MATH. SCAND. 58 (1986), 311–322

PROPER HOLOMORPHIC MAPPINGS

JOHN E. FORNÆSS[1] AND ERIK LØW

The original theorem of Fefferman [5] states that a biholomorphic map between strictly pseudoconvex domains with C^∞ boundaries extends to a C^∞ diffeomorphism of the closures. This result has been generalized to proper maps and also to more general domains in a series of papers (see for example [2], [3]). Furthermore, it has been simplified and partly localized (see [6], [7]). This paper is a contribution to this simplification and localization process and all our methods are elementary, in the sense that purely one variable techniques are used. We are only considering strictly pseudoconvex domains with C^∞ boundaries.

Central to our approach is the paper of Webster–Nirenberg–Yang [7], where it is proved that a biholomorphic map between local open pieces of strictly pseudoconvex boundaries which is bicontinuous up to the boundary will be C^∞ up to the boundary if it satisfies an additional condition, which they call Condition A. Their proof of this is completely elementary and uses their version of the reflection method. After first proving that a locally proper holomorphic map will extend Hölder $\frac{1}{2}$ continuously up to the boundary in Theorem 1, we prove in Theorem 2 that condition A is automatically satisfied in the situation of their result, but only on a dense open subset. Finally, in Theorem 3, we modify a technique of Alexander [1] to show that a global proper holomorphic map f between strictly pseudoconvex domains is locally biholomorphic on a dense open subset of the boundary, thus obtaining C^∞ extendability of f to a dense open subset of the boundary.

Let $\Omega \subset\subset \mathbf{C}^n$ be a domain with C^∞ boundary Γ and let n_p be the unit outward vector field to Γ at p. Then, for small δ, the function $\varphi_\delta(p) = p - \delta n_p$ is a C^∞ diffeomorphism of Γ onto

$$\Gamma_\delta = \{z \in \Omega;\ \text{the distance from } z \text{ to } \Gamma \text{ is } \delta\}.$$

[1]) Supported by NSF grant DMS 8401273.
Received February 6, 1986.

We denote this distance by $d(z)$. Also, let $n_z = n_p$ if $\varphi_\delta(p) = z$ for some δ. If $\Omega,\Omega' \subset\subset \mathbf{C}^n$ are domains with C^∞-smooth strictly pseudoconvex boundaries Γ,Γ' and $f\colon \Omega \to \Omega'$ is proper and holomorphic, then the Henkin–Vormor theorem states that f is boundary distance preserving, i.e.

$$C_1 d(z) \leqq d'(z') \leqq C_2 d(z)$$

for some constants C_1 and C_2, with $z' = f(z)$. From this and properties of the Kobayashi metric on strictly pseudoconvex domains, it follows easily that f extends to a Hölder $\frac{1}{2}$ continuous function on $\bar{\Omega}$. We first prove a local version of the Henkin–Vormor theorem.

THEOREM 1. *Let $\Omega,\Omega' \subset\subset \mathbf{C}^n$ be strictly pseudoconvex domains, $p \in \partial\Omega$, U an open neighbourhood of p and suppose $f\colon U \cap \Omega \to \Omega'$ is holomorphic such that $f(z_n) \to \Gamma'$ whenever $z_n \to \Gamma$. Then there is an open neighbourhood V of p such that f is boundary distance preserving in $V \cap \Omega$ and f extends to a Hölder $\frac{1}{2}$-continuous function on $\overline{V \cap \Omega}$.*

PROOF. Shrinking U if necessary, we may assume the fibers are discreet, since $f^{-1}(w)$ is a variety, which will meet Γ unless it is discreet, by strict pseudoconvexity.

It follows that f is open and for every $z \in U$ and neighbourhood N of z, there is an open neighbourhood $D \subset N$ of z such that $f\colon D \to f(D)$ is proper, hence a branched covering. (See 15.1 of Rudin [8].)

Let $V \subset\subset U$ be a sufficiently small open neighbourhood of p and let ρ be a C^∞ plurisubharmonic function in a neighbourhood of $\bar{\Omega}$ such that $-1 \leqq \rho < 0$ on Ω, $\rho \equiv 0$ in $V \cap \Gamma$ and $\rho \equiv -1$ on $\bar{\Omega} \setminus U$. Now define on Ω':

$$\rho'(z') = \begin{cases} \max\{\rho(z);\, f(z) = z'\} & \text{if } z' \in f(U) \\ -1 & \text{otherwise.} \end{cases}$$

Now $-1 \leqq \rho' < 0$ in Ω'. We claim that ρ' is plurisubharmonic. It is easy to see that ρ' is continuous. Clearly ρ' has the subaveraging property for all $z' \notin f(U)$. If $z' \in f(U)$, choose $z \in U$ with $f(z) = z'$ and $\rho'(z') = \rho(z)$, and a neighbourhood D of z such that $f\colon D \to f(D)$ is proper. Then the critical set of $f|_D$,

$$B = \{z';\, z' = f(z) \text{ for some } z \in D \text{ with } Jf(z) = 0\}$$

is a zero-variety in $f(D)$ and $f\colon D \setminus f^{-1}(B) \to f(D) \setminus B$ is an unbranched covering. Hence

$$\rho''(w) = \max\{\rho(z);\, w = f(z) \text{ for some } z \in D\}$$

is plurisubharmonic in $f(D) \setminus B$, hence in $f(D)$ by the removable singularity theorem and $\rho''(z') = \rho'(z')$, so ρ' has the subaveraging property at z', since $\rho' \geqq \rho''$, By the Hopf lemma, there is a constant $C > 0$ such that $\rho'(z') < -Cd'(z')$. Now, for z sufficiently close to p, $z' = f(z)$.

$$d(z) \cong -\rho(z) \geqq -\rho'(z') \geqq Cd'(z')$$

so f is boundary distance decreasing ($\cong$ means that the quotient is bounded above and below). The rest follows by the usual localization of the Kobayashi metric argument, (see for example [4]).

In [7], Nirenberg, Webster, and Yang proved the following theorem by means of the reflection method.

THEOREM. *Let $\Omega, \Omega' \subset\subset \mathbf{C}^n$ be domains with strictly pseudoconvex C^∞ boundaries Γ, Γ' relatively open subsets of the boundaries and V, V' open subsets of $\mathbf{C}^n$ such that $V \cap \partial\Omega = \Gamma$ and $V' \cap \partial\Omega' = \Gamma'$. Let $f: V \cap \Omega \to V' \cap \Omega'$ be biholomorphic and suppose f extends bicontinuously to $(V \cap \Omega) \cup \Gamma$, taking Γ into Γ'. Furthermore, suppose f satisfies*

CONDITION A. *The normal component of $f_*^{-1} n_{z'}$ is bounded away from zero*; i.e.

$$\langle f_*^{-1} n_{z'}, n_z \rangle \geqq C$$

for some positive constant C.

Then f is C^∞-smooth in $(V \cap \Omega) \cup \Gamma$.

They then obtain an elementary proof of Fefferman's theorem by proving Condition A in case $f: \Omega \to \Omega'$ is biholomorphic. The proof of Contidion A is lengthy and tricky however. We shall now give a simple proof that Condition A holds on a dense open subset of the boundary.

Let Ω, Ω', Γ, Γ', V and V' be as above and $f: V \cap \Omega \to V' \cap \Omega'$ biholomorphic such that $z_n \to \Gamma$ if and only if $z'_n \to \Gamma'$. Then by Theorem 1, f extends bicontinuously to $(V \cap \Omega) \cup \Gamma$, taking Γ into Γ'. Furthermore,

THEOREM 2. *There is an open dense subset $\Gamma_1 \subset \Gamma$ such that f* is *C^∞-smooth on $(V \cap \Omega) \cup \Gamma_1$.*

PROOF. It is sufficient to prove that for each $p \in \Gamma$ and neighbourhood N of p in Γ, there are (relatively) open sets $W \subset N$ and $U \subset \mathbf{C}^n$ such that $U \cap \partial\Omega = W$ and f satisfies Condition A in $U \cap \Omega$. The situation is symmetric in f and f^{-1}, so we may verify the condition for f rather than f^{-1}.

According to ([5, Lemma 2, p. 17]) each $p' \in \partial\Omega'$ has local coordinates $\varphi_{p'}$ such that $\varphi_{p'}(\partial\Omega')$ has third order contact with a fixed ball B at p'. It is convenient to assume that B is the ball with center at the origin and radius $\sqrt{n}$ and $\varphi_{p'}(p') = p'' = (1, 1, \ldots, 1)$. A local defining function is then

$$\rho_{p''}(z'') = |z''|^2 - n + O(|z'' - p''|^4)$$

and the unit normal vector field $n_{z''}$ to $\varphi_{p''}(\partial\Omega')$ is obtained locally on the boundary from $\nabla\rho_{p''}$, so

$$(*) \qquad n_{z''} = \frac{\tilde{n}_{z''}}{|\tilde{n}_{z''}|}, \quad \tilde{n}_{z''} = 2z'' + O(|z'' - p''|^3).$$

For inside points

$$z''' = z'' - dn_{z''} = z'' - d\frac{z''}{|z''|} + O(|z'' - p''|^3)$$

we have

$$n_{z'''} = n_{z''} = \frac{z'' + O(|z'' - p''|^3)}{|z'' + O(|z'' - p''|^3)|} = \frac{z''' + O(|z''' - p''|^3)}{|z''' + O(|z''' - p''|^3)|},$$

so $(*)$ is also valid there. The map $\varphi_{p'}$ depends smoothly on p' and is uniformly Lipschitzian and uniformly boundary distance preserving, i.e. with constants independent of p'. Also, the normal component of $\varphi_{p'*}(n_{z'})$ is uniformly bounded below. If we consider the family of functions $f_p = \varphi_{p'} \circ f$, $p \in \Gamma$ and $p' = f(p)$, then f_p is uniformly Hölder $\frac{1}{2}$ continuous and boundary distance preserving. Furthermore, it is sufficient to prove that for each $p \in \Gamma$ and neighbourhood N of p in Γ there is a nonempty open set $W \subset N$ such that the normal component of $f_{q*}(n_z)$ is uniformly bounded below whenever $q \in W$ and z is on the normal line to $\partial\Omega$ at q, with $|q - z| < r$: From the estimates on the Kobayashi metric, it follows that the tangential component of $f_*(n_z)$ is $O(d^{-\frac{1}{2}})$ and this contributes $O(d^{\frac{1}{2}})$ to the normal component of $f_{q*}(n_z)$. We shall actually prove a uniform lower bound valid for all $p \in \Gamma$ on the normal line with $|p - z| < r(p)$. We can only prove, however, that $r(p)$ is bounded below on small open sets arbitrarily close to any given point. This is the reason why the conclusion of the theorem is only that f extends smoothly to a dense open set of Γ.

Now, let $p \in \Gamma$ and $p' = f(p)$. We are now working in local coordinates $\varphi_{p'}$ and we use the notations, f, z', p' instead of f_p, z'', p'', so

$$\rho_{p'}(z') = |z'|^2 - n + O(|z' - p'|^4) \quad \text{and} \quad n_{z'} = 2z' + O(|z' - p'|^3).$$

Consider for small r the closed disc Δ_p with center at $p - rn_p$ and radius rn_p. We denote the boundary points of this disc by $re^{i\theta}$, p corresponding to $\theta = 0$. Then $d(re^{i\theta}) \cong r\theta^2$. Also, let $p'(\theta) = f(re^{i\theta})$. Since $|p - re^{i\theta}| = O(r\theta)$ and f is Hölder $\frac{1}{2}$, $|p' - p'(\theta)| = O(r^{\frac{1}{2}}\theta^{\frac{1}{2}})$ and since f is boundary distance preserving

$$r\theta^2 \cong d(re^{i\theta}) \cong d'(p'(\theta)) = -\rho'_{p'}(p'(\theta)) = n - |p'(\theta)|^2 + O(r^2\theta^2).$$

Hence

$$(1) \qquad |p'(\theta)|^2 = n + O(r\theta^2).$$

Notice also that since $-\rho'_{p'}(p'(\theta)) \geqq Cr\theta^2$, we have $|p'(\theta)|^2 \leqq n - Cr\theta^2$ for small r. (We use C to denote some strictly positive constant, possibly different in different expressions.) Thus f maps Δ_p into B. Scaling, we may assume $r = 1$. We identify Δ_p with the closed unit disc $\Delta \subset \mathbf{C}$. If $f = (f_1, \ldots, f_n)$ we may also assume that for some uniform constant $C > 0$,

$$(2) \qquad |f_1(e^{i\pi})| \leqq 1 - C.$$

Also Δ_p is chosen so small that the components of f are uniformly bounded away from zero on Δ_p. Let $z \in \Omega$ be the point corresponding to the point $1 - \delta \in \Delta$, δ small and positive and let $z' = f(z)$. Then

$$\begin{aligned}\langle f_* n_z, n_z \rangle &\cong (f_* n_z, \tilde{n}_{z'}) = \langle f_* n_{z'}, 2z' + O(|z' - p'|^3)\rangle \\ &= \langle f_* n_z, 2z' + O(\delta^{\frac{3}{2}})\rangle = \langle f_* n_z, 2z' \rangle + O(\delta) \\ &= \frac{d}{d\delta}|f|^2 + O(\delta),\end{aligned}$$

since the derivative mapping $f_* = O(\delta^{-\frac{1}{2}})$. We will therefore study the right-hand side of this equation. For any continuous function ψ on $T = \partial\Delta$, let $\tilde{\psi}$ denote the harmonic extension of ψ to $\bar{\Delta}$; that is $\tilde{\psi}$ is the Poisson integral of ψ. At the point $1 - \delta$ the Poisson kernel is given by

$$P(\delta,\theta) = \frac{(2-\delta)\delta}{\delta^2 + 2(1-\delta)(1-\cos\theta)} \cong \frac{\delta}{\delta^2 + \theta^2} =: Q(\delta,\theta).$$

We shall use the kernel Q (the Poisson kernel for the upper half plane) instead of P. For simplicity, assume at first that $n = 2$, so $f = (f_1, f_2)$ and define $k(\theta)$ by

$$|f_1(e^{i\theta})|^2 = 1 + k(\theta).$$

Since f is Hölder $\frac{1}{2}$ continuous, $k(\theta) = O(\theta^{\frac{1}{2}})$. By (1)

$$|f_2(e^{i\theta})|^2 = 1 - k(\theta) - \psi(\theta)$$

with $\psi(\theta) = O(\theta^2)$ and $\psi \geqq 0$ since f maps Δ into B. Since $\log|f_i|^2$ are harmonic functions, we have in Δ:

$$\begin{aligned} \log|f_1|^2 &= (\log(1+k))^\sim = \tilde{k} - \tfrac{1}{2}(k^2)^\sim + (O(\theta^{\frac{3}{2}}))^\sim \\ \log|f_2|^2 &= (\log(1-k-\psi))^\sim = -\tilde{k} - \tfrac{1}{2}(k^2)^\sim + (O(\theta^{\frac{3}{2}}))^\sim. \end{aligned}$$

At the points $1-\delta \in \Delta$ we have

$$|\tilde{k}| \leqq C_1 \left| \int_{-\pi}^{\pi} \frac{\delta}{\delta^2+\theta^2} |k(\theta)| d\theta \right| \leqq C_2 \left[\int_{|\theta|<\delta} \frac{1}{\delta} + \int_{|\theta|\geqq\delta} \delta\theta^{-\frac{3}{2}} d\theta \right] = O(\delta^{\frac{1}{2}}).$$

Similarly, $(k^2)^\sim = O(\delta \log \delta)$ and $(O(\theta^{\frac{3}{2}}))^\sim = O(\delta)$. Hence

$$\begin{aligned} |f_1|^2 &= \exp\left((\log(1+k))^\sim\right) = 1 + \tilde{k} - \tfrac{1}{2}(k^2)^\sim + O(\delta) \\ |f_2|^2 &= \exp\left((\log(1-k-\psi))^\sim\right) = 1 - k - \tfrac{1}{2}(k^2)^\sim + O(\delta) \\ |f_1|^2 + |f_2|^2 &= 2 - (k^2)^\sim + O(\delta). \end{aligned}$$

Since f is boundary distance preserving, $|f_1|^2 + |f_2|^2 \geqq 2 - C\delta$, hence $(k^2)^\sim = O(\delta)$. But

$$(k^2)^\sim \cong \int_{-\pi}^{\pi} \frac{\delta k^2}{\delta^2+\theta^2} d\theta \geqq \tfrac{1}{2}\delta \int_{|\theta|\geqq\delta} \frac{k^2}{\theta^2} d\theta,$$

so

$$I_p = \int_{-\pi}^{\pi} \frac{k^2}{\theta^2} d\theta = O(1)$$

and since $k(\pi) \leqq -C$ by (2), we have

$$I_p \cong 1. \tag{3}$$

Now,

$$\frac{d}{d\delta}|f|^2 = \left(\frac{d}{d\delta}(\log(1+k))^{\sim}\right)(1+\tilde{k}+O(\delta))$$

$$+\left(\frac{d}{d\delta}(\log(1-k-\psi))^{\sim}\right)(1-\tilde{k}+O(\delta))$$

$$(4) \qquad = (1-\tilde{k}+O(\delta))\left[\frac{d}{d\delta}(\log(1-k-\psi))^{\sim}+\frac{d}{d\delta}(\log(1+k))^{\sim}\right]$$

$$+2\tilde{k}\frac{d}{d\delta}(\log(1+k))^{\sim}$$

$$= (1+O(\sqrt{\delta}))\frac{d}{d\delta}(\log(1-k^2-\varphi))^{\sim}+2\tilde{k}\frac{d}{d\delta}(\log(1+k))^{\sim}$$

with $\varphi = (1+k)\psi$, so $\varphi = O(\theta^2)$ and $\varphi \geqq 0$.

Let P_δ denote the derivative of $P(\delta,\theta)$ with respect to δ. Then

$$P_\delta = 2\frac{(2-2\cos\theta)(1-\delta+\frac{1}{2}\delta^2)-\delta^2}{(\delta^2+(1-\delta)(2-2\cos\theta))^2}.$$

It is easy to see that $P_\delta \geqq -2/\delta^2$ and that there is a constant $C > 0$ such that $P_\delta \geqq C/\theta^2$ when $|\theta| \geqq 2\delta$ and δ is small. (The constant $C = \frac{16}{75}$ works for $\delta \leqq \frac{1}{2}$ for instance.) Hence

$$\frac{d}{d\delta}(\log(1-k^2-\varphi))^{\sim} = \int_{-\pi}^{\pi}\log(1-k^2-\varphi)P_\delta$$

$$\leqq C\int_{|\theta|\geqq 2\delta}\log(1-k^2-\varphi)\frac{1}{\theta^2}-2\int_{|\theta|\leqq 2\delta}\log(1-k^2-\varphi)\frac{1}{\delta^2}$$

$$(5) \qquad \leqq -C\int_{|\theta|\geqq 2\delta}(k^2+\varphi)\frac{1}{\theta^2}+2C'\int_{|\theta|\leqq 2\delta}(k^2+\varphi)\frac{1}{\delta^2}$$

$$\leqq -C\int_{|\theta|\geqq 2\delta}\frac{k^2}{\theta^2}+2C'\int_{|\theta|\leqq 2\delta}\frac{k^2}{\theta^2}+O(\delta)$$

where we have used the inequality $-C'x \leqq \log 1-x \leqq -x$, valid for small positive x.

Now, choose some $\varepsilon > 0$ such that $\int_{|\theta| \leqq \varepsilon} k^2/\theta^2 = o(1)$, i.e. some small number. Then for $\delta \subset\subset \varepsilon$ we have by the Cauchy–Schwarz inequality

$$\left| \int_{|\theta| \leqq \delta} k \right| \leqq \left(\int_{|\theta| \leqq \delta} \frac{k^2}{\theta^2} \right)^{\frac{1}{2}} \left(\int_{|\theta| \leqq \delta} \theta^2 \right)^{\frac{1}{2}} = o(1) \cdot \delta^{\frac{3}{2}}$$

and

$$\left| \int_{|\theta| \geqq \delta} \frac{k}{\theta^2} \right| \leqq \left| \int_{\delta \leqq |\theta| \leqq \varepsilon} \frac{k}{\theta^2} \right| + C \leqq \left(\int_{\delta \leqq |\theta| \leqq \varepsilon} \frac{k^2}{\theta^2} \right)^{\frac{1}{2}} \cdot \left(\int_{\delta \leqq |\theta| \leqq \varepsilon} \frac{1}{\theta^2} \right)^{\frac{1}{2}}$$
$$+ C = o(1)\delta^{-\frac{1}{2}} + C = o(1)\delta^{-\frac{1}{2}}.$$

Hence

$$|\tilde{k}| \leqq \left| \int_{-\pi}^{\pi} \frac{\delta k}{\theta^2 + \delta^2} \right| \leqq \frac{1}{\delta} \int_{|\theta| \leqq \delta} |k| + \delta \int_{|\theta| \geqq \delta} \frac{|k|}{\theta^2} = o(1)\delta^{\frac{1}{2}}$$

and

$$\left| \frac{d}{d\delta} (\log(1+k))^{\sim} \right| = \left| \int_{-\pi}^{\pi} \log(1+k) P_\delta' \right| \leqq C' \int_{-\pi}^{\pi} |k| \, |P_\delta'|$$
$$\leqq C' \left(\int_{|\theta| \leqq \delta} \frac{|k|}{\delta^2} + C'' \int_{|\theta| \geqq \delta} \frac{|k|}{\theta^2} \right) = o(1)\delta^{-\frac{1}{2}}.$$

So $|\tilde{k}(d/d\delta)(\log(1+k))^{\sim}| = o(1)$. Now (3), (4), and (5) give

$$\frac{d}{d\delta} |f|^2 \leqq C \quad \text{for small } \delta,$$

which is the inequality we wanted to prove. We do not know, however, that ε can be chosen so that $\int_{|\theta| \leqq \varepsilon} k^2/\theta^2 = o(1)$ uniformly, i.e. independent of $p \in \Gamma$, even locally. Now, if $p \in \Gamma$, choose a neighbourhood N of p in Γ such that the functions $k_q(\theta)$, $q \in N$ are well defined and continuous in (q,θ). By (3),

$$M = \sup \left\{ I_q = \int_{\pi}^{\pi} \frac{k_q^2}{2} ; q \in N \right\} < \infty.$$

If we pick $q_0 \in N$ such that $I_{q_0} = M - o(1)$ and $\varepsilon > 0$ such that

$$\int_{|\theta| \leqq \varepsilon} \frac{k_{q_0}^2}{\theta^2} = o(1),$$

then

$$\int_{|\theta| \geqq \varepsilon} \frac{k_{q_0}^2}{\theta^2} = M - o(1)$$

and by continuity there is a neighbourhood $W \subset N$ of q_0 such that

$$\int_{|\theta| \geqq \varepsilon} \frac{k_q^2}{\theta^2} = M - o(1)$$

for all $q \in W$, hence

$$\int_{|\theta| \leqq \varepsilon} \frac{k_q^2}{\theta^2} = o(1)$$

by definition of M.

The general case is a straightforward generalization of the case $n = 2$. We now have

$$|f_i(e^{i\theta})|^2 = 1 + k_i(\theta) \qquad i \leqq n-1, \quad k_i = O(\theta^{\frac{1}{2}})$$

$$|f_n(e^{i\theta})|^2 = 1 - \sum_{i=1}^{n-1} k_i(\theta) - \psi \quad \psi \geqq 0, \qquad \psi = O(\theta^2)$$

hence at the points $1 - \delta \in \Delta$ we have

$$|f_i|^2 = \exp\left((\log(1 + k_i))^\sim\right) = 1 + (k_i)^\sim - \tfrac{1}{2}(k_i^2)^\sim + O(\delta),$$

$$|f_n|^2 = \exp\left((\log(1 - \textstyle\sum k_i - \psi))^\sim\right) = 1 - \sum (k_i)^\sim - \tfrac{1}{2}((\sum k_i)^2)^\sim + O(\delta),$$

$$|f|^2 = n - \tfrac{1}{2}\left((\textstyle\sum k_i^2)^\sim + ((\sum k_i)^2)^\sim\right) + O(\delta).$$

Letting $k^2 := \frac{1}{2}(\sum k_i^2 + (\sum k_i)^2)$, we have

$$I_p = \int_{-\pi}^{\pi} \frac{k^2}{\theta^2} d\theta = 1,$$

as before. Now

$$\frac{d}{d\delta}|f|^2 = (1 + 0(\sqrt{\delta}))\frac{d}{d\delta}(\log[(1 - \textstyle\sum k_i) \cdot (1 + k_1) \dots (1 + k_{n-1}))^\sim - \varphi]$$

$$+ 2\sum (k_i)^\sim \frac{d}{d\delta}(\log(1 + k_i))^\sim$$

with $\varphi = \psi \prod (1 + k_i)$, so $\varphi \geqq 0$ and $\varphi = O(\theta^2)$. The last sum here is $o(1)$ if $\delta \ll \varepsilon$ and

$$\int_{|\theta| \leqq \varepsilon} \frac{k^2}{\theta^2} = o(1).$$

Now $(1-\sum k_i)(1+k_1)\dots(1+k_{n-1}) = 1-k^2+O(k^3)$, and we may assume Δ_p is chosen small enough to guarantee that

$$1-2k^2 \leqq (1-\sum k_i)\prod(1+k_i) \leqq 1-\tfrac{1}{2}k^2$$

which yields (5). The rest of the proof is as before.

THEOREM 3. *Let $\Omega, \Omega' \subset\subset \mathbf{C}^n$ be strictly pseudoconvex with C^∞ smooth boundaries and $f\colon \Omega \to \Omega'$ holomorphic and proper. Then there is a dense open subset U of $\partial\Omega$ such that f extends smoothly to $\Omega \cup U$.*

PROOF. The proof is essentially that of Alexander ([1]). We refer to the presentation of this in Rudin ([8, Theorem 15.4.2]).

The map f is boundary distance preserving and extends Hölder $\frac{1}{2}$ continuously to $\bar{\Omega}$. We assume the multiplicity of f is m and let $m(p)$ be the number of points in $f^{-1}f(p))$. We will show that $m(p)=m$ for almost all $p \in \partial\Omega$. This will imply the theorem, because if $E=\{p \in \partial\Omega;\, m(p)=m\}$, then by Step 3 in Rudin, there is an open set $W \subset \partial\Omega$ containing E such that each $p \in W$ has an open neighbourhood such that the hypotheses in Theorem 2 hold, hence f extends smoothly to a dense open subset U of W, which is dense in $\partial\Omega$.

To prove that $m(p) \geqq m$ for almost all $p \in \partial\Omega$, let $z_1(w),\dots,z_m(w)$ be the points in $f^{-1}(w)$ for every regular value w and Λ a linear functional on $\mathbf{C}^n$ that separates these points for some w. Then

$$h(w) = \prod_{i<j}(\Lambda z_i(w) - \Lambda z_j(w))^2$$

is a nonzero bounded holomorphic function in Ω'. Let $F(z)=h(f(z))$ and let E_1 be the set of $p \in \partial\Omega$ at which the radial limit $\lim_{\delta\to 0}F(p-\delta n_p)$ exists and is nonzero. Then E_1 has full measure and if $p \in E_1$, there are $r>0$ and $\varepsilon>0$ such that $|F(p-\delta n_p)|>\varepsilon$, when $\delta<r$. Hence, if $z=p-\delta n_p$, $f(z)$ is a regular value and there are points $z_1,\dots,z_m \in \Omega$ with $f(z_i)=f(z)$ and $|z_i-z_j|>\varepsilon'$, when $i \neq j$. Now, if $\delta_k \searrow 0$ and $z_{1,k},\dots,z_{m,k}$ are the corresponding points, then there exists a subsequence and m distinct points $p_1,\dots,p_m \in \partial\Omega$ such that $\lim_{k\to\infty} z_{i,k}=p_i$, hence $f(p_i)=f(p)$.

To prove that $m(p) \leqq m$ for almost all $p \in \partial\Omega$, let $p \in \partial\Omega$, $p'=f(p)$ and introduce local coordinates at p' such that Ω' is locally contained in the unit ball B of $\mathbf{C}^n$ and $p'=e_1=(1,0,\dots,0)$. Let $z=p-\delta_{n_p}$ and $w=f(z)$. Since $d(z) \cong d'(w)$ and since f is Hölder $\frac{1}{2}$ continuous, $d'(w) \cong 1-|w|^2$ and we may apply the Julia–Carathéodory theorem (which follows from the Schwarz lemma, Rudin [8, Theorem 8.5.6]) to see that there is a constant $L>0$ such that

$$w = (1-L\delta+o(\delta),\, o(\delta^{\frac{1}{2}})). \tag{1}$$

Let φ be a countable collection of linear functionals Λ on $\mathbf{C}^n$ such that every finite subset of $\mathbf{C}^n$ is separated by some $\Lambda \in \varphi$. For $t \in \mathbf{C}$ and regular values $w \in \Omega'$ define

$$Q_\Lambda(t,w) = \prod_{i=1}^{m} (t - \Lambda z_i(w)) = t^m + \sum_{k=0}^{m-1} g_{k,\Lambda}(w) t^k.$$

The functions $g_{k,\Lambda}$ are the elementary symmetric polynomials in $\Lambda z_i(w)$ and hence extend to bounded holomorphic functions in Ω'. Let $f_{k,\Lambda}(z) = g_{k,\Lambda}(f(z))$ and let E_2 be the set of $p \in \partial\Omega$ at which the radial limit $\lim_{\delta\to 0} f_{k,\Lambda}(p - \delta n_p)$ exists for all k and Λ. E_2 has full measure. Let $p \in E_2$. We claim that $m(p) \leqq m$. In local coordinates around p' as above it follows that

$$\alpha_{k.\Lambda} = \lim_{\delta\to 0} g_{k,\Lambda}(w(\delta))$$

exists for all Λ and k along the curve $w(\delta) = f(p - \delta n_p) = (1 - L\delta + o(\delta), o(\delta^{\frac{1}{2}}))$. Now, if $q \in \partial\Omega$ and $f(q) = p'$, then the image of the radius at q is the curve

$$w'(\delta) = f(q - \delta n_q) = (1 - k\delta + o(\delta), o(\delta^{\frac{1}{2}}))$$

and it follows that $g_{k,\Lambda}$ has the same limit $\alpha_{k,\Lambda}$ along this curve. Now

$$\begin{aligned} 0 &= Q_\Lambda((q - \delta n_q), f(q - \delta n_q)) \\ &= [\Lambda(q - \delta n_q)]^m + \sum_{k=0}^{m-1} g_{k,\Lambda}(w'(\delta))[\Lambda(q - \delta n_q)]^k \xrightarrow[\delta\to 0]{} (\Lambda q)^m + \sum_{k=0}^{m-1} \alpha_{k,\Lambda}(\Lambda q)^k. \end{aligned}$$

Hence for each Λ, Λq is a root of the polynomial $t^m + \sum_{k=0}^{m-1} \alpha_{k.\Lambda} t^k$, so Λq can have at most m different values. If we had $m+1$ different points q such that $f(q) = p'$, then some $\Lambda \in \varphi$ would separate these points, i.e. Λq would have had $m+1$ different values, which is impossible.

REFERENCES

1. H. Alexander, *Proper holomorphic mappings in* $\mathbf{C}^n$, Indiana Univ. Math. J. 26 (1977), 137–146.
2. S. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J. 49 (1982), 385–396.
3. K. Diederich and J. E. Fornæss, *Boundary regularity of proper holomorphic mappings*, Invent. Math. 67 (1982), 363–384.

4. K. Diederich, J. E. Fornæss, and R. P. Pflug, *Convexity in complex analysis* (to appear).
5. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. 26 (1974), 1–65.
6. L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–474.
7. L. Nirenberg, S. Webster, and P. Yang, *Local boundary regularity of holomorphic mappings*, Comm. Pure Appl. Math. 33 (1980), 305–338.
8. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, (Grundlehren Math. Wiss. 241), Springer-Verlag, Berlin - Heidelberg - New York, 1980.

DEPARTMENT OF MATHEMATICS
PRINCETON UNIVERSITY
PRINCETON, N. J. 08544
U.S.A.