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ON INFINITE PERIODIC RINGS

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A ring R is said to be *periodic* if for each x in R the set of powers $\{x^i \mid i = 1, 2, 3, \ldots\}$ is a finite set. By a trivial subring (ideal) of R we mean a subring (ideal) S of R with $S^2 = 0$. Further a ring R is *orthogonally finite* if R has no infinite set of mutually orthogonal idempotents. In examining the structure of infinite periodic rings, T. J. Laffey has established

THEOREM 1 [2, Theorem]. *Let R be an infinite periodic ring which is orthogonally finite and with all trivial subrings finite. Then R has a commutative ideal I such that R/I is finite.*

Now the existence of a cofinite commutative ideal in an infinite periodic ring does not preclude the existence of infinite trivial subrings and does not force the ring to be orthogonally finite. As an example we can consider the commutative ring $R = B \oplus N$, where B is an infinite Boolean ring and N is any infinite Abelian group with trivial multiplication.

A second example of a ring with unit can be obtained by letting B be an infinite Boolean ring with unit and letting R be the set of all matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in B \right\}.$$

Then R is commutative with unit,

$$N = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| \, b \in B \right\}$$

is an infinite trivial subring which is not a direct summand of R. Further R is not orthogonally finite since

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in B \right\}$$

is isomorphic to B.

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Thus we are led to seek a characterization of the rings considered by Laffey. We provide a complete description of these rings.

THEOREM 2. *A ring R is an orthogonally finite periodic ring with no infinite trivial subring if and only if $R = F \oplus S$, where F is a finite ring and S is a finite direct sum of periodic fields.*

PROOF. It is clear that any ring which decomposes as stated is periodic, orthogonally finite, and has no infinite trivial subrings. For the other implication we proceed by making use of Laffey's methods. Thus let

$$H = \{x \in R \mid RxR \text{ is finite}\} \quad \text{and} \quad I = \{a \in R \mid aH = Ha = 0\}.$$

Then H is a finite ideal, R/H is reduced and commutative and R/I is a finite ring; see [2; Lemmas 3,4 and Theorem 1]. Let $A = I \cap H$; then A is a finite trivial ideal of R. Note that if $A = I$ then $I \subseteqq H$, hence I is finite and so R is finite. We have

$$I/A = I/(I \cap H) \approx (I+H)/H$$

and so I/A is a commutative reduced ring. Since I/A is periodic and reduced each nonzero ideal contains a nonzero idempotent. If $x \in I$ *with* $x - x^2 \in A$, then $x(x-x^2) = 0$ so that $x^4 = x^3 = x^2$. Thus $e = x^2$ is an idempotent in I with $e - x \in A$, so idempotents lift from I/A to I. If x and y are both idempotent *modulo* A and orthogonal *modulo* A, then $x^2 y^2 = 0$ since $xy \in A$. Hence x^2 and y^2 are orthogonal idempotents in I. It follows that an infinite set of orthogonal idempotents in I/A lifts to an infinite set of orthogonal idempotents in I. By assumption no such set exists and it follows that I/A is a finite product of periodic fields; see, e.g., [1, p. 74]. We now have that I/A has a unit element and so I/A is a ring direct summand of R/A, say $R/A = I/A \oplus K/A$, where K is an ideal of R. Also $K/A \approx R/I$ is a finite ring. Next decompose I/A as $I/A = T/A \oplus G/A$, where G/A is the sum of the finite periodic fields occurring in the desomposition of I/A as a direct sum of fields. This gives $R/A = T/A \oplus F/A$, where F/A is finite and T/A is a finite sum of infinite periodic fields. Since $A^2 = 0$ we may consider A as a left and right T/A-module. But T/A has no finite factors; hence T/A acts trivially on A; i.e., $TA = AT = 0$. There exists an idempotent $e \in T$ with $e + A =$ identity of T/A. Then for all $x \in T$, $x - ex \in A$, so $0 = (x-ex)e = xe - exe$. Thus $ex = exe$; similarly, $exe = xe$, hence $ex = exe = xe$ for all $x \in T$. Because $T = Te \oplus T(1-e)$ we have

$$A = T(1-e) = \{x - xe \mid x \in T\}.$$

Finally,

$$R = T + F = (Te \oplus A) + F = Te + (A + F) = Te + F,$$

since $A \subseteqq F$. Now $Te \cap F \subseteqq T \cap F = A$, hence $x \in Te \cap F$ implies $x = xe \in Ae = 0$. Letting $S = Te$ we get $R = S \oplus F$, the desired decomposition.

The class of periodic rings is part of the larger class of strongly π-regular rings: a ring R is *strongly π-regular* if for each x in R there exists $y \in R$ and a positive integer $n \geqq 1$ such that $x^n = x^{n+1}y = yx^{n+1}$. Algebraic algebras over a field are examples of strongly π-regular rings which are not necessarily periodic. From [2; Lemma 3] we see that if R is a strongly π-regular ring with no infinite trivial subring then R/H is a reduced strongly π-regular ring and hence is an Abelian regular ring. With this in mind we can give the appropriate analogue of Theorem 2 for strongly π-regular rings.

THEOREM 3. *A ring R is an orthogonally finite strongly π-regular ring with no infinite trivial subring if and only if $R = F \oplus S$, where F is finite and S is a finite product of division rings.*

PROOF. The proof is essentially the same for Theorem 2 and so will not be included.

These results suggest the possibility of characterizing orthogonally finite rings with no infinite trivial subrings and this we do next. Recall that a ring is *connected* if its only idempotents are 0 and 1.

THEOREM 4. *A ring R is an orthogonally finite ring with no infinite trivial subring if and only if*

(i) *N, the nil radical of R, is finite; and*
(ii) *$R/N = F \oplus A \oplus B$, where F is finite, A and B are reduced rings having no nonzero finite ideals, A is a ring with unit which is a finite direct sum of connected rings, and B is a ring with no nonzero idempotents.*

PROOF. Suppose R has the structure in (i) and (ii). If W is a trivial subring of R, then $(W + N)/N \subseteqq F$ by (ii). Hence $W + N$ is finite by (i), so W is finite. Also an orthogonal set of idempotents in R gives an orthogonal set of idempotents *modulo* N. Because A is reduced all its idempotents are central. Thus R/N has only a finite set of idempotents; hence so must R. For the converse we again consider Laffey's ideal H. We have H finite and R/H reduced. Thus $N \subseteqq H$, so N is a finite ideal of R. By passing to R/N we assume that $N = 0$. The ideal H is a semisimple finite ring, so that

$R = H \oplus K$. Further no nonzero ideal of K is finite, since $H \cap K = 0$, and $K \approx R/H$ is reduced. In K all idempotents are central. If K has no nonzero idempotents, we are done. Otherwise let $e_1 \neq 0$ be an idempotent in K and write $K = Ke_1 \oplus K(1-e_1)$. If $K(1-e_1)$ has no nonzero idempotents we let $B = K(1-e_1)$. Otherwise select $e_2 \neq 0$ an idempotent in $K(1-e_1)$. Then

$$K(1-e_1) = Ke_2 \oplus K[e_1 + e_2 - e_1 e_2)].$$

Continuing, orthogonal finiteness implies that for some $n \geqq 1$,

$$K = (Ke_1 \oplus \cdots \oplus Ke_n) \oplus B,$$

where B has no nonzero idempotents. Finally, each Ke_i splits into a finite sum of connected rings, completing the proof.

We conclude with an example which shows that no further splitting can be expected in Theorem 4. Consider Z_2 as a Z-bimodule and let R be the set of all matrices

$$\left\{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \middle| \, a \in Z, \; x \in Z_2 \right\}$$

with ordinary matrix addition and multiplication although a product $a \cdot x$ with $a \in Z$, $x \in Z_2$ is the module action of Z on Z_2. It is easily checked that R is indecomposable with unit, orthogonally finite with no infinite trivial subring,

$$N = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| \, x \in Z_2 \right\}$$

is not a summand of R and $R/N \approx Z$.

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