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A COMPARISON BETWEEN THE CLOSED MODULAR IDEALS IN $l^1(w)$ AND $L^1(w)$

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Abstract.

The Beurling algebras $l^1(w)$ and $L^1(w)$ are defined as generalizations of $l^1(\mathsf{N})$ and $L^1(\mathsf{R}_+)$, respectively, using a submultiplicative weight w on R_+. For non-radical weights w one can find a one-to-one correspondence between the closed ideals in $l^1(w)$ with hulls not containing a specified line and the closed modular (regular) ideals in $L^1(w)$ with hulls contained in a specified strip. The corresponding quotient Banach algebras are proved isomorphic, and consequently their dual spaces, that is, the annihilators of the respective ideals, are isomorphic Banach spaces.

This is used to show that the problems of determining the closed modular ideals in $l^1(w)$ and $L^1(w)$ are equivalent.

0. Introduction.

An algebra is a Banach algebra if a norm is defined on it such that it is a Banach space and the multiplication is jointly continuous. We will only deal with algebras over the complex field C, and as far as we are concerned, all algebras, and in particular all Banach algebras, are assumed commutative. For standard results in the elementary theory of Banach algebras, we refer to [11], [12], and [15].

This introduction has the following structure. First we introduce quite a few notions; after that we mention some results that are known previously in the literature before we finally sketch our main theorem.

Suppose $w\colon [0,\infty) \to (0,\infty)$ is a continuous submultiplicative (weight) function:

$$w(x+y) \leqq w(x)w(y) \quad \text{for } x,y \in [0,\infty).$$

Let $M(w) = M(w,\mathsf{R}_+)$ be the space of regular Borel measures μ on $\mathsf{R}_+ = [0,\infty)$ which are absolutely continuous (with respect to the Lebesgue measure) on $\mathsf{R}_+ \setminus \mathsf{N}$ $(\mathsf{N} = \{0,1,2,\ldots\})$ and satisfy

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$$\|\mu\|_w = \int_0^\infty w(x)|d\mu|(x) < \infty.$$

Equipped with the norm $\|\cdot\|_w$, $M(w)$ is a Banach space.

All measures in $M(w)$ are tacitly assumed extended to $\mathbf{R}$ such that

$$|\mu|((-\infty, 0)) = 0 \quad \text{for } \mu \in M(w).$$

By the submultiplicativity of w, $M(w)$ is a Banach algebra under convolution, which we denote by $*$:

$$(\mu * \nu)(E) = \int_{-\infty}^\infty \mu(E - x)d\nu(x) \quad \text{for } \mu, \nu \in M(w).$$

The unit in $M(w)$ is the Dirac measure δ_0 at 0.

It is well known that the limit

$$\alpha = \lim_{x \to +\infty} x^{-1} \log w(x)$$

exists in $[-\infty, \infty)$. If $\alpha = -\infty$, $M(w)$ contains only one maximal ideal. We are interested in non-radical weights, that is, weights w for which $\alpha > -\infty$.

By replacing w by $x \mapsto e^{-\alpha x} w(x)$, $x \in \mathbf{R}_+$, we can suppose that $\alpha = 0$. Then $w(x) \geqq 1$ for $x \in \mathbf{R}_+$.

Let Π_- denote the closed lower half plane $\{z \in \mathbf{C} : \operatorname{Im} z \leqq 0\}$. The Fourier transform

$$\hat{\mu}(z) = \int_0^\infty e^{-itz} d\mu(t), \quad z \in \Pi_-,$$

defines a continuous injective homomorphism from $M(w)$ into $A(\Pi_-)$, the algebra of bounded holomorphic functions in Π_-° which extend continuously to the boundary, supplied with the uniform norm.

Every measure in $M(w)$ is the sum of an absolutely continuous measure and a singular measure with support on $\mathbf{N}$.

The absolutely continuous measures in $M(w)$ form a closed ideal and will be identified in an obvious way with $L^1(w) = L^1(w, \mathbf{R}_+)$, the space of (equivalence classes of) locally integrable functions f on $\mathbf{R}_+$ for which

$$\|f\|_w = \int_0^\infty |f(x)| w(x)dx < \infty.$$

The convolution of measures in $L^1(w)$ is then ordinary convolution of functions. It is well known that we can identify the Gelfand (or carrier) space of $L^1(w)$ with Π_-, so that the Gelfand transform on $L^1(w)$ coincides with the Fourier transform

$$\hat{f}(z) = \int_0^\infty f(t)e^{-itz}\,dt, \quad z \in \Pi_-.$$

The Fourier transform maps $L^1(w)$ into the closed ideal $A_0(\Pi_-)$ in $A(\Pi_-)$ consisting of those functions $f(z)$ in $A(\Pi_-)$ which tend to 0 as $|z| \to \infty$ within Π_-.

Since $L^1(w)$ is a closed ideal in $M(w)$, Π_- will be regarded as an open subset of the maximal ideal space $\mathcal{M}_M$ of $M(w)$ (Proposition 1.6). The Fourier transform of a measure μ in $M(w)$ is then the restriction to Π_- of the Gelfand transform of μ.

The singular measures in $M(w)$, which are supported on N, form a closed unital subalgebra, which we will identify with $l^1(w) = l^1(w, \mathsf{N})$, the space of complex-valued sequences $(a_n)_0^\infty$ such that

$$\sum_{n=0}^{\infty} |a_n|\, w(n) < \infty.$$

The choice of the norm is obvious, and the multiplication is sequence convolution. We will alternatively regard $l^1(w)$ as an algebra of sequences and an algebra of singular measures supported on N.

The reader may have anticipated that $M(w)$ was introduced to give us a convenient setting for $L^1(w)$ and $l^1(w)$.

We wish to point out the following useful observation.

If $(v_n)_0^\infty$ is any submultiplicative sequence of positive numbers, there exists a continuous submultiplicative weight $v\colon \mathsf{R}_+ \to (0,\infty)$ such that

$$v(n) = v_n, \quad n = 0, 1, 2, \dots .$$

v is obtained by interpolating $\log v_n$ linearly on each component of $\mathsf{R}_+ \setminus \mathsf{N}$:

$$v(n+t) = v_n^{1-t}\, v_{n+1}^t \quad \text{for } n \in \mathsf{N} \text{ and } t \in [0,1].$$

We leave the necessary verifications to the interested reader.

It should be observed that two continuous submultiplicative weights $w_1, w_2\colon [0,\infty) \to (0,\infty)$ which have the same values on N give rise to equivalent norms if the sequence

$$\{w_1(n)/w_1(n+1)\}_0^\infty = \{w_2(n)/w_2(n+1)\}_0^\infty$$

is bounded.

The maximal ideal space of $l^1(w)$ is topologically the closed unit disc D. For our purposes, however, it will be more suitable to use a somewhat different set.

Let $L = (\{\pi\} \times i(-\infty, 0]) \cup \{\infty\}$ be a compactified half-line, and let $\mathscr{D}$ denote the compactified half-strip $((-\pi, \pi] \times i(-\infty, 0]) \cup \{\infty\}$. The function $\varphi(z) = e^{-iz}$ extended to be 0 at ∞, which maps $\mathscr{D} \setminus L$ conformally onto $D \setminus [-1, 0]$, is a bijection $\mathscr{D} \to D$. Let the topology on $\mathscr{D}$ be defined by saying that φ is a homeomorphism. Clearly, we can use $\mathscr{D}$ as the maximal ideal space of $l^1(w)$, with the Gelfand transform on $l^1(w)$ identified with the Fourier transform restricted to $\mathscr{D} \setminus \{\infty\}$ and extended continuously to ∞.

Summing up, $M(w)$ is the direct sum in the Banach space sense of its two closed subalgebras $L^1(w)$ and $l^1(w)$.

An ideal I in an arbitrary algebra is called *modular* (regular) if A/I has a unit. If A itself has a unit, all ideals are modular.

It is well known and easy to show that a closed subspace of $L^1(w)$ is an ideal if and only if it is invariant under all right translations, since $L^1(w)$ contains approximate identities.

The dual space of $L^1(w)$ is identified with $L^\infty(w)$, the space of (equivalence classes of) functions g on R_+ such that $g/w \in L^\infty(\mathsf{R}_+)$. The norm of $g \in L^\infty(w)$ is of course $\|g/w\|_{L^\infty}$. In bracket notation,

$$\langle f, g \rangle = \int_0^\infty f(t) g(t)\, dt \quad \text{for } f \in L^1(w)$$

and $g \in L^\infty(w)$.

For $g \in L^\infty(w)$ and $f \in L^1(w)$, the function

$$g * \check{f}(x) = \int_0^\infty g(x+t) f(t)\, dt, \quad x \geqq 0,$$

belongs to $L^\infty(w)$ and satisfies

$$\langle h * f, g \rangle = \langle h, g * \check{f} \rangle \quad \text{for all } h \in L^1(w).$$

Thus the mapping $(g, f) \mapsto g * \check{f}$ determines $L^\infty(w)$ as a module over $L^1(w)$. $g * \check{f} = 0$ is equivalent to saying that g annihilates the closed ideal generated by f.

A weak $*$ closed subspace of $L^\infty(w)$ is the annihilator of a closed ideal, or, equivalently, a closed right translation invariant subspace, if and only if it is invariant under all left translations.

These observations have their obvious analogues for $l^1(w)$. For the sake of completeness, we shall write them down.

We identify the dual space of $l^1(w)$ with $l^\infty(w)$, the Banach space of complex-valued sequences $(g_n)_0^\infty$ such that $(g_n/w(n))_0^\infty \in l^\infty(\mathbf{N})$; the norm is obvious. In bracket notation,

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f_n g_n$$

for $f = (f_n)_0^\infty \in l^1(w)$ and $g = (g_n)_0^\infty \in l^\infty(w)$.

For arbitrary elements $f = (f_n)_0^\infty \in l^1(w)$ and $g = (g_n)_0^\infty \in l^\infty(w)$, the sequence defined by

$$g * \check{f}(n) = \sum_{k=0}^{\infty} g_{n+k} f_k, \quad n \in \mathbf{N},$$

belongs to $l^\infty(w)$; as is the case for $L^1(w)$, the mapping $(g, f) \mapsto g * \check{f}$ determines $l^\infty(w)$ as a module over $l^1(w)$. The condition $g * \check{f} = 0$ is equivalent to saying that g annihilates the closed ideal generated by f.

The Dirac measure δ_1 at 1 generates $l^1(w)$. Hence a closed subspace of $l^1(w)$ is an ideal if and only if it is invariant under right translations, that is, convolution by δ_1. It follows that a weak $*$ closed subspace A of $l^\infty(w)$ is the annihilator of a closed ideal if and only if it is invariant under all left translations, that is,

$$g * \check{\delta}_1 \in A \quad \text{if } g \in A.$$

For an arbitrary family $\mathscr{F}$ of measures in $M(w)$, let

$$h_M(\mathscr{F}) = \bigcap_{\mu \in \mathscr{F}} \{m \in \mathscr{M}_M : \hat{\mu}(m) = 0\},$$

where $\mathscr{M}_M$ is the maximal ideal space of $M(w)$ and $\hat{\mu}$ denotes the Gelfand transform of μ. We call $h_M(\mathscr{F})$ the hull of $\mathscr{F}$ in $M(w)$. If $\mathscr{F}$ is an ideal in $M(w)$, its hull is empty if and only if $\mathscr{F} = M(w)$.

For an arbitrary ideal I in $L^1(w)$, its hull $h_L(I)$ is the set of maximal modular ideals containing I, or, explicitly,

$$h_L(I) = h_M(I) \cap \Pi_-.$$

The hull $h_l(I)$ of an arbitrary ideal I in $l^1(w)$ is the set

$$h_l(I) = \bigcap_{f \in I} \{z \in \mathscr{D} : \hat{f}(z) = 0\},$$

where the Fourier transform restricted to $\mathscr{D} \setminus \{\infty\}$ is extended continuously to ∞. In this case, too, the hull $h_l(I)$ can be interpreted as the set of maximal ideals in $l^1(w)$ which contain I.

A closed ideal in $l^1(w)$, the hull of which is a single point in $\mathscr{D}$, is called primary. For the case $L^1(w)$, we say that a closed modular ideal I is primary at $z_0 \in \Pi_-$ if $h_L(I) = \{z_0\}$. Of course, this terminology can be extended to $M(w)$, too.

For $z_0 \in \Pi_-^\circ$, we define the closed ideals

$$I_n(z_0) = \{\mu \in M(w) \colon \hat{\mu}^{(k)}(z_0) = 0 \ \text{ for } k = 0, 1, \dots, n\}$$

in $M(w)$ for $n = 0, 1, 2, \dots$. It is well known that all primary ideals at $z_0 \in \Pi_-^\circ$ in $L^1(w)$ are of the form $I_n(z_0) \cap L^1(w)$, and that in $l^1(w)$, all primary ideals at $z_0 \in \mathscr{D}^\circ \setminus \{\infty\}$ are of the form $I_n(z_0) \cap l^1(w)$ (see [7, Theorem 3]). Likewise, the primary ideals at ∞ in $l^1(w)$ are well known, since $\infty \in \mathscr{D}^\circ$: The ideals $I_n(\infty) = \{\mu \in l^1(w) \colon |\mu|[0, n] = 0\}$, $n = 0, 1, \dots$, are primary ideals at ∞, and every primary ideal at ∞ in $l^1(w)$ coincides with one of them.

Since the mapping which maps $f \in L^1(w)$ onto $e^{ity} f(t)$, which belongs to $L^1(w)$, is an isometric automorphism of $L^1(w)$ for $y \in \mathsf{R}$, and the Fourier transform of the function $t \mapsto e^{ity} f(t)$, $t \geqq 0$, is $z \mapsto \hat{f}(z - y)$, $z \in \Pi_-$, we can always assume that for any primary ideal at $z_0 \in \partial\Pi_- = \mathsf{R}$, z_0 is chosen to be 0. The same argument for $l^1(w)$ shows that a primary ideal at some point on the boundary $\partial\mathscr{D} = \mathscr{D} \cap \mathsf{R}$ can be assumed to be primary at 0.

The structure of the primary ideals at 0 in $L^1(w)$ and $l^1(w)$ is much more complicated than for interior points. If w grows faster than polynomially, we get the ideals

$$I_n(0) = \{\mu \in M(w) \colon \hat{\mu}^{(k)}(0) \ \text{ for } k = 0, 1, \dots, n\}$$

for $n = 0, 1, \dots$, which are primary ideals at 0 in $M(w)$, and $I_n(0) \cap L^1(w)$ and $I_n(0) \cap l^1(w)$ are primary ideals at 0 in $L^1(w)$ and $l^1(w)$, respectively. Yngve Domar [7] has shown that if w is monotonically increasing and the function $x \mapsto w(x)(1 + |x|)^{-q}$, $x \geqq 0$, is bounded away from 0 and submultiplicative for all $q \in \mathsf{N}$, then each primary ideal at 0 in $L^1(w)$ either coincides with one of $I_n(0) \cap L^1(w)$, $n = 0, 1, \dots$, or is contained in

$$\bigcap_{n \in \mathsf{N}} I_n(0) \cap L^1(w).$$

The corresponding results hold for $l^1(w)$.

There is a chain of primary ideals which comes from $A(\Pi_-)$. A closed ideal in $A(\Pi_-)$ is primary at 0 if it is contained in the maximal ideal at 0 but in no other maximal ideal. The Beurling–Rudin theorem (see [13]) describes the closed ideals in the disc algebra $A(D)$. Applying this to $A_0(\Pi_-)$ and using [7, Theorem 1], we find that

$$J_\alpha = \{f \in A(\Pi_-) : f(0) = 0 \text{ and } \limsup_{y \to 0^+} y \log |f(-iy)| \leqq -\alpha\}, \quad \alpha \geqq 0,$$

are the primary ideals at 0 in $A(\Pi_-)$. Therefore

$$J_\alpha(M) = \{\mu \in M(w) : \hat{\mu} \in J_\alpha\}$$

is a closed ideal in $M(w)$. Put

$$J_\alpha(L) = J_\alpha(M) \cap L^1(w) \quad \text{and} \quad J_\alpha(l) = J_\alpha(M) \cap l^1(w),$$

which consequently are closed ideals in $L^1(w)$ and $l^1(w)$, respectively.

For the algebra $l^1(w)$, the knowledge about the primary ideals at 0 appears to be very limited. In the case $w \equiv 1$, Feldman [8] proved that all primary ideals at 0 in $l^1(w)$ are of the form $J_\alpha(l)$ for some $\alpha \geqq 0$. For weights w of polynomial growth, Colin Bennett and John Gilbert [2] described the primary ideals at 0; $J_\alpha(l)$ is a primary ideal at 0 for $\alpha \geqq 0$. Feldman [8] also showed that the annihilator of the primary ideal $J_\alpha(l)$ consists of those sequences $(g(n))_0^\infty$ in $l^\infty(w)$ which are restrictions to N of entire functions of order $\frac{1}{2}$ and type α. An entire function g is said to be of order $\frac{1}{2}$ and type α if

$$|g(z)| \leqq C_\varepsilon e^{(\alpha+\varepsilon)|z|^{1/2}} \quad \text{for all } \varepsilon > 0.$$

We are better off in the case $L^1(w)$. V. P. Gurariĭ has written a series of papers on the subject; most of the results can be found in [9]. We mention two of his results. Under some slight regularity conditions on the weight w, $J_\alpha(L)$ is a primary ideal at 0 if

$$\int_1^\infty x^{-3/2} \log w(x)\,dx < \infty; \tag{0.1}$$

otherwise $J_\alpha(L) = \{0\}$. If (0.1) holds, the annihilator of $J_\alpha(L)$ consists of those functions in $L^\infty(w)$ which are restrictions to R_+ of entire functions of order $\frac{1}{2}$ and type α.

In Section 4, we will show that there is a natural one-to-one correspondence between the closed ideals I in $l^1(w)$ for which

$$h_l(I) \subset \mathscr{D} \setminus L \tag{0.2}$$

and those closed modular ideals J in $L^1(w)$ which satisfy

$$h_L(J) \subset \mathscr{D} \setminus L. \tag{0.3}$$

We also indicate why (0.2–0.3) can be assumed without loss of generality if we want to study closed modular ideals. Further, it will be shown that the corresponding quotient algebras $l^1(w)/I$ and $L^1(w)/J$ are isomorphic.

It follows in particular that we can apply Gurariĭ's investigations concerning the primary ideals in $L^1(w)$ to the algebra $l^1(w)$, too.

It should be stressed that no special regularity or growth condition need be imposed on the weight in order to obtain our results.

1. Preliminaries.

We shall need a few basic results about Banach spaces and Banach algebras. For any Banach space A, A^* denotes its dual, that is, the space of continuous linear functionals on A; for a subspace E of A, $E^\perp$ denotes its annihilator.

In the following two propositions, B is a Banach space, and A is a Banach subspace of B, which means that A is a subspace of B having a norm which is stronger than the norm of B and makes A into a Banach space. By the closed graph theorem, a subspace can have (within equivalence) at most one Banach subspace norm.

PROPOSITION 1.1. *Let F be a closed subspace of B, and let E be a closed subspace of A which is contained in $F \cap A$. Then the following two conditions are equivalent*:

(a) *$F^\perp|_A$ is weak $*$ dense in $E^\perp$* $(\subset A^*)$.
(b) *$E = F \cap A$.*

PROOF. Since A is a Banach subspace of B, the restrictions of functionals in B^* to A are in A^*.

First we show that $F^\perp|_A$ is weak $*$ dense in $(F \cap A)^\perp$ $(\subset A^*)$. If ${}^\perp(F^\perp|_A)$ denotes the annihilator in A of $F^\perp|_A$, this amounts to showing that

$${}^\perp(F^\perp|_A) = F \cap A.$$

Now

$$\begin{aligned} {}^{\perp}(F^{\perp}|_A) &= \{x \in A : \langle x, y\rangle = 0 \quad \text{for all } y \in F^{\perp}\} \\ &= \{x \in B : \langle x, y\rangle = 0 \quad \text{for all } y \in F^{\perp}\} \cap A \\ &= F \cap A, \end{aligned}$$

which proves the assertion.

Thus (b) $\Rightarrow$ (a). Assume (a) holds. By the above argument,

$$E^{\perp} \subset (F \cap A)^{\perp}.$$

But since E is a closed subspace of $F \cap A$, equality must hold, and we conclude that

$$E = F \cap A,$$

which is (b).

PROPOSITION 1.2. *Let E and F be two closed subspaces of B such that E is contained in F and $E \cap A = F \cap A$. Then the following conditions are equivalent*:

(a) *Every functional in $(E \cap A)^{\perp}$ $(\subset A^*)$ has a unique extension in B^* which annihilates E.*

(b) *$E = F$, and the mapping $x + E \cap A \mapsto x + E$, $x \in A$, defines a Banach space isomorphism between $A/E \cap A$ and B/E; consequently, $B = A + E$.*

PROOF. First we show the implication (b) $\Rightarrow$ (a), which is a rather easy task. Assume (b) holds, and denote by ψ the Banach space isomorphism $A/E \cap A \to B/E$ defined by $x + E \cap A \mapsto x + E$, $x \in A$. By elementary functional analysis, the dual spaces of the quotient spaces B/E and $A/E \cap A$ can be identified with the annihilators $E^{\perp}$ and $(E \cap A)^{\perp}$ $(\subset A^*)$, respectively.

Since ψ is a Banach space isomorphism, its adjoint mapping $\psi^* : E^{\perp} \to (E \cap A)^{\perp}$ $(\subset A^*)$, which maps every functional in $F^{\perp}$ onto its restriction to A, is a Banach space isomorphism. This proves (a).

The less trivial part is the implication (a) $\Rightarrow$ (b). So, assume (a) holds. Let $\varphi : A/E \cap A \to B/E$ denote the continuous linear mapping $x + E \cap A \mapsto x + E$, $x \in A$. Then (a) asserts that the adjoint mapping $\varphi^* : E^{\perp} \to (E \cap A)^{\perp}$ $(\subset A^*)$, which restricts the functionals in $E^{\perp}$ to A, is bijective. Since φ^* is continuous, an application of the open mapping theorem shows that φ^* is a Banach space isomorphism. By elementary duality theory, which can be found in [14, Corollary 4.12, Theorem 4.15],

φ is a Banach space isomorphism $A/E \cap A \to B/E$. Hence $B = A + E$. We will now show that $F = E$. Let x be an arbitrary element in F. Since $B = A + E$, there exists a $y \in A$ such that

$$x - y \in E.$$

Hence, $y \in (F + E) \cap A = F \cap A = E \cap A$. We conclude that $F = E$.

REMARK 1.3. If A and B are Banach algebras such that the injection mapping $A \to B$ is a homomorphism and if moreover, E and F are closed ideals, the isomorphism in (b) of the previous proposition is a Banach algebra isomorphism.

DEFINITION 1.4. If B is a Banach algebra with Gelfand space $\mathcal{M}$, and I is an ideal in B, the hull of I is the set

$$h_B(I) = \bigcap_{x \in I} \{m \in \mathcal{M} : \hat{x}(m) = 0\},$$

which is closed in the Gelfand topology.

REMARK 1.5. This generalizes the definitions of hulls in $M(w)$, $L^1(w)$, and $l^1(w)$ in the introduction.

PROPOSITION 1.6. *Let I be a closed ideal in the Banach algebra B with Gelfand space $\mathcal{M}$. Then:*

(a) *The Gelfand space of I is homeomorphic to $\mathcal{M} \setminus h_B(I)$.*
(b) *The Gelfand transform in I of $x \in I$ is the restriction of the Gelfand transform $\hat{x}$ in B to $\mathcal{M} \setminus h_B(I)$.*
(c) *The Gelfand space of the quotient algebra B/I is homeomorphic to $h_B(I)$.*
(d) *The Gelfand transform of $x + I \in B/I$ is the restriction of $\hat{x}$ to $h_B(I)$.*

This is Theorem 7.3.1 in [11].

PROPOSITION 1.7. *Let A be a closed ideal in a Banach algebra B. Then $I \mapsto I \cap A$ defines a bijection from the set of closed modular B-ideals I, satisfying*

$$h_B(I) \cap h_B(A) = \varnothing,$$

onto the set of closed modular A-ideals. The inverse mapping is given by

$$J \mapsto \{b \in B : bA \subset J\}.$$

For every I in the family above, the mapping $x + I \cap A \mapsto x + I$, $x \in A$, is a Banach algebra isomorphism of $A/I \cap A$ onto B/I; hence every functional in A^* which annihilates $I \cap A$ has a unique extension in B^* which annihilates I.

PROOF. This follows from Theorems 1 and 2 in [7], combined with Proposition 1.2.

We wish to describe the space $\mathcal{M}_M$ of (non-trivial) complex homomorphisms on $M(w)$. Clearly, they are determined by their behaviour on $L^1(w)$ and $l^1(w)$.

PROPOSITION 1.8. *If m is a non-trivial complex homomorphism on $M(w)$, then either*

(a) $m \in \Pi_-$, *that is* $\hat{\mu}(m) = m(\mu) = \hat{\mu}(z) = \int_0^\infty e^{-itz}\, d\mu(t)$, $\mu \in M(w)$, *for some* $z \in \Pi_-$, *or*
(b) $\hat{\mu}(m) = m(\mu) = \sum_{n=0}^\infty z^n \mu(\{n\})$ *for some z in the closed unit disc D.*

PROOF. Observe that (a) and (b) do indeed define complex homomorphisms. The restriction of m to $L^1(w)$ is of course a complex homomorphism. If the restriction is non-trivial, m must have the form (a), by Proposition 1.6 (or Proposition 1.7). If m vanishes on $L^1(w)$, it has to be non-trivial on $l^1(w)$, so it is of the form (b).

2. Closed modular ideals in $L^1(w)$.

Let a be the function

$$a(x) = i e^{-t}, \quad t \geqq 0,$$

which clearly belongs to $L^1(w)$. The Fourier transform of a is

$$\hat{a}(z) = (z - i)^{-1}, \quad z \in \Pi_-.$$

If I is a closed modular ideal in $L^1(w)$, its hull $h_L(I)$ is a compact nonempty subset of Π_-, by Proposition 1.6. We denote by 1 the unit in the quotient Banach algebra $L^1(w)/I$, and the multiplication in $L^1(w)/I$ is denoted by juxtaposition.

An entire function g is said to be of finite exponential type if

$$|g(z)| \leqq C e^{A|z|}, \quad z \in \mathbf{C}, \tag{2.1}$$

for some constants C and A. The infimum of all admissible A in (2.1) is called the type of g.

For a definition of the indicator diagram of a function of finite exponential type we refer to Boas [4, Chapter 5]. The conjugate indicator diagram is the image of the indicator diagram under complex conjugation.

THEOREM 2.1. *Let I be a closed modular ideal in $L^1(w)$, and suppose $g \in L^\infty(w)$ annihilates I. Then*

$$z \mapsto \langle e^{z(1-i(a+I)^{-1})}, g\rangle, \quad z \in \mathbf{C}, \tag{2.2}$$

is an entire function of finite exponential type, and

$$g(t) = \langle e^{t(1-i(a+I)^{-1})}, g\rangle \tag{2.3}$$

almost everywhere on $\mathbf{R}_+$. By (2.3), *it is consistent to denote the entire function in* (2.2) *by g, too. The conjugate indicator diagram of g is contained in the closed convex hull of $-ih_L(I)$.*

PROOF. Since I is modular, $h_L(I)$ is a compact subset of Π_-, and the expression $(a+I)^{-1}$ is a well-defined element in $L^1(w)/I$. We first show that (2.3) holds.

A simple calculation shows that

$$\left(a - \frac{1}{z-i} + I\right)^{-1} = i - z - i(z-i)^2 \int_0^\infty e^{itz}\, e^{t(1-i(a+I)^{-1})}\, dt \tag{2.4}$$

for $z \in \mathbf{C} \setminus (\Pi_- \cup \{i\})$. The integral on the right hand side of (2.4) is well defined, since

$$\| e^{t(1-i(a+I)^{-1})} \| \leqq C(\varepsilon) e^{\varepsilon t} \quad \text{for every } \varepsilon > 0$$

by the spectral radius formula.

Let J be the closed $M(w)$-ideal

$$J = \{\mu \in M(w) \colon \mu * L^1(w) \subset I\}.$$

By Proposition 1.7, g has a unique extension in $(M(w))^*$ which annihilates J. Call this extension h. Since the canonical mapping $L^1(w)/I \to M(w)/J$ is a Banach algebra isomorphism,

$$\langle \delta_0, h\rangle = \langle 1, g\rangle,$$

where δ_0 is the Dirac measure at 0, thus the unit in $M(w)$.

Taking Fourier transforms, it is easy to see that

$$(i-z)\delta_0 - i(z-i)^2 c_z$$

is the inverse of $(a - \delta_0/(z-i))$, where c_z is the $L^1(w)$ function

$$c_z(t) = e^{itz}, \quad t \geqq 0,$$

for $z \in \mathbf{C} \setminus (\Pi_- \cup \{i\})$; consequently,

$$\begin{aligned}\left\langle\left(a-\frac{\delta_0}{z-i}\right)^{-1}, h\right\rangle &= (i-z)\langle\delta_0, h\rangle - i(z-i)^2\langle c_z, g\rangle \\ &= (i-z)\langle\delta_0, h\rangle - i(z-i)^2\int_0^\infty g(t)e^{itz}\,dt \\ &= (i-z)\langle 1, g\rangle - i(z-i)^2\int_0^\infty g(t)e^{itz}\,dt\end{aligned} \tag{2.5}$$

for $z \in \mathbf{C} \setminus (\Pi_- \cup \{i\})$.

Clearly,

$$\begin{aligned}\left\langle\left(a-\frac{\delta_0}{z-i}\right)^{-1}, h\right\rangle &= \left\langle\left(a-\frac{\delta_0}{z-i}+J\right)^{-1}, h\right\rangle \\ &= \left\langle\left(a-\frac{1}{z-i}+I\right)^{-1}, g\right\rangle\end{aligned} \tag{2.6}$$

for $z \in \mathbf{C} \setminus (\Pi_- \cup \{i\})$.

Applying g to (2.4) and observing that (any) $g \in L^\infty(w)$ is uniquely determined by the function

$$\mathscr{G}(z) = \int_0^\infty g(t)e^{itz}\,dt, \quad z \in \mathbf{C} \setminus \Pi_-,$$

we find that by (2.5–2.6)

$$g(t) = \langle e^{t(1-i(a+I)^{-1})}, g\rangle$$

almost everywhere on $\mathbf{R}_+$. Since

$$\|e^{z(1-i(a+I)^{-1})}\| \leqq \exp(|z| \cdot \|1-i(a+I)^{-1}\|) \quad \text{for } z \in \mathbf{C},$$

g is an entire function of finite exponential type.

Clearly, $\mathscr{G}$ is holomorphic in $\mathbf{C} \setminus \Pi_-$, and (2.4) defines a Banach algebra valued holomorphic function in $\mathbf{C} \setminus (h_L(I) \cup \{i\})$; consequently, $\mathscr{G}$ extends to a holomorphic function in $\mathbf{C} \setminus h_L(I)$ by (2.4–2.6). Now

$$z \mapsto \mathscr{G}(iz), \quad z \in \mathbf{C} \setminus (-ih_L(I))$$

is the Borel transform of g, and hence, the conjugate indicator diagram of g is contained in the closed convex hull of $-ih_L(I)$. The proof is finished.

REMARK 2.2. Keep the notation from Theorem 2.1 and assume that

$$h_L(I) \subset \mathscr{D} \setminus L.$$

It is then well known that the conclusion of Theorem 2.1 implies that there exist an $\varepsilon > 0$ and a constant $C = C(g)$ such that

$$|g(z)| \leqq C e^{(\pi-\varepsilon)|z|} \quad \text{for } z \in i\mathbf{R}.$$

An application of the Banach-Steinhaus theorem now shows that there exists a constant $K = K(I)$ such that

$$\| e^{z(1-i(a+I)^{-1})} \| \leqq K e^{(\pi-\varepsilon)|z|}, \quad z \in i\mathbf{R}.$$

We will now make some interesting observations which will not be used in any later context. The reader only interested in the main result can move on to Proposition 2.3.

Let g and I be as described in Theorem 2.1. Then

$$|g(z)| \leqq \| e^{z(1-i(a+I)^{-1})} \| \cdot \| g \|.$$

Since the functions

$$z \mapsto \cos \lambda \sqrt{z}, \quad z \in \mathbf{C},$$

which are entire functions of exponential type 0 for $\lambda \in \mathbf{C}$, are all bounded by 1 on $\mathbf{R}_+$ for real λ, but

$$|\cos \lambda \sqrt{z}| \to \infty \quad \text{as } \lambda \to +\infty$$

for fixed $z \in \mathbf{C} \setminus \mathbf{R}_+$, they cannot all belong to $I^\perp$ (λ real).

Let $g(1-i(a+I)^{-1})$ denote the functional in $I^\perp$ determined by the relation

$$\langle f+I, g(1-i(a+I)^{-1})\rangle = \langle (f+I)(1-i(a+I)^{-1}), g\rangle$$

for $f \in L^1(w)$, which satisfies

$$\| g(1-i(a+I)^{-1}) \| \leqq \| g \| \cdot \| 1-i(a+I)^{-1} \|.$$

Differentiating (2.2) with respect to z, we find that

$$\begin{aligned} g'(z) &= \langle (1-i(a+I)^{-1}) e^{z(1-i(a+I)^{-1})}, g\rangle \\ &= \langle e^{z(1-i(a+I)^{-1})}, g(1-i(a+I)^{-1})\rangle \end{aligned}$$

for $z \in \mathbf{C}$, which means that $g' \in I^\perp$ and $g' = g(1-i(a+I)^{-1})$ as functionals in $I^\perp$.

This shows that $I^{\perp}$ is invariant under differentiation, a result which should be compared with Theorem 3.34 in Domar's thesis [6] for regular algebras. On the other hand, if we have a weak $*$ closed subspace E of $L^{\infty}(w)$ such that differentiation is a bounded operator on E, E is the annihilator of a closed ideal in $L^1(w)$. Observe that this follows if we can show that E is left translation invariant. By assumption, there exists a constant C such that

$$\|g'\| \leqq C\|g\| \quad \text{for all } g \in E.$$

By Taylor's formula and a simple estimate of the remainder, we see that g can be extended to an entire function of exponential type $\leqq C$, and hence

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} g^{(n)}$$

defines a translation by x units of g. Therefore E is translation invariant, and the assertion follows. It is worth mentioning that one can show that the corresponding closed ideal is modular.

Differentiating (2.2) iteratively, we obtain the formula

$$g^{(n)}(z) = \langle (1 - i(a+I)^{-1})^n e^{z(1-i(a+I)^{-1})}, g \rangle, \quad z \in \mathbf{C},$$

for $n = 0, 1, \ldots$.

If $h_L(I) \subset \Pi_- \setminus \{0\}$, the above formula has a natural extension (among many), which defines fractional differentiation of $g \in I^{\perp}$:

$$g^{(\alpha)}(z) = e^{i\pi\alpha} \langle (i(a+I)^{-1} - 1)^{\alpha} e^{z(1-i(a+I)^{-1})}, g \rangle, \quad z \in \mathbf{C},$$

for $\alpha \in \mathbf{C}$. Here the power is defined using the principal branch of the logarithm. It is worth noting that

$$\alpha \mapsto g^{(\alpha)}(z), \quad \alpha \in \mathbf{C},$$

is an entire function of finite exponential type.

PROPOSITION 2.3. *Let I be a closed modular ideal in $L^1(w)$, and suppose $g \in L^{\infty}(w)$ annihilates I. Proposition* 1.7 *gives a unique extension of g in $(M(w))^*$ which annihilates the closed ideal*

$$J = \{\mu \in M(w) : \mu * L^1(w) \subset I\}.$$

If we denote this extension by h, we have

$$\langle \delta_n, h \rangle = \langle e^{n(1-i(a+I)^{-1})}, g \rangle = g(n) \quad \textit{for } n = 0, 1, \ldots.$$

PROOF. Let $\{e_k\}_0^\infty$ be a sequence of approximate identities in $L^1(w)$ with support shrinking to $\{0\}$. Then $e_k * \delta_n \in L^1(w)$ and

$$\langle e_k * \delta_n, g\rangle \to g(n) \quad \text{as } k \to \infty.$$

Evidently, $\{e_k + I\}$ is a sequence of approximate identities $i\, L^1(w)/I$. Since I is modular, $e_k + I$ tends to the unit in $L^1(w)/I$ as $k \to \infty$, and consequently

$$e_k + J \to \delta_0 + J \quad \text{as } k \to \infty$$

in the norm of $M(w)/J$. Therefore

$$\langle e_k * \delta_n, g\rangle = \langle e_k * \delta_n, h\rangle \to \langle \delta_n, h\rangle \quad \text{as } k \to \infty.$$

which proves the proposition.

3. Closed ideals in $l^1(w)$.

Let I be a closed non-zero ideal in $l^1(w)$. Then $h_l(I) \cap \mathscr{D}^\circ$ consists of isolated points, and $h_l(I) \cap \partial\mathscr{D}$ is a closed subset of $\partial\mathscr{D} = \mathscr{D} \cap \mathbf{R}$ with linear measure zero. The easiest way of seeing this is to observe that the function

$$\hat{\mu}(i \log z) = \sum_{n=0}^{\infty} \mu(\{n\}) z^n, \quad z \in D \setminus [-1,0],$$

extends continuously to a function in the disc algebra $A(D)$. Here log is the principal branch of the logarithm.

Since $\infty \in \mathscr{D}^\circ$, an application of Šilov's idempotence theorem (see [7]) shows that we can assume without loss of generality that

$$h_l(I) \subset \mathscr{D} \setminus \{\infty\}.$$

It is easy to check that the mapping which maps $\mu \in l^1(w)$ onto the measure $\nu \in l^1(w)$ with

$$\nu(\{n\}) = z_0^n \mu(\{n\}) \quad \text{for } n \in \mathbf{N}$$

is an isometric automorphism of $l^1(w)$ for z_0 of modulus 1. Hence if $h_l(I) \subset \mathscr{D} \setminus \{\infty\}$, we can assume without loss of generality that

$$h_l(I) \subset \mathscr{D} \setminus L.$$

We now state the following

THEOREM 3.1. *Let I be a closed ideal in $l^1(w)$ such that*

$$h_l(I) \subset \mathscr{D} \setminus L.$$

Suppose

$$g = (g_n)_0^\infty \in l^\infty(w)$$

annihilates I. Then there exists a unique entire function of finite exponential type, which we also denote by g, with indicator diagram contained in $-i(\mathscr{D} \setminus L)$ such that

$$g(n) = g_n \quad \text{for } n \in \mathbf{N}. \tag{3.1}$$

The entire function is given by the formula

$$g(z) = \langle (\delta_1 + I)^z, g \rangle, \quad z \in \mathbf{C}, \tag{3.2}$$

where the power is defined, as usual, by the principal branch of the logarithm.

PROOF. The uniqueness follows from Carlson's theorem [4, Theorem 9.2.1]. Clearly, (3.2) defines an entire function, and the estimate

$$|g(z)| \leqq \|g\| \cdot e^{|z| \, \|\log(\delta_1 + I)\|}, \quad z \in \mathbf{C},$$

shows that it is of finite type. It is trivially observed that (3.1) holds.

It remains to show that the indicator diagram of g is contained in $-i(\mathscr{D} \setminus L)$. After an elèmentary calculation, we see that the Borel transform of g is the function

$$\mathscr{G}(z) = \langle (z - \log(\delta_1 + I))^{-1}, g \rangle,$$

which is defined and holomorphic for $z \in \mathbf{C} \setminus (-i\, h_l(I))$. By assumption,

$$h_l(I) \subset \mathscr{D} \setminus L,$$

and consequently, the conjugate indicator diagram is a compact subset of $-i(\mathscr{D} \setminus L)$. After complex conjugation, we see that this holds for the indicator diagram, too, and the theorem is established.

4. The main result.

To prepare ourselves, let us first make some remarks. Recall that by Theorem 2.1 and Proposition 2.3, a functional $g \in M(w)^*$ annihilating a closed ideal J in $M(w)$ such that

$$h_M(J) \subset \Pi_-$$

can be identified with an entire function with specified growth whose restriction to $\mathbf{R}_+$ belongs to $L^\infty(w)$. Moreover, the norm of g is the $L^\infty(w)$

norm of (the restriction to R_+ of) the corresponding entire function. Before we can state our main result, we shall need the following lemma.

LEMMA 4.1. *Let J be an arbitrary closed ideal in $M(w)$ such that*

$$h_M(J) \subset \mathscr{D} \setminus L,$$

and let g be any functional in $M(w)^*$ which annihilates J. Then there exists a constant $C = C(J)$ such that

$$\|g\|_{L^\infty(w)} \leqq C \|g|_{\mathsf{N}}\|_{l^\infty(w)}.$$

PROOF. We argue by contradiction. So, assume that the conclusion of the lemma is false. Then there exists a sequence $\{g_n\}_0^\infty$ of functionals in $J^\perp$ such that

$$\|g_n\|_{L^\infty(w)} = 1, \quad n = 0, 1, \ldots,$$

and

$$\|g_n|_{\mathsf{N}}\|_{l^\infty(w)} \to 0.$$

Since

$$|g_n(z)| \leqq \|e^{z(1-i(a+J)^{-1})}\| \quad \text{for } z \in \mathsf{C}, \tag{4.1}$$

the sequence $\{g_n\}_0^\infty$ forms a normal family, by Arzela–Ascoli's theorem.

The Bolzano–Weierstrass theorem combined with the equicontinuity of $\{g_n\}_0^\infty$ shows that it is sufficient to deal with the following two cases, which are not necessarily mutually exclusive.

CASE 1. There exists a $t_0 \in \mathsf{R}_+$ such that

$$|g_n(t_0)|/w(t_0) \to 1 \quad \text{as } n \to \infty.$$

By normality, there exists a subsequence $\{g_{n_j}\}_0^\infty$ which converges uniformly on compact sets to an entire function g, which cannot vanish identically, since $|g(t_0)| = 1$. By assumption

$$\|g_n|_{\mathsf{N}}\|_{l^\infty(w)} \to 0 \quad \text{as } n \to \infty,$$

and consequently

$$g(k) = \lim_{j\to\infty} g_{n_j}(k) = 0 \quad \text{for } k \in \mathsf{N}.$$

Evidently, g also has the bound (4.1), and consequently,

$$|g(z)| \leqq C e^{(\pi-\varepsilon)|z|}, \quad z \in i\mathsf{R},$$

for some $\varepsilon > 0$ and some constant C, by Remark 2.2. Carlson's theorem (see [4, Theorem 9.2.1]) now implies that g must vanish identically, which is a contradiction.

CASE 2. There exists a sequence $\{\xi_n\}_0^\infty$ of points in $\mathbf{R}_+$ tending to $+\infty$ such that

$$|g_n(\xi_n)|/w(\xi_n) \to 1 \quad \text{as } n \to \infty.$$

For $n \in \mathbf{N}$, put

$$F_n(z) = g_n(\xi_n + z)/w(\xi_n), \quad z \in \mathbf{C}.$$

A careful look at the statement of Theorem 2.1 reveals that

$$\| e^{t(1-i(a+J)^{-1})} \| \leqq w(t) \quad \text{for } t \in \mathbf{R}_+.$$

Consequently, we have the estimate

$$
\begin{aligned}
|F_n(z)| &\leqq \| e^{(\xi_n+z)(1-i(a+J)^{-1})} \| / w(\xi_n) \\
&\leqq \| e^{z(1-i(a+J)^{-1})} \| \cdot \| e^{\xi_n(1-i(a+J)^{-1})} \| / w(\xi_n) \qquad (4.2)\\
&\leqq \| e^{z(1-i(a+J)^{-1})} \| \quad \text{for } z \in \mathbf{C},
\end{aligned}
$$

which shows that $\{F_n\}_0^\infty$ forms a normal family. Hence there exists a subsequence $\{F_{n_j}\}_0^\infty$ which converges, uniformly on compact sets, to an entire function F. Since

$$|F(0)| = \lim_{j \to \infty} |g_{n_j}(\xi_{n_j})|/w(\xi_{n_j}) = 1,$$

F does not vanish identically.

Obviously, F also has the bound (4.2). Let ξ be some cluster point of the bounded sequence $\{\xi_{n_j} - [\xi_{n_j}]\}_0^\infty$. Here $[x]$ denotes the largest integer $\leqq x$. By taking a thinner subsequence, we can assume without loss of generality that ξ is the only cluster point.

Since the convergence of $\{F_{n_j}\}_0^\infty$ is uniform on compact sets, and the Cauchy estimates show that the same holds for the derivatives, our assumption

$$\| g_n|_{\mathbf{N}} \|_{l^\infty(w)} \to 0 \quad \text{as } n \to \infty$$

implies that

$$F(k - \xi) = \lim_{j \to \infty} g_{n_j}([\xi_{n_j}] + k)/w([\xi_{n_j}] + k) \cdot w([\xi_{n_j}] + k)/w(\xi_{n_j}) = 0 \text{ for } k \in \mathbf{Z}_+.$$

Again Carlson's theorem or perhaps more accurately, Theorem 9.3.4 [see 4] shows that F must vanish identically, which is our desired contradiction.

The proof of the lemma is now finished.

REMARK 4.2. In the proof of the lemma, we borrowed some arguments from the proof of Theorem 3a in Shmuel Agmon's paper [1]. Unfortunately, Agmon's results do not seem to imply Lemma 4.1 since he allows the constant C which appears in the formulation of our Lemma 4.1 to depend on the entire function g.

We now state our main result.

THEOREM 4.3. *For any closed ideal I in $l^1(w)$ with $h_l(I) \subset \mathscr{D} \setminus L$, let J_0 be the closed ideal in $M(w)$ generated by I.*

(a) *There exist two unique closed ideals J and K in $M(w)$ such that*

$$\begin{aligned} J_0 \quad &= J \cap K, \\ h_M(J) \; &= h_l(I), \\ h_M(K) &= h_M(J_0) \setminus h_l(I). \end{aligned}$$

This defines a mapping from the closed ideals I in $l^1(w)$ with $h_l(I) \subset \mathscr{D} \setminus L$ to the closed ideals J in $M(w)$ with $h_M(J) \subset \mathscr{D} \setminus L$.

(b) *The mapping in* (a) *is injective, since $J \cap l^1(w) = I$.*

(c) *The mapping in* (a) *is onto, and by* (b) *it is a bijection.*

(d) *The mapping*

$$\varphi : \mu \mapsto \int_0^\infty e^{t \log(\delta_1 + I)} d\mu(t), \quad \mu \in M(w),$$

is a continuous homomorphism from $M(w)$ onto $l^1(w)/I$ with kernel J. Here log *is the principal branch of the logarithm. On $l^1(w)$, φ is the canonical quotient mapping $l^1(w) \to l^1(w)/I$.*

(e) *The mapping $\mu + I \mapsto \mu + J$, $\mu \in l^1(w)$, defines a Banach algebra isomorphism of $l^1(w)/I$ onto $M(w)/J$; its inverse is the quotient mapping*

$$M(w)/J \to l^1(w)/I$$

induced by φ.

REMARK 4.4. The adjoint mapping

$$\varphi^* : I^\perp \to (M(w))^*$$

extends the elements of $I^{\perp}$ to functionals in $(M(w))^*$ that annihilate the closed ideal J. A functional $g \in I^{\perp}$ is mapped onto

$$\varphi^* g(t) = \langle e^{t\log(\delta_1 + I)}, g\rangle, \quad t \in \mathsf{R}_+,$$

which extends to an entire function of finite exponential type, by Theorem 3.1. It is clear in what way $\varphi^* g$ is an element of $(M(w))^*$:

$$\langle \mu, \varphi^* g\rangle = \int_0^{\infty} \varphi^* g(t) d\mu(t) \quad \text{for } \mu \in M(w).$$

PROOF OF THEOREM 4.3. We first prove (a). Proposition 1.8 tells us what the complex homomorphisms on $M(w)$ look like, and we have previously mentioned that the closed lower half plane Π_- is an open subset of the Gelfand space of $M(w)$. It is now easy to see that

$$h_M(J_0) \cap \Pi_- = \bigcup_{k \in \mathsf{Z}} (h_l(I) + 2\pi k),$$

and since by assumption $h_l(I) \subset \mathscr{D} \setminus L$, $h_l(I)$ is an open and closed subset of $h_M(J_0)$. (a) now follows from a form of Šilov's idempotence theorem which can be found in Domar's paper [7, Theorem 2].

We proceed to prove (d). Since the estimate

$$\begin{aligned} \| e^{t\log(\delta_1 + I)} \| &\leqq \| (\delta_1 + I)^n \| \cdot \| e^{(t-n)\log(\delta_1 + I)} \| \\ &\leqq w(n) \cdot \| e^{(t-n)\log(\delta_1 + I)} \| \end{aligned}$$

holds for $n \in \mathsf{N}$, we see that the continuous function

$$t \mapsto \| e^{t\log(\delta_1 + I)} \|, \quad t \in \mathsf{R}_+,$$

belongs to $L^{\infty}(w)$, by choosing n as the smallest integer $\geqq t$. This shows that φ is a well-defined continuous linear mapping. A straightforward calculation shows that φ is a homomorphism. To finish the proof of (d), it remains to show that the kernel of φ is J.

Evidently, the restriction of φ to $l^1(w)$ is the canonical homomorphism $l^1(w) \to l^1(w)/I$, and since φ is a continuous homomorphism, $\operatorname{Ker} \varphi$ is a closed ideal in $M(w)$ which contains I, and consequently it contains J_0, too.

As a first step, we show that $\operatorname{Ker} \varphi \supset J$. Denote by $\tilde{\varphi}$ the continuous isomorphism

$$M(w)/\operatorname{Ker} \varphi \to l^1(w)/I$$

induced by φ. By the open mapping theorem, $\tilde{\varphi}$ is a Banach algebra isomorphism. Consequently, the adjoint mapping $\varphi^*: I^{\perp} \to (M(w))^*$

is a Banach space (module) isomorphism from $I^{\perp}$ onto its image $\operatorname{im}\varphi^* = (\operatorname{Ker}\varphi)^{\perp}$.

Since the only complex homomorphisms in $\operatorname{im}\varphi^* = (\operatorname{Ker}\varphi)^{\perp}$ correspond to points in $h_M(J) = h_l(I)$, it follows that $\operatorname{Ker}\varphi \supset J$. Our next step is to show that $\operatorname{Ker}\varphi = J$. Recall that the annihilator $J^{\perp}$ of J consists of the restrictions to R_+ of some entire functions of finite exponential type with conjugate indicator diagrams contained in the closed convex hull of $-i\,h_M(J)$, by Theorem 2.1 and Proposition 2.3.

Since $h_M(J) \subset \mathscr{D} \setminus L$, Carlson's theorem ([4, Theorem 9.2.1]) shows that these entire functions are uniquely determined by their values on N.

If we start with an arbitrary $g \in J^{\perp}$, then its restriction to N (regarded as an entire function) belongs to $l^{\infty}(w)$ and annihilates I. By Theorem 3.1, the extension $\varphi^*(g|_{\mathsf{N}})$ of $g|_{\mathsf{N}}$ is the restriction to R_+ of an entire function of finite exponential type with indicator diagram contained in $-i(\mathscr{D} \setminus L)$. Since $\varphi^*(g|_{\mathsf{N}})$ and g have the same values on N, Carlson's theorem ([4, Theorem 9.2.1]) shows that

$$\varphi^*(g|_{\mathsf{N}}) = g;$$

hence $\operatorname{im}\varphi^* \supset J^{\perp}$, and consequently $\operatorname{Ker}\varphi \subset J$. Since we previously showed that $\operatorname{Ker}\varphi \supset J$, this proves (d).

As we have remarked before, the restriction of φ to $l^1(w)$ is the canonical homomorphism $l^1(w) \to l^1(w)/I$, and consequently the inverse of the Banach algebra isomorphism $\tilde{\varphi}$ is the mapping $\mu + I \mapsto \mu + J$, $\mu \in l^1(w)$. This proves (e).

Now is the time to show that (b) holds, that is, that $J \cap l^1(w) = I$. Evidently, $J \cap l^1(w) \supset I$. Since the restriction of $J^{\perp}$ to $l^1(w)$ is $I^{\perp}$, an application of Proposition 1.1 shows that $J \cap l^1(w) = I$, which is (b).

The only thing that remains for us to do is to show that (c) holds.

To this end, let J be an arbitrary closed ideal in $M(w)$ such that $h_M(J) \subset \mathscr{D} \setminus L$. Denote the closed $l^1(w)$-ideal $J \cap l^1(w)$ by I.

It will be sufficient to show that

$$J^{\perp}|_{l^1(w)} = I^{\perp}.$$

To see this, observe that then $h_l(I) = h_M(J) \subset \mathscr{D} \setminus L$, so that the homomorphism φ is well-defined and that, by the uniqueness property of the entire functions associated to the annihilators, φ^* is the inverse of the restriction mapping. Then

$$\operatorname{im}\varphi^* = (\operatorname{Ker}\varphi)^{\perp} = J^{\perp},$$

and consequently, $\operatorname{Ker}\varphi = J$, which does it.

Let $v: l^1(w)/I \to M(w)/J$ be the canonical homomorphism

$$\mu + I \mapsto \mu + J, \quad \mu \in l^1(w).$$

Its adjoint mapping $v^*: J^\perp \to I^\perp$ restricts the functionals in $J^\perp$ to $l^1(w)$. By Theorem 4.14 [see 14], $\operatorname{im} v^* = J^\perp|_{l^1(w)}$ is weak $*$ closed if and only if it is norm closed.

Let $g \in J^\perp$ be arbitrary, and denote its extension (see Theorem 2.1 and Proposition 2.3) to an entire function of finite exponential type with indicator diagram contained in $-i(\mathscr{D} \setminus L)$ by g, too. By Lemma 4.1, there exists a constant C not depending on the chosen $g \in J^\perp$ such that

$$\|g\|_{L^\infty(w)} \leqq C \|g|_{\mathsf{N}}\|_{l^\infty(w)}.$$

Hence $J^\perp|_{l^1(w)}$ is norm-closed, and therefore weak $*$ closed, too.

By Proposition 1.1, $J^\perp|_{l^1(w)} = I^\perp$, which completes the proof of the theorem.

REMARK 4.5. Combining Theorem 4.3 with Proposition 1.7, we have a Banach algebra isomorphism from $l^1(w)/I$ onto $L^1(w)/J \cap L^1(w)$. Another consequence is that the set of closed modular ideals in $L^1(w)$ with hulls in $\mathscr{D} \setminus L$ and the set of closed ideals in $l^1(w)$ with hulls in $\mathscr{D} \setminus L$ can be identified.

In Section 3, we mentioned that when trying to describe the closed ideals in $l^1(w)$ one can without loss of generality restrict one's attention to closed ideals with hulls in $\mathscr{D} \setminus L$. Similarly, if J is a closed modular ideal in $L^1(w)$, we can without loss of generality restrict our attention to ideals with $h_L(J) \subset \mathscr{D} \setminus L$. To see this, remember that $h_L(J) \setminus \Pi_-^\circ$ consists of isolated points and that $h_L(J) \cap \partial\Pi_- = h_L(J) \cap \mathsf{R}$ is a closed subset of R with linear measure zero, and apply Šilov's idempotence theorem in the form found in [7, Theorem 2]. The last step is to remember that the automorphism mentioned in the introduction translates the Fourier transforms. There is a different argument which is less special, since it does not use any special information about the hulls, but we have to sacrifice something: we have to change our weight function. Let

$$w_t(x) = w(tx), \quad x \in \mathsf{R}_+, \text{ for } t > 0.$$

Then the function w_t is submultiplicative, making $M(w_t)$ a Banach algebra.

The mapping T_t which takes a function $x \mapsto f(x)$, $x \in \mathsf{R}_+$, onto the function $x \mapsto tf(tx)$, $x \in \mathsf{R}_+$, is an isometric Banach algebra isomorphism of $L^1(w)$ onto $L^1(w_t)$.

By choosing t small enough one can assume that

$$h_L(T_t(J)) \subset \mathscr{D} \setminus L.$$

Theorem 4.3 then states that there is a closed ideal in $l^1(w_t)$ which corresponds to J.

5. Final remarks.

A Theorem 4.3 could also be obtained for the corresponding Beurling algebras on the additive groups R and Z; one then assumes the weight to be submultiplicative on the whole real axis. We will denote these algebras by $L^1(w, \mathsf{R})$ and $l^1(w, \mathsf{Z})$, respectively. One usually separates between the analytic case when the Gelfand space of $L^1(w, \mathsf{R})$ is a strip with width and the Gelfand space of $l^1(w, \mathsf{Z})$ is an annulus, and the non-analytic case when the Gelfand space of $L^1(w, \mathsf{R})$ is a line and that of $l^1(w, \mathsf{Z})$ is a circle.

In the analytic case, the situation concerning the hulls of ideals is almost exactly the same as in the case of algebras on the additive semigroups R_+ and N, and the technique involving Šilov's idempotence theorem to shrink the hulls can be applied.

In the non-analytic case, we can assume, without loss of generality, that

$$\lim_{|x| \to \infty} \frac{1}{x} \log w(x) = 0.$$

Following Arne Beurling, one speaks of the non-quasianalytic case when

$$\int_{-\infty}^{\infty} \frac{\log w(x)}{1+x^2} dx < +\infty,$$

and the quasianalytic case when this integral diverges. In the quasianalytic case, the hull of a modular ideal in $L^1(w, \mathsf{R})$ is a compact subset of the real axis which does not contain any interval, according to Beurling [3]. Again, the technique using Šilov's idempotence theorem works.

In the non-quasianalytic case, however, one can only use the weight modification described at the end of the last section, since $L^1(w, \mathsf{R})$ will then contain functions having Fourier transforms with arbitrarily small compact support.

The closed subspace of $L^1(w, \mathsf{R}_+)$ consisting of those functions which vanish (almost everywhere) on the interval $[0, \alpha]$ for some fixed $\alpha \geqq 0$ is a so-called primary ideal at infinity, which means that it is a closed ideal which is not contained in any maximal modular ideal. This shows that there are plenty of non-modular ideals in $L^1(w, \mathsf{R}_+)$. Some attention has been given to the problem of describing all primary ideals at infinity. Under some slight growth and regularity restrictions on the weight w, Gurariĭ [9] has shown that all primary ideals at infinity are of the above-mentioned type for some $\alpha \in [0,\infty)$. The author [10] has recently studied the corresponding problem for the group algebra $L^1(w, \mathsf{R})$ under the assumption that the weight w is of analytic type.

We will now describe the relation between quasianalytic Beurling algebras and the familiar notion of quasianalytic classes. Assume that we have a quasianalytic weight w. Call $L^1(w, \mathsf{R})$ quasianalytic in the sense of Carleman if

$$\hat{f}^{(n)}(0) = 0 \quad \text{for } n = 0, 1, \ldots, \text{ for any } f \in L^1(w, \mathsf{R})$$

implies that $f = 0$, and extend this terminology to $l^1(w)$ in the obvious fashion. It then follows from our results that $L^1(w, \mathsf{R})$ and $l^1(w, \mathsf{Z})$ are quasianalytic in the sense of Carleman simultaneously.

In a paper from 1951, Lennart Carleson [5] shows that Bernstein's approximation problem has a solution for the weight w if w is even, $\log w$ is convex in $\log x$, and

$$\int_{-\infty}^{\infty} \frac{\log w(x)}{1+x^2}\, dx = +\infty;$$

this result can be used to show that for such weights w, $L^1(w, \mathsf{R})$ is quasianalytic in the sense of Carleman.

It should also be mentioned here that there are generalizations of Theorem 4.3 to weighted algebras on R^n_+ and N^n for $n > 1$. However, in this case, the hull of a general ideal in $l^1(w, \mathsf{N}^n)$, which can be regarded as a compact subset of C^n, can be rather twisted, and there may not be any way to shrink it with Šilov's idempotence theorem.

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