MATH. SCAND. 59 (1986), 9–29

CLASSIFICATION OF ALGEBRAIC SURFACES WITH SECTIONAL GENUS LESS THAN OR EQUAL TO SIX. III: RULED SURFACES WITH $\dim \varphi_{K_X \otimes L}(X) = 2$

ELVIRA LAURA LIVORNI

Introduction.

In this paper we have considered the problem of classifying biholomorphically smooth, connected, projective, ruled, non rational surfaces X with smooth hyperplane section C such that the genus $g = g(C)$ is less than or equal to six and $\dim \varphi_{\bar{L}}(X) = 2$, where $\varphi_{\bar{L}}$ is the map associated to $\bar{L} = K_X \otimes L$. L. Roth in [12] had given a birational classification of such surfaces.

Let $L = [C]$ for some hyperplane section C. From the adjunction formula, see [5], we have that

$$2g - 2 = L \cdot (K_X + L)$$

where by K_X we denote the canonical line bundle on X. If $g = 0$ or 1, then X has been classified, see [10]. If $g = 2 \neq h^{1,0}(X)$, by [14, Lemma (2.2.2)] it follows that X is a rational surface. Thus we can assume $g \geqq 3$.

Since X is ruled, $h^{2,0}(X) = 0$ and

$$(*) \qquad \frac{L \cdot L}{8} + h^{1,0}(X) \leqq \frac{g+1}{2},$$

see [4] and [14, p. 390]. Moreover by the classification of surfaces in $\boldsymbol{P}^2$ and $\boldsymbol{P}^3$, it follows that $h^0(L) \geqq 5$. Our classification is essentially based on the adjunction process which has been introduced by the Italian school and which has been particularly studied by A. J. Sommese [14]. Let $\varphi_L = r \circ s$ be the Remmert–Stein factorization of φ_L. When $\dim \varphi_{\bar{L}}(X) = 2$, Sommese, in [14, p. 392], has proved that there exists a pair $(\hat{X}, \hat{L})$ such that:

(a) X is obtained by blowing up a finite set F of points on $\hat{X}$, $\pi: X \to \hat{X}$.
(b) Every smooth hyperplane section $C \in |L|$ is the proper transform of a hyperplane section $\hat{C} \in |\hat{L}|$.

Received July 15, 1984; in revised version May 20, 1985.

(c) $\hat{L}$ is ample and spanned off F.
(d) $\hat{L}$ is very ample if $H^1(X, L) = 0$.
(e) If $h^{1,0}(X) = 0$, then $\hat{s}$ is an embedding unless there is a smooth hyperelliptic $C \in |L|$. This can happen only in the cases (2.5.1) and (2.5.2) of [14, p. 394].

Let $L' = K_{\hat{X}} \otimes \hat{L}$ and $\varphi_{L'} = \varphi_{K_{\hat{X}} \otimes \hat{L}}$. Then $\varphi_{L'} = s$. We call $\hat{X}$ the minimal model of X relative to L. It has the property that there is no irreducible curve $\mathcal{P} \subset \hat{X}$ such that $\mathcal{P} \cdot \mathcal{P} = -1$ and $\hat{L} \cdot \mathcal{P} = 1$. We call $(\hat{X}, \hat{L})$ the minimal pair.

Moreover by the construction of $\hat{X}$ in [14] it follows that $\hat{C}$ is smooth. Our main goal is to classify the pairs $(\hat{X}, \hat{L})$.

We shall mention that our classification has a slight overlap with the classification that P. Ionescu [6] has given for projective surfaces of sectional genus less than or equal to four. We have summarized our results in Table 1, where e is, by [5], the invariant which characterizes $\mathcal{P}(E)$. We wish to thank Andrew J. Sommese for suggesting the problem and Alan Howard for helpful discussions about ruled surfaces.

0. Background material.

We have already fixed the meaning of X, L, C, $\hat{X}$, $\hat{L}$, $\hat{C}$, $\bar{L}$ and L'. We would like to fix now the following notations.

We let $d = L \cdot L$, $g = g(C) = g(L)$, $\hat{d} = \hat{L} \cdot \hat{L}$, $d' = L' \cdot L'$, $g' = g(L')$, $c_1^2 = K_X \cdot K_X$, $\hat{c}_1^2 = K_{\hat{X}} \cdot K_{\hat{X}}$.

(0.1) Proposition. *Let L be a line bundle on a smooth, connected, projective surface X. Then:*

(1) $d' = g' + g - 2$,
(2) $dd' \leqq 4(g-1)^2$,
(3) $d + d' = c_1^2 + 4(g-1)$.

The proof follows using the adjunction formula [5, p. 361].

(0.2) Proposition. *Let X be a smooth, connected, projective surface embedded by a very ample line bundle l into $\mathbf{P}^4$. Then*

$$l \cdot l(l \cdot l - 5) - 10(g(l) - 1) + 12\chi(\mathcal{O}_X) = 2c_1^2.$$

Proof. See [5, p. 434].

$\dim \varphi_{\bar{L}}(X)$	g	$h^{1,0}(X)$	$h^{2,0}(X)$	$\hat{d}$	d_1^2	$h^0(\hat{L})$	d'	g'	$h^0(K_X \otimes L)$	$(\hat{X}, \hat{L})$
2	4	1	0	9	0	6	3	1	3	$e = -1, \hat{L} \equiv \zeta_E^3$
2	5	1	0	9	0	5	7	4	4	$e = -1, \hat{L} \equiv \zeta_E^9 \otimes \mathscr{L}^{-4}$
2	6	1	0	11	0	6	9	5	5	$e = -1, \hat{L} \equiv \zeta_E^{11} \otimes \mathscr{L}^{-5}$
2	6	1	0	12	0	7	8	4	5	$e = -1, \hat{L} \equiv \zeta_E^6 \otimes \mathscr{L}^{-2}$
2	6	1	0	15	0	10	5	1	5	$e = -1, \hat{L} \equiv \zeta_E^3 \otimes \mathscr{L}$
2	6	1	0	10	−1	6	9	5	5	$e = 0, \hat{L} \equiv \zeta_E^{11} \otimes \mathscr{L}^5 \otimes [\mathscr{P}]^{-10}$ or $e = -1, \hat{L} \equiv \zeta_E^7 \otimes \mathscr{L}^{-1} \otimes [\mathscr{P}]^{-5}$
2	6	1	0	11	−1	6, 7	8	4	5	$e = -1, 0, 1$
2	6	1	0	12	−1	7, 8	7	3	5	$e = -1, 0, 1$
2	6	1	0	9	−2	6	9	5	5	$e = -1, 0, 1$
2	6	1	0	9	−1	6	10	6	5	$e = -1, 0$

Table 1.

(0.3) PROPOSITION. (Castelnuovo's inequality [2, p. 234 f f]; [4].) *If C is an irreducible curve embedded in $\boldsymbol{P}_{\mathrm{C}}^{l-1}$ and C belongs to no linear hyperplane $\boldsymbol{P}_{\mathrm{C}}^{l-2}$, then with d the degree of C and g the genus:*

$$g \leqq \left[\frac{d-2}{l-2}\right]\left(d-l+1-\left[\frac{d-l}{l-2}\right]\left(\frac{l-2}{2}\right)\right),$$

where $[\cdot]$ is the least integer function.

(0.4) PROPOSITION. *Let X be any projective, smooth surface and let*

$$0 \to E \to F \to G \to 0$$

be the short exact sequence obtained by tensoring the sequence

$$0 \to [C]^{-1} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

with a line bundle F, where C is a curve in X. Suppose that:

(a) *G is a very ample line bundle on C,*
(b) *E is very ample,*
(c) $\ker(H^0(G) \to H^1(E))$ *gives an embedding of C.*

Then F is very ample.

Since the proof is standard we will omit it.

(0.5) RULED SURFACES. Let X be a smooth, connected, projective, geometrically ruled surface, i.e. a fibration $\pi: X \to \bar{C}$, over a curve $\bar{C}$ whose fibres are $\boldsymbol{P}^1$. Then there exists a rank two vector bundle E (not unique) over $\bar{C}$ and an isomorphism $X = \boldsymbol{P}(E)$, where $\boldsymbol{P}(E)$ denotes the associated projective bundle of E. Let $\bar{g}$ be the genus of $\bar{C}$. Let σ be a minimal section of π, there is a line bundle $\underline{\mathfrak{L}}$ on $\bar{C}$ and an extension E of $\underline{\mathfrak{L}}$ by $\mathcal{O}_{\bar{C}}$

(0.5.1) $$0 \to \mathcal{O}_{\bar{C}} \to E \to \underline{\mathfrak{L}} \to 0$$

such that $X = \boldsymbol{P}(E)$ and

$$\underline{\mathfrak{L}} = \sigma^* \mathcal{O}_{\boldsymbol{P}(E)}(1) = \mathcal{O}_{\sigma(\bar{C})}(\zeta_E),$$

where ζ_E is the tautological line bundle.

$$e = -\zeta_E \cdot \zeta_E = -\deg \underline{\mathfrak{L}}$$

is an invariant of the surface X. If E is decomposable, then $e \geqq 0$ and all the values of e are possible. If E is indecomposable, then

(0.5.2) $$-\bar{g} \leqq e \leqq 2\bar{g} - 2.$$

See [5, p. 376 and 384] and [11, p. 191].

Let f be a fiber of $\pi: X \to \bar{C}$. Then every line bundle L on X is numerically equivalent to $\zeta_E^a \otimes \mathscr{L}^b$, that is $L \equiv \zeta_E^a \otimes \mathscr{L}^b$ for some integers a, b and $\mathscr{L} = \mathcal{O}_X(f)$, so

$$\deg \mathscr{L}|_{\sigma(\bar{C})} = 1,$$

$$(0.5.2) \qquad \begin{aligned} L \cdot L &= -a^2 e + 2ab \quad \text{and} \\ 2g(L) - 2 &= -a^2 e + ae + 2ab - 2b - 2a + 2a\bar{g}. \end{aligned}$$

The canonical line bundle K_X of X is $K_X \equiv \zeta_E^{-2} \otimes \mathscr{L}^{(2\bar{g}-2-e)}$. Given a line bundle $\mathfrak{A}$ on $\bar{C}$ we will denote its lift $\pi^* \mathfrak{A}$ on X again by $\mathfrak{A}$. We have the following propositions:

(0.5.4) PROPOSITION. *Let X be a geometrically ruled surface over a curve $\bar{C}$, with invariant $e \geqq 0$.*

(i) *If $Y \equiv a\zeta_E + b\mathscr{L}$ is an irreducible curve, $Y \not\equiv \zeta_E, \mathscr{L}$, then $a > 0$, $b \geqq a \cdot e$.*

(ii) *A divisor $D \equiv a\zeta_E + b\mathscr{L}$ is ample if and only if $a > 0$, $b > a \cdot e$.*

PROOF. See [5, p. 382].

(0.5.5) PROPOSITION. *Let X be a geometrically ruled surface over a curve $\bar{C}$, of genus $\bar{g}$ and invariant $e < 0$.*

(i) *If $Y \equiv a\zeta_E + b\mathscr{L}$ is an irreducible curve, $Y \not\equiv \zeta_E, \mathscr{L}$, then either $a = 1$, $b \geqq 0$ or $a \geqq 2$, $b \geqq \frac{1}{2}ae$.*

(ii) *A divisor $D \equiv a\zeta_E + b\mathscr{L}$ is ample if and only if $a > 0$, $b > \frac{1}{2}ae$.*

PROOF. See [5, p. 382].

The determination of the very ample divisors on a ruled surface with $\bar{g} \geqq 1$, is more difficult than in the case of a rational ruled surface, i.e. a Hirzebruch surface. There is moreover the following result which is stated as an exercise in [5, p. 385] and it is not too difficult to prove.

(0.5.6) PROPOSITION. *Let X be a geometrically ruled surface with invariant e over an elliptic curve ε. Let $L \equiv \zeta_E \otimes \mathscr{L}^b$. Then*

(i) *L is spanned if and only if $b \geqq e + 2$.*

(ii) *L is very ample if and only if $b \geqq e + 3$'*

(0.5.7) THEOREM. *Let $X = \mathbf{P}(E)$ be a geometrically ruled surface over an elliptic curve ε. Then $L \equiv \zeta_E^a \otimes \mathscr{L}^b$ is very ample if $a \geqq 1$ and $b \geqq \max_{1 \leqq k \leqq a} \{3 + ke\}$.*

PROOF. See [8, Theorem (1.6)].

(0.5.8) **Proposition.** *Let X be a geometrically ruled surface over a curve $\bar{C}$ with $\bar{g} = g(\bar{C})$ and invariant e. Let $L \equiv \zeta_E^a \otimes \mathscr{L}^b$ be a line bundle on X with $a > -2$. Then:*

(i) $h^1(L) = 0$ *for* $b > \begin{cases} ae + 2\bar{g} - 2 + e & \text{if } e \geqq 0 \\ \frac{1}{2}ae + 2\bar{g} - 2 & \text{if } e < 0 \end{cases}$

(ii) $h^0(L) - h^1(L) = (a+1)(b+1-\bar{g} - ae/2)$.

The proof is a direct application of the Kodaira Vanishing Theorem and the Riemann–Roch Theorem.

By a ruled surface we mean a surface birational to a geometrically ruled surface.

(0.6) **Proposition.** *Let X be a smooth, connected surface and L an ample line bundle on it. Suppose that $h^{2,0}(X) \neq 0$ and $L \cdot L = 2g-2$. Then K_X is trivial.*

Proof. Use [14, p. 382].

(0.7) **Proposition.** *Let X and L be as above. Suppose that $L \cdot L = 2g - 2$ and $h^0(L|_C) = g$, where $C \in |L|$.*
Then K_X is trivial.

Proof. Use [14, p. 382].

(0.8) **Proposition.** *Let L be an ample and spanned line bundle on a smooth, connected, projective surface X. Assume $h^0(L) \geqq 4$, $L \cdot L \geqq 5$. Then $K_X \otimes L$ is spanned.*

Proof. See [15, Theorem (0.8)].

(0.9) **Theorem.** *Let X be a smooth, connected, ruled surface and L be an ample and spanned line bundle on it. Let $C \in |L|$, $g = g(C) = g(L) = 2$. Suppose that $h^{1,0}(X) \neq 2$ and that $K_X \otimes L$ is spanned. Then X is rational.*

Proof. By the first Lefschetz Theorem, see [1] or [3], $h^{1,0}(X) \leqq 2$. Thus $h^{1,0}(X) = 0$ or 1. Consider the long cohomology sequence associated to the short exact sequence

$$0 \to K_X \to K_X \otimes L \to K_C \to 0. \tag{1.1.1}$$

The Kodaira Vanishing Theorem, [5], implies $h^1(K_X \otimes L) = 0$. By definition $h^0(K_C) = g = 2$. Since $K_X \otimes L$ is spanned, by restriction, K_C is also spanned. Therefore

$$H^0(K_X \otimes L) \xrightarrow{\alpha} H^0(K_C) \to 0$$

is exact; otherwise the image of α would have only one section and this would contradict the fact that K_C is spanned. Then by (1.1.1) it follows that $h^{1,0}(X) = 0$ and hence X is rational.

1. The case of dim $\varphi_L(X) = 2$ and $h^{1,0}(X) = 2$.

Since $h^{1,0}(X) = 2$, X is a ruled surface over a curve of genus two. By Theorem (0.9), $g \geqq 3$. Let $g = 3$ and consider the long cohomology sequence of

$$0 \to K_X \to K_X \otimes L \to K_C \to 0.$$

By the facts that

(a) $h^{2,0}(X) = 0$, since X is a ruled surface,
(b) $h^0(K_C) = g = 3$,
(c) $h^1(K_X \otimes L) = 0$ by Kodaira Vanishing Theorem,
(d) $h^{1,0}(X) = 2$ by hypothesis,

it follows that $h^0(K_X \otimes L) = 1$ which contradicts the fact that $K_X \otimes L$ is spanned by [14, p. 387]. Therefore $g \geqq 4$.

Now consider $(\hat{X}, \hat{L})$. *If* $g = 4$, by (*) it follows that $d \leqq 4$ which contradicts $h^0(L) \geqq 5$ and Castelnuovo's inequality. Therefore $g = 5, 6$. Again by (*) if $g = 5$ and $\hat{d} \geqq 2g - 1$, then $h^{1,0}(X) \leqq 1$. Thus if $g = 5$, $\hat{d} = 7$ or 8 and $h^0(\hat{L}) = 5$. If $\hat{d} = 7$ then, by degree consideration $X = \hat{X}$, $L = \hat{L}$, $d = \hat{d} = 7$, and $h^0(L) = h^0(\hat{L}) = 5$. Therefore by Proposition (0.2) we have $c_1^2 = -16$. Now applying Proposition (0.1) it follows that $d' = -7$ which gives a contradiction. Now suppose that $\hat{d} = 8$. If $X = \hat{X}$, then $h^0(L) = h^0(\hat{L}) = 5$ and by Proposition (0.2), $c_1^2 = -14$ which contradicts Proposition (0.1). If X is made by blowing up one point we get again a contradiction in the same way. Hence $g = 6$. Using the fact that $d' \geqq g - h^{1,0}(X) - 2$ and $\hat{c}_1^2 \leqq -8$ we obtain that $\hat{d} \leqq 10$. So $7 \leqq \hat{d} \leqq 10$. By Castelnuovo's inequality if $\hat{d} = 7, 8$, then $h^0(\hat{L}) = 5$. Let $\hat{d} = 7$. Then $X = \hat{X}$, $L = \hat{L}$, $h^0(L) = 5$, $d = \hat{d} = 7$. By Proposition (0.2), $\hat{c}_1^2 = -24$ which contradicts Proposition (0.1). If $\hat{d} = 8$, we get contradictions in the same way in both the cases in which $X = \hat{X}$ and X is made by blowing up one point. Therefore

$$\hat{d} = 9,\ 10,\quad h^0(\hat{L}) \geqq 5.$$

Using the fact that $\hat{c}_1^2 \leqq -8$ and Proposition (0.1), we get contradictions. Thus we can state the following theorem:

(1.1) THEOREM. *There is no smooth, connected, projective, ruled surface such that* $h^{1,0}(X) = 2$, $\dim \varphi_{\bar{L}}(X) = 2$ *and* $g \leqq 6$.

2. The case of $\dim \varphi_{\bar{L}}(X) = 2$ **and** $h^{1,0}(X) = 1$.

We would like to remind that $h^0(\hat{L}) \geqq h^0(L) \geqq 5$ and $g = 3, \ldots, 6$. By the long cohomology sequence of

$$0 \to K_X \to K_X \otimes L \to K_C \to 0, \tag{2.0.1}$$

it follows that $g = 4, 5, 6$ and by Castelnuovo's inequality:

$$g = 4 \Rightarrow d \geqq 6,$$
$$g = 5, 6 \Rightarrow d \geqq 7.$$

(2.1) LEMMA. *Let X be a smooth, connected, projective surface such that* $h^{1,0}(X) = 1$, $h^{2,0}(X) = 0$. *Let L be an ample line bundle on it. Suppose that* $K_X \otimes L$ *is ample, spanned and* $g' = g(K_X \otimes L) = 1$. *Then* $c_1^2 = 0$.

PROOF. [15, Corollary (3.4.2)], [16, Theorem (1.3)] or [7, Corollary (2.4)].

(2.2) PROPOSITION. *Let X be a smooth, connected, projective, ruled surface such that* $h^{1,0}(X) = 1$, $\dim \varphi_{\bar{L}}(X) = 2$, *and* $\hat{d} = 2g - 2$. *Then if* $(\hat{X}, \hat{L})$ *exists it has to satisfy the following invariants,*

$$g = 6, \quad \hat{d} = 10, \quad d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -1, \quad h^0(\hat{L}) = 6.$$

PROOF. Since $\hat{d} = 2g - 2$ using Clifford's Theorem, Riemann–Roch's Theorem, Proposition (0.7) and the long cohomology sequence of

$$0 \to \mathcal{O}_{\hat{X}} \to \hat{L} \to \hat{L}|_{\hat{C}} \to 0,$$

we have that $h^0(\hat{L}) \leqq g$. Therefore, using the fact that $h^0(\hat{L}) \geqq 5$ we have that $g = 5$ or 6. Assume that $g = 5$. Then $h^0(\hat{L}) = 5$. By Propositions (0.1), (2.1) and Theorem (0.9), we obtain the following invariants:

$$d' = 6, \quad g' = 3, \quad \hat{c}_1^2 = -2,$$
$$d' = 7, \quad g' = 4, \quad \hat{c}_1^2 = -1,$$
$$d' = 8, \quad g' = 5, \quad \hat{c}_1^2 = 0.$$

Since $h^0(L) \geqq 5$, by Castelnuovo's inequality $d \geqq 7$. Suppose that $X = \hat{X}$, that is $L = \hat{L}$ and $d = \hat{d} = 8$. Then by Proposition (02),

$c_1^2 = -8$ which gives a contradiction. Now suppose that X is obtained by blowing up one point on $\hat{X}$. Then $h^0(L) = 5$ and $d = 7$. By Proposition (0.2) we have that $c_1^2 = -13$ which contradicts the values that we have obtained for $\hat{c}_1^2$. Therefore $g \neq 5$. It remains to examine the case in which $g = 6$. As we have seen $h^0(\hat{L}) = 5$ or 6. Exactly as in the case $g = 5$ we obtain the following set of invariants:

$$\begin{aligned} d' &= 7, & g' &= 3, & \hat{c}_1^2 &= -3, \\ d' &= 8, & g' &= 4, & \hat{c}_1^2 &= -2, \\ d' &= 9, & g' &= 5, & \hat{c}_1^2 &= -1, \\ d' &= 10, & g' &= 6, & \hat{c}_1^2 &= 0. \end{aligned}$$

As in the case in which $g = 5$ we see that $h^0(\hat{L}) \neq 5$. Thus $h^0(\hat{L}) = 6$. Now consider the first set of invariants.

$$(K_{\hat{X}} + L') \cdot (K_{\hat{X}} + L') = -2,$$

which contradicts the fact that $K_{\hat{X}} \otimes L'$ is spanned by Proposition (0.8). Also in the last case we obtain a contradiction using the formula

$$t(2h^{1,0}(X) - 2) + \frac{t-1}{t} d = 2g - 2, \tag{2.2.1}$$

which is obtained for ruled surfaces which are minimal models using the adjunction formula and the Hurwitz formula, see [5].

Now consider the second set of invariants. By the long cohomology sequence of

$$0 \to K_{\hat{X}} \to K_{\hat{X}} \otimes L' \to K_{C'} \to 0, \tag{2.2.2}$$

we have that $h^0(K_{\hat{X}} \otimes L') = 3$. Moreover

$$(K_{\hat{X}} + L') \cdot (K_{\hat{X}} + L') = 2.$$

Since $\varphi_{K_{\hat{X}} \otimes L'}$ cannot be an embedding, it follows that it gives a 2:1 branched cover of $\boldsymbol{P}^2$. Thus we have a contradiction since 2:1 branched covers of $\boldsymbol{P}^2$ have first Betti numbers zero.

(2.3) **Theorem.** *Let $(\hat{X}, \hat{L})$ be a minimal pair of a smooth, connected, projective, ruled surface. Suppose that $(\hat{X}, \hat{L})$ satisfy the invariants:*

$$\begin{aligned} &g = 6, && \hat{d} = 10, && d' = 9, && g' = 5, \\ &\hat{c}_1^2 = -1, && h^0(\hat{L}) = 6, && h^{1,0}(X) = 1. \end{aligned}$$

Then, if $(\hat{X}, \hat{L})$ *exists, it has to be made by blowing up one point on a geometrically ruled surface over an elliptic curve such that either* $e = 0$ *and*

$$\hat{L} \equiv \zeta_E^{11} \otimes \mathscr{L}^5 \otimes [\mathscr{P}]^{-10} \quad or \quad e = -1 \quad and \quad \hat{L} \equiv \zeta_E^7 \otimes \mathscr{L}^{-1} \otimes [\mathscr{P}]^{-5},$$

where $\mathscr{P}$ *is the irreducible line on* $\hat{X}$ *that we obtain, when we blow up a point on a minimal model.*

PROOF. Since $\hat{c}_1^2 = -1$, the surface has to be made by blowing up one point over a minimal model. Hence

$$\hat{L} \equiv \zeta_E^a \otimes \mathscr{L}^b \otimes [\mathscr{P}]^r.$$

Since the surface is a minimal model relative to L and $\mathscr{P} \cdot \mathscr{P} = -1$ we have that $\hat{L} \cdot \mathscr{P} \geqq 2$, that is

$$2 \leqq (a\zeta_E + b\mathscr{L} + r\mathscr{P}) \cdot \mathscr{P} = -r.$$

Since $\dim \varphi_{\hat{L}}(X) = 2$ we have that $\hat{L} \cdot f \geqq 3$. Hence

$$3 \leqq (a\zeta_E + b\mathscr{L} + r\mathscr{P}) \cdot f = a.$$

Since ε is an elliptic curve and $\hat{L}$ is ample

$$\hat{L} \cdot \zeta_E = (a\zeta_E + b\mathscr{L} + r\mathscr{P}) \cdot \zeta_E = -ae + b \geqq 1.$$

Moreover $K_{\hat{X}} \equiv \zeta_E^{-2} \otimes \mathscr{L}^{-e} \otimes [\mathscr{P}]$, so

$$K_{\hat{X}} \otimes \hat{L} \equiv \zeta_E^{a-2} \otimes \mathscr{L}^{b-e} \otimes [\mathscr{P}]^{r+1}.$$

Therefore we have the following system:

$$r \leqq -2, \quad a \geqq 3.$$

(i) $ae - b \leqq -1$,
(ii) $\hat{d} = -a^2 e + 2ab - r^2$,
(iii) $2g - 2 = \hat{d} + ae - 2b - r$,
(iv) $d' = -a^2 e + 2ae + 2ab - 4b - r^2 - 2r - 1$,
(v) $2g' - 2 = d' + ae - 2b - r - 1$.

Using (i) and (iii) it follows that

(2.4.1) $$b \leqq -1 - r.$$

Again by (i)

$$e \leqq \frac{b-1}{a}.$$

By (2.4.1)

(2.4.2) $$e \leqq \frac{b-1}{a} \leqq \frac{-2-r}{a}.$$

Now we write (ii) as

$$10 + r^2 = -a^2 e + 2ab = a(-ae + 2b).$$

By (iii) the above equality becomes

$$-ar = 10 + r^2,$$

which implies

$$a = \frac{10 + r^2}{-r}.$$

Substituting in (2.4.2) we get

$$e \leqq \frac{-2-r}{a} = \frac{(-2-r)(-r)}{10+r^2}.$$

Since $r^2 < r^2 + 10$ and $r \leqq -2$ we have that

$$r^2 + 2r < r^2 + 10,$$

which implies that $e < 1$, that is $e = -1$ or 0. Let $e = 0$. By (v)

$$2b = -r.$$

Substituting in (iv) we obtain

$$10 = -r(a + r).$$

Therefore we have the following cases:

(A) $a = 7, \quad r = -2, \quad b = 1,$

(B) $a = 11, \quad r = -10, \quad b = 5,$

that is, either

$$\hat{L} \equiv \zeta_E^7 \otimes \mathscr{L} \otimes [\mathscr{P}]^{-2} \quad \text{or} \quad \hat{L} \equiv \zeta_E^{11} \otimes \mathscr{L}^5 \otimes [\mathscr{P}]^{-10}.$$

Since by [14, p. 393], $\hat{L}|_{\zeta_E}$ has to be very ample we see that case (A) is not possible. Now let $e = -1$. By (iii)

$$a = -2b - r.$$

Substituting in (iv)

$$br = 5.$$

Since $r \leqq -2$, and, by (i), $b \geqq -2$ we have that

$$a = 7, \quad b = -1, \quad r = -5$$

that is $\hat{L} \equiv \zeta_E^7 \otimes \mathscr{L}^{-1} \otimes [\mathscr{P}]^{-5}$.

(2.5) LEMMA. *There is no geometrically ruled surface* $(\hat{X}, \hat{L})$ *over an elliptic curve such that* $\hat{L}$ *is ample and* $\hat{d} \leqq 2g - 2$.

PROOF. Use (0.5.3) and Propositions (0.5.4) and (0.5.5).

Now we assume $\hat{d} \leqq 2g - 3$. We get the following proposition:

(2.6) PROPOSITION. *There is no smooth, connected, projective, ruled surface such that* $h^{1,0}(X) = 1$, $\dim \varphi_{\hat{L}}(X) = 2$, $\hat{d} \leqq 2g - 3$, $g = 4, 5$. *In the case in which* $g = 6$, $(\hat{X}, \hat{L})$ *has to satisfy one of the following sets of invariants:*

(1) $\hat{d} = 9, \quad h^0(\hat{L}) = 6, \quad d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -2,$

(2) $\hat{d} = 9, \quad h^0(\hat{L}) = 6, \quad d' = 10, \quad g' = 6, \quad \hat{c}_1^2 = -1.$

PROOF. Since $h^0(\hat{L}) \geqq 5$, by Castelnuovo's inequality $g \neq 4$. If $g = 5$, then $\hat{d} = 7$, $h^0(\hat{L}) = 5$, If $g = 6$, then $\hat{d} = 7, 8$, $h^0(\hat{L}) = 5$, and $\hat{d} = 9$, $h^0(\hat{L}) = 5, 6$. In the case in which $g = 5$, by degree considerations $X = \hat{X}$. Thus $\hat{L}$ is very ample and we have a contradiction using Propositions (0.2) and (0.1). Now let $g = 6$. In the case in which $\hat{d} = 7$ by Castelnuovo's inequality since $h^0(L) \geqq 5$ we have $X = \hat{X}$ and we get a contradiction as before. If $\hat{d} = 8$, then we can blow up at most one point. Thus $d = 7$, $h^0(L) = 5$ and we get again a contradiction as before. If $X = \hat{X}$, that is $d = 8$, then $\hat{L} = L$ is very ample, $h^0(L) = 5$ and we get, in the same way, a contradiction.

Now consider the case in which $\hat{d} = 9$ and $h^0(\hat{L}) = 5$. Again by Castelnuovo's inequality, since $h^0(L) \geqq 5$, we can blow up at most two points. If $d = 7, 8$ we have contradictions as before. If $X = \hat{X}$, that is $d = \hat{d} = 9$, then by Propositions (0.2) and (0.1) we have $c_1^2 = -7$, $d' = 4$ and $g' = 0$ which implies that X is rational. It remains to consider the case in which $\hat{d} = 9$ and $h^0(\hat{L}) = 6$. By Propositions (0.1), (0.7) and Theorem (0.9) we obtain the following sets of invariants:

(A) $d' = 7, \quad g' = 3, \quad \hat{c}_1^2 = -4,$
(B) $d' = 8, \quad g' = 4, \quad \hat{c}_1^2 = -3,$
(C) $d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -2,$
(D) $d' = 10, \quad g' = 6, \quad \hat{c}_1^2 = -1,$
(E) $d' = 11, \quad g' = 7, \quad \hat{c}_1^2 = 0.$

By Lemma (2.5), case (E) does not happen. By the long cohomology sequence of

$$0 \to K_{\hat{X}} \to K_{\hat{X}} \otimes L' \to K_{\hat{X}} \otimes L'|_{C'} \to 0$$

obtained by tensoring with $K_{\hat{X}}$ the short exact sequence

$$0 \to \mathcal{O}_{\hat{X}} \to L' \to L'|_{C'} \to 0,$$

where $C' \in |L'|$, we have that, in case (B),

$$h^0(K_X \otimes L') = 3.$$

Moreover

$$(K_{\hat{X}} + L') \cdot (K_{\hat{X}} + L') = \hat{c}_1^2 + 4g' - 4 - d' = 1,$$

which implies that $\hat{X} = \boldsymbol{P}^2$. Therefore case (B) can not happen either. In case (A) we get a contradiction, since

$$(K_{\hat{X}} + L') \cdot (K_{\hat{X}} + L') = -3.$$

Therefore (C) and (D) are the only possible cases.

Now consider the case in which $\hat{c}_1^2 = -1$, that is $\hat{X}$ is made by blowing up one point on a geometrically ruled surface over an elliptic curve. Then the system is:

$$r \leqq -2, \quad a \geqq 3$$

(i) $ae - b \leqq -1,$
(ii) $9 = -a^2 e + 2ab - r^2,$
(iii) $10 = 9 + ae - 2b - r,$
(iv) $10 = -a^2 e + 2ae + 2ab - 4b - r^2 - 2r - 1,$
(v) $10 = 10 + ae - 2b - r - 1.$

By (i) and (iii) it follows

(2.6.1) $$b \leqq -r - 2.$$

Again by (i)

$$e \leqq \frac{b-1}{a}.$$

Thus, using (2.6.1)

(2.6.2) $$e \leqq \frac{-r-3}{a}.$$

By (ii) and (iii) we obtain

$$a = \frac{r^2+9}{-r-1}.$$

Substituting in (2.6.2) we get

$$e \leqq \frac{r^2+4r+3}{r^2+9}.$$

By $r^2 < r^2+9$ and $r \leqq -2$ it follows that $e < 1$, that is $e = -1, 0$.

(2.7) REMARK. If there exist smooth, connected, projective, ruled surfaces $(\hat{X}, \hat{L})$ with $g = 6$ and $h^{1,0}(\hat{X}) = 1$ which satisfy the invariants:

$$\hat{d} = 9, \quad d' = 10, \quad g' = 6, \quad \hat{c}_1^2 = -1\,.$$

then $\hat{X}$ is made by blowing up one point on a geometrically ruled surface over an elliptic curve with invariant $e = -1,0$.

Now consider the case in which $\hat{c}_1^2 = -2$. In this case $\hat{X}$ is made by blowing up two points on a minimal model. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ denote the irreducible lines on $\hat{X}$ that we obtain, when we blow up two points on a minimal model.

We have that either

$$\mathscr{P}_1 \cdot \mathscr{P}_1 = \mathscr{P}_2 \cdot \mathscr{P}_2 = -1, \; \mathscr{P}_1 \cdot \mathscr{P}_2 = 0\,,$$

or

$$\mathscr{P}_1 \cdot \mathscr{P}_1 = -2, \; \mathscr{P}_2 \cdot \mathscr{P}_2 = -1, \; \mathscr{P}_1 \cdot \mathscr{P}_2 = +1,$$

In the first case we have:

$$\hat{L} \equiv \zeta_E^a \otimes \mathscr{L}^b \otimes [\mathscr{P}_1]^{r_1} \otimes [\mathscr{P}_2]^{r_2}$$

$$K_{\hat{X}} \equiv \zeta_E^{-2} \otimes \mathscr{L}^e \otimes [\mathscr{P}_1] \otimes [\mathscr{P}_2]$$

and

$$K_{\hat{X}} \otimes \hat{L} \equiv \zeta_E^{a-2} \otimes \mathscr{L}^{b-e} \otimes [\mathscr{P}_1]^{r_1+1} \otimes [\mathscr{P}_2]^{r_2+1}.$$

Thus:

$$r_1 \leqq -2, \quad r_2 \leqq -2, \quad a \geqq 3,$$

(i) $ae - b \leqq -1,$
(ii) $9 = -a^2 e + 2ab - r_1^2 - r_2^2,$
(iii) $10 = 9 + ae - 2b - r_1 - r_2,$
(iv) $9 = -a^2 e + 2ae + 2ab - 4b - r_1^2 - r_2^2 - 2r_1 - 2r_2 - 2,$
(v) $8 = 9 + ae - 2b - r_1 - r_2 - 2.$

By (i) and (iii) it follows

(2.7.1) $$b \leqq -r_1 - r_2 - 2.$$

Again by (i) we have

$$e \leqq \frac{-1+b}{a}.$$

Using (2.7.1) we get

(2.7.2) $$e \leqq \frac{-r_1 - r_2 - 3}{a}.$$

By (ii) and (iii) we obtain

$$a = \frac{9 + r_1^2 + r_2^2}{-r_1 - r_2 - 1}.$$

Substituting in (2.7.2) we get

$$e \leqq \frac{r_1^2 + r_2^2 + 2r_1 r_2 + 2r_1 + 2r_2 + 7}{r_1^2 + r_2^2 + 9}.$$

Since $r_1 \leqq -2$ for $i = 1, 2$, again by Schwartz's Lemma, it follows that

$$e \leqq 2 \cdot \frac{(r_1^2 + r_2^2 + r_1 + r_2 + \frac{7}{2})}{r_1^2 + r_2^2 + 9}.$$

Hence $e = -1, 0, 1$.

Again in the second case we get $e = -1, 0, 1$.

We would like to state the following

(2.8) Remark. If there exist smooth, connected, ruled surfaces $(\hat{X}, \hat{L})$ with $g = 6$, $\dim \varphi_L(X) = 2$, and $h^{1,0}(\hat{X}) = 1$, which satisfy the invariants:

$$\hat{d} = 9, \quad d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -2,$$

then $\hat{X}$ is made by blowing up two points on a geometrically ruled sur- over an elliptic curve with invariant $e = -1, 0, 1$.

Finally we can assume $\hat{d} \geqq 2g - 1$. Let $g = 6$. By

$$(K_{\hat{X}} + \hat{L}) \cdot (K_{\hat{X}} + \hat{L}) \geqq g + h^{2,0}(X) - h^{1,0}(X) - 2,$$

it follows that $\hat{d} \leqq 17$. By the long cohomology sequence of

$$0 \to \mathcal{O}_{\hat{X}} \to \hat{L} \to \hat{L}|_{\hat{C}} \to 0,$$

and by Riemann–Roch's Theorem, $h^1(\hat{L}) = 0$ or 1 and $h^0(\hat{L}) \geqq 6$. By the long cohomology sequence of

$$0 \to K_{\hat{X}} \to K_{\hat{X}} \otimes \hat{L} \to K_{\hat{C}} \to 0,$$

it follows that $h^0(K_{\hat{X}} \otimes \hat{L}) = 5$. In the same way we have that:

$$\text{if} \quad g = 5, \quad \hat{d} \leqq 14, \quad h^1(\hat{L}) = 0, 1, \quad h^0(\hat{L}) \geqq 5, \quad h^0(K_{\hat{X}} \otimes \hat{L}) = 4,$$
$$\text{if} \quad g = 4, \quad \hat{d} \leqq 11, \quad h^1(\hat{L}) = 0, 1, \quad h^0(\hat{L}) \geqq 5, \quad h^0(K_{\hat{X}} \otimes \hat{L}) = 3.$$

Let $g = 6$. By Propositions (0.1), (2.1) and Theorem (0.9) we have that:

$$\begin{array}{llll}
\hat{d} = 11, & d' = 7, & \hat{c}_1^2 = -2, & g' = 3, \\
\hat{d} = 11, & d' = 8, & \hat{c}_1^2 = -1, & g' = 4, \\
\hat{d} = 11, & d' = 9, & \hat{c}_1^2 = 0, & g' = 5, \\
\hat{d} = 12, & d' = 7, & \hat{c}_1^2 = -1, & g' = 3, \\
\hat{d} = 12, & d' = 8, & \hat{c}_1^2 = 0, & g' = 4, \\
\hat{d} = 13, & d' = 7, & \hat{c}_1^2 = 0, & g' = 3.
\end{array}$$

Let $g = 5$. In the same way we have:

$$\begin{array}{llll}
\hat{d} = 9, & d' = 6, & \hat{c}_1^2 = -1, & g' = 3, \\
\hat{d} = 9, & d' = 7, & \hat{c}_1^2 = 0, & g' = 4, \\
\hat{d} = 10, & d' = 6, & \hat{c}_1^2 = 0, & g' = 3, \\
\hat{d} = 12, & d' = 4, & \hat{c}_1^2 = 0, & g' = 1.
\end{array}$$

Let $g = 4$. In the same way we have:

$$\begin{array}{llll}
\hat{d} = 7, & d' = 5, & \hat{c}_1^2 = 0, & g' = 3, \\
\hat{d} = 9, & d' = 3, & \hat{c}_1^2 = 0, & g' = 1.
\end{array}$$

Now consider the cases in which $\hat{X}$ is a minimal model, i.e. $\hat{c}_1^2 = 0$. We have obtained the following cases:

$$\begin{aligned}
g = 6, \quad & \hat{d} = 11, \quad d' = 9, \quad g' = 5,\\
& \hat{d} = 12, \quad d' = 8, \quad g' = 4,\\
& \hat{d} = 13, \quad d' = 7, \quad g' = 3,\\
& \hat{d} = 15, \quad d' = 5, \quad g' = 1,\\
g = 5, \quad & \hat{d} = 9, \quad d' = 7, \quad g' = 4,\\
& \hat{d} = 10, \quad d' = 6, \quad g' = 3,\\
& \hat{d} = 12, \quad d' = 4, \quad g' = 1,\\
g = 4, \quad & \hat{d} = 7, \quad d' = 5, \quad g' = 3,\\
& \hat{d} = 9, \quad d' = 3, \quad g' = 1.
\end{aligned}$$

Let $g = 6$, $\hat{d} = 11$, $d' = 9$, $g' = 5$. By (0.5.3),

$$a = 11, \quad b = \frac{11e + 1}{2}.$$

By Propositions (0.5.4) and (0.5.5) we get that $e = -1$, $a = 11$, $b = -5$. Let $g = 6$, $\hat{d} = 12$, $d' = 8$, $g' = 4$. By (0.5.3)

$$a = 6, \quad b = \frac{6e + 2}{2}.$$

As before we get $e = -1$, $a = 6$, $b = -2$. Let $g = 6$, $\hat{d} = 13$. As before we get $a = \frac{13}{3}$ which is a contradiction. Let $g = 6$, $\hat{d} = 15$. Then:

$$\begin{aligned}
e &= 0, & a &= 3, & b &= \tfrac{5}{2}, & &\text{contradiction,}\\
e &= 1, & a &= 3, & b &= 4, & &\\
e &= 2, & a &= 3, & b &= \tfrac{11}{2}, & &\text{contradiction,}\\
e &= -1, & a &= 3, & b &= 1. & &
\end{aligned}$$

Let $g = 5$, $\hat{d} = 9$. Then

$$e = -1, \quad a = 9, \quad b = -4.$$

Let $g = 5$, $\hat{d} = 10$. Then

$$e = -1, \quad a = 5, \quad b = -\tfrac{3}{2}, \quad \text{contradiction.}$$

Let $g = 5$, $\hat{d} = 12$. Then

$$e = 0, \quad a = 3, \quad b = 2,$$
$$e = 1, \quad a = 3, \quad b = \tfrac{7}{2}, \quad \text{contradiction},$$
$$e = -1, \quad a = 3, \quad b = \tfrac{1}{2}, \quad \text{contradiction}.$$

Let $g = 4$, $\hat{d} = 7$. Then

$$e = -1, \quad a = 7, \quad b = -3.$$

Let $g = 4$, $\hat{d} = 9$. Then

$$e = 0, \quad a = 3, \quad b = \tfrac{3}{2}, \quad \text{contradiction},$$
$$e = -1, \quad a = 3, \quad b = 0.$$

Since by [14, p. 392], $\hat{L}|_{\zeta_E}$ has to be very ample, the cases $g = 6$, $\hat{d} = 15$, $\hat{L} \equiv \zeta_E^2 \otimes \mathscr{L}^4$, $e = 1$ and $g = 5$, $\hat{d} = 12$, $\hat{L} \equiv \zeta_E^3 \otimes \mathscr{L}^2$, $e = 0$, cannot happen. Since by Proposition (0.5.8) we can compute $h^0(\hat{L})$, we see that in the case $g = 4$, $\hat{d} = 7$ it follows that $h^0(\hat{L}) = 4$ which contradicts $h^0(\hat{L}) \geqq 5$. Thus we can state the following proposition:

(2.9) **Proposition.** *Let X be a smooth, connected, projective, ruled surface and L a very ample line bundle on it. Suppose that $\hat{X}$ is a minimal model, $h^{1,0}(X) = 1$, $\dim \varphi_L(X) = 2$ and $\hat{d} \geqq 2g - 1$. Then $(\hat{X}, \hat{L})$ has to be one of the following surfaces:*

(1) $e = -1$, $\hat{L} \equiv \zeta_E^{11} \otimes \mathscr{L}^{-5}$, $g = 6$, $\hat{d} = 11$, $d' = 9$, $g' = 5$, $h^0(\zeta_E^{11} \otimes \mathscr{L}^{-5}) = 6$,

(2) $e = -1$, $\hat{L} \equiv \zeta_E^{6} \otimes \mathscr{L}^{-2}$, $g = 6$, $\hat{d} = 12$, $d' = 8$, $g' = 4$, $h^0(\zeta_E^{6} \otimes \mathscr{L}^{-2}) = 7$.

(3) $e = -1$, $\hat{L} \equiv \zeta_E^{3} \otimes \mathscr{L}$, $g = 6$, $\hat{d} = 15$, $d' = 5$, $g' = 1$, $h^0(\zeta_E^{3} \otimes \mathscr{L}) = 10$.

(4) $e = -1$, $\hat{L} \equiv \zeta_E^{9} \otimes \mathscr{L}^{-4}$, $g = 5$, $\hat{d} = 9$, $d' = 7$, $g' = 4$, $h^0(\zeta_E^{9} \otimes \mathscr{L}^{-4}) = 5$.

(5) $e = -1$, $\hat{L} \equiv \zeta_E^{3}$, $g = 4$, $\hat{d} = 9$, $d' = 3$, $g' = 1$, $h^0(\zeta_E^{3}) = 6$.

Remark. We do not know if those $\hat{L}$ are very ample.

Now consider the case when $\hat{X}$ is made by blowing up one point over a geometrically ruled surface over an elliptic curve, i.e. when $\hat{c}_1^2 = -1$. We have to examine the following cases:

(A) $g = 6, \quad \hat{d} = 11, \quad d' = 8, \quad g' = 4,$

(B) $g = 6, \quad \hat{d} = 12, \quad d' = 7, \quad g' = 3,$

(C) $g = 5, \quad \hat{d} = 9, \quad d' = 6, \quad g' = 3.$

In case (C) as usual we compute that $h^0(K_{\hat{X}} \otimes L') = 2$ and

$$(K_{\hat{X}} + L') \cdot (K_{\hat{X}} + L') = 1,$$

which gives a contradiction, since if $\dim \varphi_{K_{\hat{X}} \otimes L'}(\hat{X}) = 1$, then

$$(K_{\hat{X}} + L') \cdot (K_{\hat{X}} + L') = 0.$$

Thus we have to examine the usual systems in cases (A) and (B).

$$r \leqq -2, \quad a \geqq 3,$$

(i) $ae - b \leqq -1,$
(ii) $\hat{d} = -a^2 e + 2ab - r^2,$
(iii) $2g - 2 = \hat{d} + ae - 2b - r,$
(iv) $d' = -a^2 e + 2ae + 2ab - 4b - r^2 - 2r - 1,$
(v) $2g' - 2 = d' + ae - 2b - r - 1.$

As usual $e = -1, 0, 1$. Consider case (A). By (v) we get

(2.9.1) if $e = 0, \quad 2b = -r + 1,$

(2.9.2) if $e = -1, \quad 2b = -a - r + 1,$

(2.9.3) if $e = 1, \quad 2b = a - r + 1.$

Substituting (2.9.1), (2.9.2), and (2.9.3) in (ii) or (iv) we get:

$$11 = -ae + a - e^2 .$$

Consider case (B). By (v) we get

(2.9.4) if $e = 0, \quad 2b = -r + 2,$

(2.9.5) if $e = -1, \quad 2b = -a - r + 2,$

(2.9.6) if $e = 1, \quad 2b = a - r + 2.$

Substituting (2.9.4), (2.9.5), and (2.9.6) in (ii) we get:

$$12 = -ar + 2a - r^2.$$

We can state the following lemma.

(2.10) Lemma. *Let $(\hat{X}, \hat{L})$ be a minimal pair of a smooth, connected, projective, ruled surface such that $\hat{X}$ is made by blowing up one point over a geometrically ruled surface over an elliptic curve with invariant e. Suppose that $\hat{d} \geqq 2g - 1$. Then $g = 6$ and $(\hat{X}, \hat{L})$ has to be one of the following:*

(1) $e=-1,\quad 0,1,\quad \hat{d}=11,\quad d'=8,\quad g'=4,$

(2) $e=-1,\quad 0,1,\quad \hat{d}=12,\quad d'=7,\quad g'=3.$

Now consider the case in which $\hat{X}$ is made by blowing up two points over a geometrically ruled surface over an elliptic curve, that is $\hat{c}_1^2=-2$. We have that

$$g=6,\quad \hat{d}=11,\quad d'=7,\quad g'=3.$$

As in the previous case we have that

$$h^0(K_{\hat{X}}\otimes L')=2 \text{ and } (K_{\hat{X}}+L')\cdot(K_{\hat{X}}+L')=-1$$

which gives a contradiction, since $K_{\hat{X}}\otimes L'$ is spanned by Proposition (0.8). We can finally state the following theorem.

(2.11) THEOREM. *Let $(\hat{X},\hat{L})$ be a minimal pair of a smooth, connected, projective, ruled surface such that $\hat{d}\geqq 2g-1$. Then the pair $(\hat{X},\hat{L})$, if it exists, has to satisfy one of the following sets of invariants:*

(1) $g=6,\quad \hat{d}=11,\quad d'=8,\quad g'=4,\quad \hat{c}_1^2=-1,\quad e=-1,0,1,$

(2) $g=6,\quad \hat{d}=12,\quad d'=7,\quad g'=3,\quad \hat{c}_1^2=-1,\quad e=-1,0,1,$

where e is the invariant of the minimal model.

Moreover if $\hat{X}$ is a minimal model then it has to be one of the following:

(3) $g=6,\quad e=-1,\quad \hat{L}\equiv\zeta_E^{11}\otimes\mathscr{L}^{-5},\quad \hat{d}=11,\quad d'=9,\quad g'=5,$

(4) $g=6,\quad e=-1,\quad \hat{L}\equiv\zeta_E^{6}\otimes\mathscr{L}^{-2},\quad \hat{d}=12,\quad d'=8,\quad g'=4,$

(5) $g=6,\quad e=-1,\quad \hat{L}\subset\zeta_E^{3}\otimes\mathscr{L},\quad \hat{d}=15,\quad d'=5,\quad g'=1,$

(6) $g=5,\quad e=-1,\quad \hat{L}\equiv\zeta_E^{9}\otimes\mathscr{L}^{-4},\quad \hat{d}=9,\quad d'=7,\quad g'=4,$

(7) $g=4,\quad e=-1,\quad \hat{L}\equiv\zeta_E^{3},\quad \hat{d}=9,\quad d'=3,\quad g'=1.$

REFERENCES

1. A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. 69 (1959), 713–717.
2. H. F. Baker, *Principles of geometry*, V; *Analytical principles of the theory of curves*, Cambridge University Press, Cambridge, 1933.
3. R. Bott, *On a theorem of Lefschetz*, Michigan Math. J. 6 (1959), 211–216.
4. P. A. Griffiths and J. Harris, *Principles of algebraic geometry*, J. Wiley and Sons, New York, 1978.
5. R. Hartshorne, *Algebraic geometry* (Grad. Texts in Math. 52), Springer-Verlag, Berlin - Heidelberg - New York, 1977.
6. P. Ionescu, *An enumeration of all smooth protective varieties of degree 5 and 6*, Preprint Series in Mathematics NR 74, Increst Bucarest, 1981.
7. A. Lanteri and M. Palleschi, *About the adjunction process for polarized algebraic surfaces*, J. Reine Angew. Math. 352 (1984), 15–23.

8. E. L. Livorni, *Classification of algebraic surfaces with sectional genus less than or equal to six* II: Ruled surfaces with $\dim \varphi_{K_X \otimes L}(X) = 1$, Canad. J. Math. 38 (1986).
9. M. Maruyama, *On classification of ruled surfaces* (Lectures in Math., Kyoto Univ. 3), Kinokuniya Book-Store Co. Ltd., Tokyo, 1970.
10. M. Nagata, *On rational surfaces*, I, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. 32 (1960), 351–370.
11. M. Nagata, *On self-intersection number of a section on a ruled surface*, Nagoya Math. J. 37 (1970), 191–196.
12. L. Roth, *On the projective classification of surfaces*, Proc. London Math. Soc. (2) 42 (1937), 142–170.
13. I. R. Šafarevič, *Algebraic surfaces*, Trudy Mat. Inst. Steklov 75 (1965), pp. 1–215 (in Russian). (Proc. Steklov Inst. Math. 75 (1965), translation by American Mathematical Society, R. I., 1967.)
14. A. J. Sommese, *Hyperplane sections of projective surfaces*, I: *The adjunction mapping*, Duke Math. J. Vol. 46 (1979), 377–401.
15. A. J. Sommese, *The birational theory of hyperplane sections of projective threefolds*, Preprint, University of Notre Dame, 1981.
16. A. J. Sommese, *Ample divisors on normal Gorestein surfaces*, Abh. Math. Sem. Univ. Hamburg 55 (1985), 151–170.
17. A. Van de Ven, *On the 2-connectedness of very ample divisors on a surface*, Duke Math. J. 46 (1979), 403–407.

UNIVERSITÁ DEGLI STUDI DELL' AQUILA DEGLI ABRUZZI
ISTITUTO DI MATEMATICA
VIA ROMA 33
67100 L'AQUILA
ITALY