quad = \int_0^t ds\, W(\rho, s, s(h_0\varphi)'(\xi) + Y'_\xi(m-1, \varphi, \rho, s, 0, \xi), \xi)\varphi(\xi) \\ &\quad = \int_0^t ds\, W(\rho, s, sh'_0(\xi) + Y'_\xi(m-1, \varphi, \rho, s, 0, \xi), \xi) \quad \text{for } \xi \text{ in } B \\ &\quad = \int_0^t ds\, W(\rho, s, sh'_0(\xi) + Y'_\xi(m-1, \rho, s, \xi), \xi) \\ &\qquad \text{for } \xi \text{ in } B \text{ by induction hypothesis} \\ &\quad = Y(m, \rho, t, \xi) \quad \text{for } \xi \text{ in } B. \end{aligned}$$

Now clearly $D_\xi^\alpha Y(m, \varphi, \rho, t, 0, \xi) = D_\xi^\alpha Y(m, \rho, t, \xi)$ for ξ in B.

(ii) (For the positive sign only.). Clearly it suffices to show that $\lim_{t\to\infty} D_\xi^\alpha[Y(m,1,t,\xi) - Y(m,\rho,t,\xi)]$ exists for each α, m, and ξ in a fixed open set B such that $\bar{B}$ is compact in G. The proof is, as usual, by induction on m. For $m = 0$ it is clear. Assume the result to be true for $m-1$. Now put

$$a = \inf\{|h'_0(\xi)| : \xi \in \bar{B}\}$$

so that $a > 0$. Now let φ in $C_0^\infty(G)$ be such that $0 \leqq \varphi \leqq 1$ and $\varphi = 1$ on B. Now

$$\begin{aligned} &|sh'_0(\xi) + Y'_\xi(m-1, 1, s, \xi)| \\ &\qquad \geqq sa - |Y'_\xi(m-1, 1, s, \xi)| \qquad \text{for } \xi \text{ in } B \\ &\qquad \geqq sa - K(m-1, \varphi, 1)\langle s \rangle^{1-\delta_0} \quad \text{by (i) and Lemma 4.1 (i)} \end{aligned}$$

Choose $t_0 \geqq 0$ large so that for $s \geqq t_0$ we have

$$sa - K(m-1,\varphi,1)\langle s\rangle^{1-\delta_0} \geqq 2 + 2\rho^{-1}.$$

Then we get with χ_0 as in (3.1),

$$(4.11)\ \chi_0(sh_0'(\xi) + Y_\xi'(m-1,1,s,\xi)) = 1 = \chi_0(\rho[sh_0'(\xi) + Y_\xi'(m-1,1,s,\xi)]) \quad \text{for } \xi \text{ in } B,\ s \geqq t_0 \geqq 0.$$

Also note that for x, y, ξ in R^n and $s \geqq 0$ we get

$$\begin{aligned} &W(1,s,x,\xi) - W(\rho,s,y,\xi) \\ (4.12)\quad &= W(1,s,x,\xi) - W(\rho,s,x,\xi) + W(\rho,s,x,\xi) - W(\rho,s,y,\xi) \\ &= [\chi_0(x) - \chi_0(\rho x)]\chi_0(x \log\langle s\rangle/\langle s\rangle) W_L(x,\xi) + \\ &\quad + \textstyle\int_0^1 d\theta (\nabla_x W)(\rho,s,\theta x + (1-\theta)y,\xi)\cdot(x-y). \end{aligned}$$

Now let $t \geqq t_0$ and ξ be in B. Then by (4.11) and (4.12) we get

$$\begin{aligned} &Y(m,1,t,\xi) - Y(m,\rho,t,\xi) \\ &= \textstyle\int_0^{t_0} ds[W(1,s,sh_0'(\xi) + Y_\xi'(m-1,1,s,\xi),\xi) - \\ &\quad - W(\rho,s,sh_0'(\xi) + Y_\xi'(m-1,1,s,\xi),\xi)] + \\ (4.13)\quad &+ \textstyle\int_0^t ds \int_0^1 d\theta[(\nabla_x W)(\rho,s,sh_0'(\xi) + \theta Y_\xi'(m-1,1,s,\xi) + \\ &+ (1-\theta) Y_\xi'(m-1,\rho,s,\xi),\xi)]\cdot \\ &\cdot [Y_\xi'(m-1,1,s,\xi) - Y_\xi'(m-1,\rho,s,\xi)]. \end{aligned}$$

Now the result follows by (4.13) and the induction hypothesis.

(iii) Follows from Lemma 4.1 (ii) by noting

$$\begin{aligned} &|Y_\xi'(m_0-1,\varphi,\rho,t,x,\xi) - Y_\xi'(m_0-1,\varphi,\rho,t,0,\xi)| \\ &= |\textstyle\int_0^1 d\theta x \cdot [\nabla_x Y_\xi'](m_0-1,\varphi,\rho,t,\theta x,\xi)|. \end{aligned}$$

5. Phase space decomposition and approximate propagators.

This section closely follows the ideas from [8]. In this section we shall not distinguish between an integral operator and the kernel of the integral operator, i.e. I will denote the kernel $I(q,x)$ as well the operator given by

$$(If)(q) = \int dx\, I(q,x) f(x).$$

Choose η in $C^\infty(\mathsf{R}^n)$ so that

$$(5.1)\qquad 0 \leqq \eta \leqq 1,\quad \eta = 1 \text{ for } |x| \geqq 2,\quad 0 \text{ for } |x| \leqq 1.$$

Choose and fix σ_0 in $(0,1)$. For this σ_0 choose $\psi_\pm$ in $C^\infty[-1,1]$ so that

$$(5.2)\qquad \begin{aligned} &0 \leqq \psi_{\pm} \leqq 1, && \psi_{+} + \psi_{-} = 1 \\ &\psi_{+} = 1 \text{ on } [\sigma_0, 1], && 0 \text{ on } [-1, -\sigma_0]. \end{aligned}$$

Choose any real valued γ in $C_0^\infty(G)$ and fix it throughout this section. For $r \geqq 1$ define $g_{\pm}(r,.,.)$ by

$$(5.3)\qquad g_{\pm}(r,x,\xi) = \eta(x/r)\gamma(\xi)\psi_{\pm}(x \cdot h_0'(\xi)/[|x|\,|h_0'(\xi)|]).$$

For the given γ choose a real valued φ in $C_0^\infty(G)$ such that $\varphi = 1$ in an open neighbourhood of $\operatorname{supp}\gamma$. This forces

$$(5.4)\qquad \varphi(\xi)\gamma(\xi) = \gamma(\xi).$$

For this φ choose ρ_0 in $(0,1]$ so that, by Lemma 4.1 (ii),

$$(5.5)\qquad \sup_{t,x,\xi}\left[\sum_{|\alpha|=|\beta|=1} |D_\xi^\alpha D_x^\beta Y(m_0,\varphi,\rho_0,t,x,\xi)|^2\right]^{\frac{1}{2}} \leqq \frac{1}{2}.$$

For $r \geqq 1$, $\pm t$ in $[0,\infty)$ define an operator $T_{\pm}(r,t)$ on $\mathscr{S}$ by

$$(5.6)\quad [T_{\pm}(r,t)f](q) = \int d\xi\, dx f(x) g_{\pm}(r,x,\xi)\exp(i[q\cdot\xi - X(\varphi,\rho_0,t,x,\xi)].$$

Now we have

LEMMA 5.1.

(i) $T_{\pm}(r,t)\mathscr{S} \subseteqq \mathscr{S}$,

(ii) $\sup_{r\geqq 1, t\gtrless 0} \|T_{\pm}(r,t)\| < \infty$,

(iii) $T_{+}(r,0) + T_{-}(r,0) = (2\pi)^n \gamma(P)\eta(Q/r)$,

(iv) $\|T_{\pm}(r,0) - T_{\pm}^*(r,0)\| \leqq Kr^{-1}$.

PROOF. For (ii), (iii) and (iv), the proof is similar to the proof of Lemma 7.1 of [13]. For (i) note that for f in $\mathscr{S}$ we get

$$[T_{\pm}(r,t)f]^{\wedge}(\xi) = (2\pi)^{-(\frac{1}{2})n}\int dx f(x) g_{\pm}(r,x,\xi)\exp[-iX(\varphi,\rho_0,t,x,\xi)].$$

Note that $[T_{\pm}(r,t)f]^{\wedge} \in C_0^\infty(G)$.

LEMMA 5.2.

(i) *Let* $\theta \in C_0^\infty(\mathrm{R}^n)$ *be such that* $0 \leqq \theta \leqq 1$, $\theta = 1$, *for* $|x| \leqq \frac{1}{2}$, 0 *for* $|x| \geqq 1$. *Then there exists* $b > 0$ *and* $r_0 > 0$ *(depending on* φ*) such that for* $r \geqq r_0$

$$\|\theta(Q/[b(r+|t|)])T_{\pm}(r,t)\| \leqq K\langle t\rangle^{-2\lambda}\langle r\rangle^{-2(1-\lambda)}$$

for each λ *in* $[0,1]$. *Here* K *is independent of* λ.

From now on we choose $r \geqq r_0$ *always and* $\lambda \in [0,1]$.

(ii) $\| \theta(Q/[b(r+|t|)]) W_L(Q,P)\varphi(P) T_\pm(r,t) \| \leqq K\langle t\rangle^{-2\lambda} r^{-2(1-\lambda)}$,

(iii) $\| W_L(Q,P)\varphi(P) T_\pm(r,t) \| \leqq K\langle t\rangle^{-\delta_0}\langle r\rangle^{-\delta_0}$,

(iv) $\| \langle Q\rangle^{-2} T_\pm(r,t) \| \leqq K\langle t\rangle^{-2\lambda}\langle r\rangle^{-2(1-\lambda)}$,

(v) $\| \langle Q\rangle^{-\sigma} T_\pm(r,t) \| \leqq K\langle t\rangle^{-\sigma\lambda}\langle r\rangle^{-\sigma(1-\lambda)}$ *for each* σ *in* $[0,2]$,

(vi) $\| \langle Q\rangle^{-2} W_L(Q,P)\varphi(P) T_\pm(r,t) \| \leqq K\langle t\rangle^{-2\lambda}\langle r\rangle^{-2(1-\lambda)}$,

(vii) $\lim_{r\to\infty} \int_0^{\pm\infty} dt \, \| W_S T_\pm(r,t) \| = 0$,

(viii) $\lim_{r\to\infty} \int_0^{\pm\infty} dt \, \| [W_L(Q,P) - W(\varphi,\rho_0,t,Q,P)] T_\pm(r,t) \| = 0$.

(ix) *Let* $I_\pm(r,t)$ *be the operator with the kernel*

$$[I_\pm(r,t)](q,x) = \int d\xi \exp(i[q\cdot\xi - X(\varphi,\rho_0,t,x,\xi)]) g_\pm(r,x,\xi)$$
$$[W(\varphi,\rho_0,t,X'_\xi(\varphi,\rho_0,t,x,\xi),\xi) - W(\varphi,\rho_0,t,x+t(h_0\varphi)'(\xi)$$
$$+ Y'_\xi(m_0-1,\varphi,\rho_0,t,x,\xi),\xi)].$$

Then

$$\lim_{r\to\infty} \int_0^{\pm\infty} dt \, \| I_\pm(r,t) \| = 0.$$

(x) *Define* $J_\pm(r,t)$ *by*

$$[J_\pm(r,t)](q,x) = \int d\xi \exp(i[q,\xi - X(\varphi,\rho_0,t,x,\xi)] g_\pm(r,x,\xi)$$
$$[W(\varphi,\rho_0,t,q,\xi) - W(\varphi,\rho_0,t,X'_\xi(\varphi,\rho_0,t,x,\xi),\xi)].$$

Then

$$\sup_{t\gtrless 0} \| J_\pm(r,t) \| \leqq K\langle r\rangle^{-\delta_0/2}.$$

(xi) $\| J_\pm(r,t) \| \leqq K\langle t\rangle^{-1-\delta_0}$,

(xii) $\lim_{r\to\infty} \int_0^{\pm\infty} dt \, \| J_\pm(r,t) \| = 0$.

PROOF. We prove all the results for the positive sign only.

(i) The proof is similar to that of Lemma 7.2 (i) of [13].

(ii) Similar to (i).

(iii) Note that $\langle Q\rangle^{3\delta_0} W_L(Q,P)\varphi(P)$ is a bounded operator. Write

$$W_L(Q,P) = \{[1-\theta(Q/[b(r+t)])]\langle Q\rangle^{-3\delta_0}\}\langle Q\rangle^{3\delta_0} W_L(Q,P)\varphi(P) + \theta(Q/[b(r+t)]) W_L(Q,P)\varphi(P). \tag{5.7}$$

Clearly

$$\|[1-\theta(Q/[b(r+t)])]\langle Q\rangle^{-3\delta_0}\| \leqq K(r+t)^{-3\delta_0}.$$

Now the result follows from (5.7) by using Lemma 5.1 (ii) and Lemma 5.2 (ii).

(iv) Follows from (i) and Lemma 5.1 (ii) by using the identity $1 = 1-\theta+\theta$.

(v) Follows from (iv) and Lemma 5.1 (ii) by using interpolation [17].

(vi) Similar to (iv) by using (ii) instead of (i).

(vii) Since $\varphi(P)\gamma(P) = \gamma(P)$ we see that

$$(5.8) \qquad W_S T_+(r,t) = \{W_S\varphi(P)\langle Q\rangle^{1+\varepsilon_0}\}\{\langle Q\rangle^{-1-\varepsilon_0}T_+(r,t)\}.$$

By the short range assumption A3, the operator $W_S\varphi(P)\langle Q\rangle^{1+\varepsilon_0}$ is bounded. Now the result follows from (5.8) by using (v) and choosing λ in $(0,1)$ so that $(1+\varepsilon_0)\lambda > 1$.

(viii) Let b be as in (i). Choose $t_0 \geqq 0$ so that $\log\langle t_0\rangle = 4/b$. Then using $\varphi(P)\gamma(P) = \gamma(P)$ we easily see that

$$\begin{aligned}
&[W_L(Q,P) - W(\varphi,\rho_0,t,Q,P)]T_+(r,t)\\
&\quad = \langle Q\rangle^2[1-\chi_0(\rho_0 Q)]\langle Q\rangle^{-2}W_L(Q,P)\varphi(P)T_+(r,t) +\\
&\qquad + \chi_0(\rho_0 Q)[1-\chi_0(Q\log\langle t\rangle/\langle t\rangle)]F(t\leqq t_0)W_L(Q,P)\varphi(P)T_+(r,t) +\\
&\qquad + \chi_0(\rho_0 Q)[1-\chi_0(Q\log\langle t\rangle/\langle t\rangle)]F(t\geqq t_0)\cdot\\
&\qquad\quad \cdot\,\theta(Q/[b(r+t)])W_L(Q,P)\varphi(P)T_+(r,t).
\end{aligned}$$

The result will follow if we can show that each term on the R.H.S. has the corresponding property. For the first term note that $\langle Q\rangle^2[1-\chi_0(\rho_0 Q)]$ is bounded and apply (vi) with λ in $(\frac{1}{2},1)$. For the second term note that χ_0 is bounded, the t-integration is over a finite interval and apply (iii). For the third term apply (ii) with λ in $(\frac{1}{2},1)$.

(ix) The proof is similar to the proof of Lemma 7.2 (v) of [13].

(x) Define $L_+(r,t)$ by

$$\begin{aligned}
&[L_+(r,t)](q,x)\\
&\quad = \int d\xi \exp(i[q\cdot\xi - X(\varphi,\rho_0,t,x,\xi)])g_+(r,x,\xi)W(\varphi,\rho_0,t,X'_\xi(\varphi,\rho_0,t,x,\xi),\xi)
\end{aligned}$$

so that

$$(5.9) \quad J_+(r,t) = \chi_0(\rho_0 Q)\chi_0(Q\log\langle t\rangle/\langle t\rangle)W_L(Q,P)\varphi(P)T_+(r,t) - L_+(r,t).$$

As in the proof of Lemma 7.2 (vi) of [13] we get

(5.10) $$\| L_+(r,t) \| \leqq K\langle r\rangle^{-\delta_0/2}$$

Now the result follows from (5.9), (5.10) and (iii).

(xi) Similar to the proof of Lemma 7.2 (vii) of [13].
(xii) Follows from (x) and (xi) by the Lebesgue dominated convergence theorem.

LEMMA 5.3.

(i) $\lim_{r\to\infty} \int_0^{\pm\infty} dt \left\| \frac{d}{dt} V_t^* T_\pm(r,t) \right\| = 0$

(ii) $\Omega_\pm(r) = \operatorname{s-lim}_{t\to\pm\infty} V_t^* T_\pm(r,t)$ *exist for* $r \geqq r_0$, r_0 *as in Lemma* 5.2 (i).

(iii) $\lim_{r\to\infty} \sup_{t\gtrless 0} \left\| V_t T_\pm(r,0) - T_\pm(r,t) \right\| = 0.$

(iv) $\lim_{r\to\infty} \| T_\pm(r,0) - \Omega_\pm(r) \| = 0.$

(v) $\lim_{r\to\infty} \sup_{t\gtrless 0} \| T_\pm^*(r,0) V_t^* f \| = 0$ *for each* f *in* $L^2(\mathsf{R}^n)$.

(vi) $\lim_{r\to\infty} \sup_{t\gtrless 0} \| T_\pm(r,0) V_t^* f \| = 0$ *for each* f *in* $L^2(\mathsf{R}^n)$.

PROOF. Similar to the proof of Lemma 7.3 of [13]. One has to use Lemma 5.1 (i) for (i).

LEMMA 5.4.

(i) $\omega_\pm(r) = \operatorname{s-lim}_{t\to\pm\infty} \exp[iX(1,t,P)] T_\pm(r,t)$ *exist for* $r \geqq r_0$.

Here r_0 *is as in Lemma* 5.2 (i) *and* $X(1,t,\xi)$ *as in* (3.7).

(ii) $\Omega_\pm(r) = \Omega_\pm \omega_\pm(r)$ *for* $r \geqq r_0$, r_0 *as in* (i).

(iii) $\lim_{r\to\infty} \| (1 - \Omega_\pm \Omega_\pm^*) T_\pm(r,0) \| = 0,$

(iv) $\lim_{r\to\infty} \sup_{t\gtrless 0} \| (1 - \Omega_\pm \Omega_\pm^*) \gamma(P) \eta(Q/r) V_t f \| = 0$ *for* f *in* $L^2(\mathsf{R}^n)$,

for each γ *in* $C_0^\infty(G)$ *and* η *as in* (5.1).

PROOF. (i) Since $\varphi(P) T_\pm(r,t) = T_\pm(r,t)$, by Lemma 4.2 (ii) it suffices to show the existence of $\operatorname{s-lim}_{t\to\pm\infty} \exp[iX(\rho_0,t,P)] T_\pm(r,t)$.

Since $\varphi = 1$ on an open neighbourhood of γ we easily see by Lemma 4.2 (i) that

$$\exp[iX(\rho_0,t,P)]\,T_\pm(r,t) = \exp[iX(\varphi,\rho_0,t,0,P)]\,T_\pm(r,t).$$

Now the proof is similar to that of Lemma 7.4 (i) of [13] by using Lemma 4.2 (iii).

(ii) Since $\varphi(P)\gamma(P) = \gamma(P)$ we easily see that $\varphi(P)\omega_\pm(r) = \omega_\pm(r)$. Now the result follows by (i) and Theorem 2.3 (i).

(iii) The proof is similar to the proof of Lemma 7.4 (iii) of [13]. By Lemma 5.3 (iv) we get

$$0 = \lim_{r\to\infty} \| \Omega_\pm^* T_\pm(r,0) - \Omega_\pm^* \Omega_\pm(r) \|$$

$$= \lim_{r\to\infty} \| \Omega_\pm^* T_\pm(r,0) - E_{ac}(H_0)\omega_\pm(r) \|$$

by (ii) and Theorem 2.3 (ii). Hence

$$0 = \lim_{r\to\infty} \| \Omega_\pm \Omega_\pm^* T_\pm(r,0) - \Omega_\pm E_{ac}(H_0)\omega_\pm(r) \|$$

$$(5.11) \qquad = \lim_{r\to\infty} \| \Omega_\pm \Omega_\pm^* T_\pm(r,0) - \Omega_\pm(r) \|$$

by Theorem 2.3 (i) and Lemma 5.4 (ii).

Now the result follows from (5.11) and Lemma 5.3 (iv).

(iv) follows from (iii), Lemma 5.3 (vi) and Lemma 5.1 (iii).

The rest of this section is not necessary for this article. However, it shall be useful for improving the contents of [15].

More precisely, let

$$Y(0,\rho,t,x,\xi) = 0,$$

$$(5.12) \qquad Y(m,\rho,t,x,\xi) = \int_0^t ds\, W(\rho,s,x+sh_0'(\xi) + Y_\xi'(m-1,\rho,s,x,\xi),\xi) \quad \text{for } m = 1,\ldots m_0,$$

$$(5.13) \qquad X(\rho,t,x,\xi) = x\cdot\xi + th_0(\xi) + Y(m_0,\rho,t,x,\xi).$$

Then as in Lemma 4.2 (i) we can see that if a real valued φ in $C_0^\infty(G)$ is 1 on an open set B, for ξ in B

$$(5.14) \qquad D_\xi^\alpha Y(m,\varphi,\rho,t,x,\xi) = D_\xi^\alpha Y(m,\rho,t,x,\xi),$$

$$(5.15) \qquad X(\varphi,\rho,t,x,\xi) = X(\rho,t,x,\xi).$$

It is clear that when ρ_0 of (5.5) is replaced by any ρ in $(0,\rho_0]$ all the results of this section are valid. Note that φ is chosen so that $\varphi = 1$ on an open neighbourhood of $\operatorname{supp}\gamma$. Thus we have proved the following theorem.

THEOREM 5.5. *Let the assumptions* A1 *and* A2 *hold. Choose a real valued* γ *in* $C_0^\infty(G), \eta, \psi_\pm$ *as in* (5.1), (5.2). *Define* $S_\pm(\gamma,\rho,r,t)$ *for* $r \geqq 1$, $\pm t$ *in* $[0,\infty)$, ρ *in* $(0,1]$ *by*

$$S_\pm(\gamma,\rho,r,t)(q,x)$$
$$= \int d\xi\, \gamma(\xi)\eta(x/r)\psi_\pm(x\cdot h_0'(\xi)/|x|\,|h_0'(\xi)|)\exp(i[q\cdot\xi - X(\rho,t,x,\xi)]).$$

Then there exists $\rho_0 > 0$ *(depending on* γ*), so that for* ρ *in* $(0,\rho_0]$ *the following hold:*

(i) $S_\pm(\gamma,\rho,r,t)\mathscr{S} \subseteqq \mathscr{S}$,

(ii) $\sup_{r\geqq 1, t\gtrless 0} \|S_\pm(\gamma,\rho,r,t)\| < \infty$,

(iii) $S_+(\gamma,\rho,r,0) + S_-(\gamma,\rho,r,0) = (2\pi)^n\gamma(P)\eta(Q/r)$,

(iv) $\|S_\pm(\gamma,\rho,r,0) - S_\pm^*(\gamma,\rho,r,0)\| \leqq K_\rho r^{-1}$.

(v) *With* θ *as in Lemma* 5.2 (i) *there exists* b, r_0 *depending on* γ,ρ *such that for* $r \geqq r_0$

$$\|\theta(Q/[b(r+|t|)])S_\pm(\gamma,\rho,r,t)\| \leqq K(\rho)\langle t\rangle^{-2\lambda}\langle r\rangle^{-2(1-\lambda)}$$

for each λ *in* $[0,1]$.

From now on $r \geqq r_0$ *and* $\lambda \in [0,1]$.

(vi) $\|\langle Q\rangle^{-2}S_\pm(\gamma,\rho,r,t)\| \leqq K(\rho)\langle r\rangle^{-2(1-\lambda)}\langle t\rangle^{-2\lambda}$,

(vii) $\displaystyle\lim_{r\to\infty}\int_0^{\pm\infty} dt\left\|[H_0 + W_L(Q,P)]S_\pm(\gamma,\rho,r,t) - i\frac{\partial}{\partial t}S_\pm(\gamma,\rho,r,t)\right\| = 0$,

(viii) $\omega_\pm(\gamma,\rho,r,t) = \operatorname*{s-lim}_{t\to\pm\infty}\exp[iX(1,t,P)]S_\pm(\gamma,\rho,r,t)$ *exists.*

6. Proof of Theorems 2.3 (vi), (vii), 2.4.

Of Theorems 2.3 (vi), (vii) we prove only (vi) since the other one has a similar proof. This also we do only for the positive sign. Thus we need to prove

$$M(1,+\infty,H) \ominus \operatorname{Range}\Omega_+$$

(6.1) $$= \{f \in M(1,+\infty,H)\colon \lim_{t\to\infty} \|\gamma(P)V_t f\| = 0 \text{ for each } \gamma \text{ in } C_0^\infty(G)\}.$$

Let $f \in$ L.H.S. of (6.1) and $\gamma \in C_0^\infty(G)$. Then with η as in (5.1)

$$\|\gamma(P)V_t f\|^2 = \langle(1-\Omega_+\Omega_+^*)|\gamma|^2(P)\eta(Q/r)V_t f, V_t f\rangle + \\ + \langle[1-\eta(Q/r)]V_t f, |\gamma|^2(P)V_t f\rangle.$$

In the last equality use Lemma 5.4 (iv) for the first term of R.H.S. and $f \in M(1,+\infty,H)$ to the second term to see that $\lim_{t\to\infty}\|\gamma(P)V_t f\| = 0$. Thus L.H.S. $\subseteqq$ R.H.S. in (6.1).

Let $f \in$ R.H.S. of (6.1). To show $f \perp \operatorname{Range}\Omega_+$ it is enough to show, by density arguments, that $f \perp \Omega_+ g$ for each g in

$$D = \{g \in \mathscr{S} : \hat{g} \in C_0^\infty(G)\}.$$

For g in D put $h = \Omega_+ g$ and choose γ in $C_0^\infty(G)$ so that $\gamma(P)g = g$. Then since

$$\lim_{t\to\infty} \|V_t h - \exp[-iX(1,t,P)]g\| = 0 \quad \text{and} \quad \gamma(P)g = g$$

we see that

(6.2) $$\lim_{t\to\infty} \|[1-\gamma(P)]V_t h\| = 0.$$

Clearly

$$\begin{aligned} |\langle f,h\rangle| &= \lim_{t\to\infty} |\langle V_t f, V_t h\rangle| \\ &\leqq \lim_{t\to\infty} \|f\|\,\|[1-\gamma(P)]V_t h\| + \lim_{t\to\infty}\|h\|\,\|\gamma(P)V_t f\| \\ &= 0 \end{aligned}$$

by (6.2) and the condition on f. Thus R.H.S. $\subseteqq$ L.H.S. in (6.1). This completes the proof.

Proof of Theorem 2.4. (i). By Theorem 2.2 (ii) and Theorem 2.3 (iv) we get that

$$\operatorname{Range}\Omega_\pm \subseteqq H_{ac}(H) \subseteqq M(1,\pm\infty,H) \subseteqq H_c(H) \subseteqq M(2,\pm\infty,H).$$

So it suffices to show $M(2, \pm\infty, H) \subseteqq \mathrm{Range}\,\Omega_{\pm}$ and this we do for the positive sign only. We show that $M(2, +\infty, H) \ominus \mathrm{Range}\,\Omega_{+} = \{0\}$.

Note that for $\varphi \in C_0^\infty(\mathsf{R} \setminus \bar{C}_v)$ if we put $\gamma(\xi) = \varphi(h_0(\xi))$, then $\gamma \in C_0^\infty(G)$. Let $f \in M(2, +\infty, H) \ominus \mathrm{Range}\,\Omega_{+}$ so that by Theorem 2.3 (vii) we get

$$(6.3) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, \| \varphi(H_0) V_t f \| = 0 \quad \text{for } \varphi \text{ in } C_0^\infty(\mathsf{R} \setminus \bar{C}_v).$$

Since $\varphi(H) - \varphi(H_0)$ is compact for φ in $C_0^\infty(\mathsf{R} \setminus \bar{C}_v)$ we get

$$(6.4) \qquad \lim_{r \to \infty} \| [\varphi(H) - \varphi(H_0)] F(|Q| \geqq r) \| = 0.$$

Since $f \in M(2, +\infty, H)$ we see by (6.4) that

$$(6.5) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, \| [\varphi(H) - \varphi(H_0)] V_t f \| = 0 \quad \text{for } \varphi \text{ in } C_0^\infty(\mathsf{R} \setminus \bar{C}_v).$$

By (6.3) and (6.5) we conclude $\varphi(H) f = 0$ for each φ in $C_0^\infty(\mathsf{R} \setminus \bar{C}_v)$. Since $\bar{C}_v$ is countable $f \in H_p(H)$, So we get

$$(6.6) \qquad \lim_{r \to \infty} \sup_{t \geqq 0} \| F(|Q| \geqq r) V_t f \| = 0.$$

Since $f \in M(2, +\infty, H)$, by (6.6) we get that $f = 0$. Thus $M(2, +\infty, H) \ominus \mathrm{Range}\,\Omega_{+} \subset \{0\}$ completing the proof.

(ii). By (i) we get $E = E_p(H) = 1 - \Omega_{\pm} \Omega_{\pm}^*$. By Lemma 5.4 (iii) and Lemma 5.1 (iii) we see that with η as in (5.1)

$$\lim_{r \to \infty} \| E \varphi(H_0) \eta(Q/r) \| = 0, \quad \text{for each } \varphi \text{ in } C_0^\infty(\mathsf{R} \setminus \bar{C}_v).$$

Since $\varphi(H_0)[1 - \eta(Q/r)]$ is compact and $\varphi(H) - \varphi(H_0)$ is compact for φ in $C_0^\infty(\mathsf{R} \setminus \bar{C}_v)$, we get the compactness of $E\varphi(H)$ for each φ in $C_0^\infty(\mathsf{R} \setminus \bar{C}_v)$. Now the result is clear.

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