MATH. SCAND. 59 (1986), 30–40

REGULAR h-RANGES AND WEAKLY PLEASANT h-BASES

CHRISTOPH KIRFEL AND ERNST S. SELMER

1. Introduction.

For the definition of the ordinary h-range $n_h(A_k)$ and the regular h-range $g_h(A_k)$, we refer to Selmer [7]. There we introduce

$$h_0 = h_0^{(k)} = \min\{h \in \mathsf{N} \mid n_h(A_k) \geqq a_k\},$$

as the smallest h for which A_k is "admissible". For regular h-ranges, we shall similarly denote the smallest admissible h by $\tilde{h}_0 = \tilde{h}_0^{(k)} \geqq h_0^{(k)}$. In [7], we also define the important "stabilization" bound $h = h_1 \geqq h_0 - 1$, as the minimal h for which

$$(1.1) \qquad n_{h+1}(A_k) = n_h(A_k) + a_k, \quad h \geqq h_1.$$

A basis A_k is called *pleasant* if one "minimal" representation always coincides with the unique regular representation. It is clear that

$$A_k \text{ pleasant } \Rightarrow n_h(A_k) = g_h(A_k), \quad \forall h \in \mathsf{N}.$$

This implication *cannot be reversed.* A basis A_k such that

$$(1.2) \qquad n_h(A_k) = g_h(A_k), \quad \forall h \in \mathsf{N},$$

will be called *weakly pleasant.*

Any basis $A_2 = \{1, a_2\}$ is pleasant. For $A_3 = \{1, a_2, a_3\}$, we put $a_3 = qa_2 - s, 0 \leqq s < a_2$. It is known that A_3 is pleasant if and only if

$$(1.3) \qquad q > s.$$

For non-pleasant A_3, we have $n_h(A_3) > g_h(A_3)$ for $h \geqq h_0 = h_0^{(3)} = \tilde{h}_0^{(3)} = a_2 + [a_3/a_2] - 2$, so a weakly pleasant basis A_3 is automatically pleasant.

It was proved by Zöllner [9] that

$$(1.4) \qquad k \geqq 4, \ A_k \text{ pleasant} \Rightarrow \{1, a_2, a_3\} \text{ pleasant.}$$

Received June 25, 1985.

Several years ago, Selmer was able to weaken the condition to "A_k weakly pleasant", by showing that

(1.5) $\{1, a_2, a_3\}$ non-pleasant $\Rightarrow$ $n_h(A_k) > g_h(A_k)$, $h \geqq h_0^{(3)}$.

Since this result is superseded by Theorem 3 below, we shall not give Selmer's original proof, which has appeared in [8].

Later, Zöllner [10] generalized (1.4) to

(1.6) $k \geqq 4$, A_k pleasant $\Rightarrow$ $\{1, a_2, a_i\}$ pleasant, $3 \leqq i \leqq k$.

He pointed out that this cannot be reversed: $\{1, 2, 4\}$ and $\{1, 2, 5\}$ are pleasant, $\{1, 2, 4, 5\}$ not. The basis element a_2 is essential: $\{1, 2, 3, 4\}$ is pleasant, $\{1, 3, 4\}$ not.

One main object of the present paper is again to weaken the condition of (1.6) to "A_k weakly pleasant". This problem is solved in Section 4, due to Kirfel. His results are based on Theorem 1, established by Selmer several years ago. Section 3 on weakly pleasant bases is also due to Selmer.

2. The regular h-range.

By regular representations, we first use the largest basis element a_k as often as possible, then a_{k-1} as often as possible, etc. This implies that $g_{\tilde{h}_0}(A_k)$ contains just *one* addend a_k, which is removed in $g_{\tilde{h}_0 - 1}(A_k)$. In general, it is clear that

(2.1) $$g_{h+1}(A_k) = g_h(A_k) + a_k, \quad h \geqq \tilde{h}_0 - 1 = \tilde{h}_1.$$

This should be compared with (1.1) for ordinary h-ranges.

The first explicit determination of $g_h(A_k)$ was given by Hofmeister [1, Satz 1]. His proof was extremely short and almost impossible to understand. Readable but less accessible versions appeared in [3], [4], [5].

We shall give an *alternative formulation*, which is easier to use and perhaps simpler to prove. In addition, we get an explicit expression for $\tilde{h}_0$.

We perform the divisions

(2.2) $$\begin{aligned} a_3 &= f_2 a_2 + r_2, & 0 &\leqq r_2 < a_2 \\ a_4 + r_2 &= f_3 a_3 + r_3, & 0 &\leqq r_3 < a_3 \\ \cdots\cdots &\quad \cdots\cdots & &\cdots\cdots \\ a_{i+1} + r_{i-1} &= f_i a_i + r_i, & 0 &\leqq r_i < a_i \\ \cdots\cdots &\quad \cdots\cdots & &\cdots\cdots \\ a_k + r_{k-2} &= f_{k-1} a_{k-1} + r_{k-1}, & 0 &\leqq r_{k-1} < a_{k-1}. \end{aligned}$$

Since $a_1 = 1$, we can also formally put $f_1 = a_2$, $r_1 = 0$.

THEOREM 1.

$$\tilde{h}_0 = \tilde{h}_0^{(k)} = a_2 + f_2 + f_3 + \ldots + f_{k-1} - k + 1, \tag{2.3}$$

$$\begin{aligned} g_{\tilde{h}_0}(A_k) &= (a_2 - 2) + (f_2 - 1)a_2 + \ldots + (f_{k-1} - 1)a_{k-1} + 1 \cdot a_k \\ &= 2a_k - (r_{k-1} + 2). \end{aligned} \tag{2.4}$$

PROOF. From (2.1), we get

$$g_h(A_k) = (h - \tilde{h}_0)a_k + g_{\tilde{h}_0}(A_k), \quad h \geqq \tilde{h}_0 - 1. \tag{2.5}$$

By induction, it follows immediately from (2.2) that

$$(a_2 - 2) + (f_2 - 1)a_2 + \ldots + (f_i - 1)a_i = a_{i+1} - (r_i + 2) < a_{i+1} \tag{2.6}$$

for $i = 1, 2, \ldots, k-1$. This shows that the representation (2.4) is *regular*, and also proves the last equality of (2.4). In particular,

$$n = (a_2 - 1) + (f_2 - 1)a_2 + \ldots + (f_{k-1} - 1)a_{k-1} < a_k$$

is a regular representation whose coefficient sum equals the $\tilde{h}_0$ of (2.3). Since $n < a_k$, this $\tilde{h}_0$ is consequently a *lower bound* for $\tilde{h}_0^{(k)}$.

Now Theorem 1 is trivially correct for $k = 2$, and we may prove it by *induction*, assuming it to be correct for A_{k-1}. To prove (2.3), we must then show that

$$h' = \tilde{h}_0^{(k-1)} + (f_{k-1} - 1)$$

equals $\tilde{h}_0^{(k)}$. Combining (2.5) and (2.4), both with A_k replaced by A_{k-1}, we get

$$g_{h'}(A_{k-1}) = (a_2 - 2) + (f_2 - 1)a_2 + \ldots + (f_{k-2} - 1)a_{k-2} + f_{k-1}a_{k-1}. \tag{2.7}$$

From (2.6) for $i = k-1$, it follows that

$$g_{h'}(A_{k-1}) = a_k + a_{k-1} - (r_{k-1} + 2) \geqq a_k - 1,$$

showing that h' is also an *upper bound* for $\tilde{h}_0^{(k)}$. This completes the proof of (2.3).

Substituting $h' = \tilde{h}_0 = \tilde{h}_0^{(k)} \geqq \tilde{h}_0^{(k-1)}$ in (2.7), and again using (2.5) for A_{k-1}, we get

$$g_{\tilde{h}_0 - 1}(A_{k-1}) = (a_2 - 2) + (f_2 - 1)a_2 + \ldots + (f_{k-1} - 1)a_{k-1}.$$

Then using (2.5) for A_k, with $h = \tilde{h}_0 - 1$, we finally get (2.4), and Theorem 1 is proved.

It is also easily seen that our result and the original formulation of Hofmeister are *equivalent*.

3. Weakly pleasant bases.

These are defined by (1.2). It is of course particularly interesting to determine those bases which are weakly pleasant *without* being pleasant. The first such example, $A_5 = \{1, 2, 5, 7, 10\}$, was discovered by E. Deinert and reproduced in Hofmeister's lecture notes [3]. In a later set of notes [4], Hofmeister put the name "schwach angenehm" to the bases satisfying (1.2). So far, no theoretical study of such bases has appeared.

The weakly pleasant bases (*including* the pleasant ones) are characterized by the following

THEOREM 2. *The basis A_k is weakly pleasant if and only if the following three conditions are satisfied:*

$$\tilde{h}_0^{(k)} = h_0^{(k)} = h_0, \quad (say) \tag{3.1}$$

$$n_{h_0-1}(A_{k-1}) = g_{h_0-1}(A_{k-1}), \tag{3.2}$$

$$h_1 = h_0 - 1. \tag{3.3}$$

PROOF. The necessity of (3.1) is clear, since

$$\tilde{h}_0^{(k)} > h_0^{(k)} = h_0 \;\Rightarrow\; n_{h_0}(A_k) \geqq a_k > g_{h_0}(A_k). \tag{3.4}$$

The condition (3.2) is part of (1.2), since $n_{h_0-1}(A_k) = n_{h_0-1}(A_{k-1})$ and $g_{h_0-1}(A_k) = g_{h_0-1}(A_{k-1})$.

We know by (2.1) that $g_h(A_k)$ is *stabilized* from $h = \tilde{h}_0^{(k)} - 1 = h_0 - 1$, and will thus "keep pace" with $n_h(A_k)$ for $h \geqq h_0 - 1$ if and only if (3.3) is satisfied, cf. (1.1).

The proof of Theorem 2 will then be complete if we can show that $n_h(A_k) = g_h(A_k)$ for $h < h_0 - 1$. And this follows from (3.2) and the general result

$$n_{h+1}(A_k) - g_{h+1}(A_k) \geqq n_h(A_k) - g_h(A_k), \quad h \geqq 1. \tag{3.5}$$

To prove this, we first note that the alternative to $=$ in (1.1) is $>$, since

$$n_{h+1}(A_k) \geqq n_h(A_k) + a_k, \quad h \geqq h_0 - 1. \tag{3.6}$$

If then $h \geqq \tilde{h}_0^{(k)} - 1$ $(\geqq h_0^{(k)} - 1)$, the correctness of (3.5) is an immediate consequence of (2.1) and (3.6).

Let next $h < \tilde{h}_0^{(k)} - 1$. We may assume $h \geqq \tilde{h}_0^{(3)} = h_0^{(3)}$, since always $n_h(A_2) = g_h(A_2)$. Then $\tilde{h}_0^{(3)} < \tilde{h}_0^{(k)} - 1$, and we can find an i with $3 \leqq i < k$ such that

$$\tilde{h}_0^{(i)} - 1 \leqq h < \tilde{h}_0^{(i+1)} - 1.$$

Then $h + 1 < \tilde{h}_0^{(i+1)}$, hence by (2.1):

$$g_{h+1}(A_k) = g_{h+1}(A_i) = g_h(A_i) + a_i = g_h(A_k) + a_i.$$

As before, $h \geqq h_0^{(i)} - 1$. It is even possible that $h \geqq h_0^{(j)} - 1$ with $j > i$, but in any case (3.6) gives

$$n_{h+1}(A_k) \geqq n_h(A_k) + a_i,$$

and we are through.

To show that the three conditions (3.1–3.3) are *independent*, we list the following bases A_4 which fail to satisfy just *one* of the conditions (in turn), and which are not weakly pleasant:

A_4	h_0	$\tilde{h}_0$	h_1	
$\{1,3,4,7\}$	2	3	1	$n_1(A_3) = g_1(A_3) = 1,\ n_2(A_4) = 8 > g_2(A_4) = 5$
$\{1,3,4,10\}$	3	3	2	$n_2(A_3) = 8 > g_2(A_3) = 5$
$\{1,2,4,5\}$	2	2	2	$n_1(A_3) = g_1(A_3) = 2,\ n_2(A_4) = 10 > g_2(A_4) = 7.$

If the partial basis A_{k-1} is *pleasant*, the conditions (3.1–3.2) are automatically satisfied, and we are left with the condition (3.3), $h_1^{(k)} = h_0^{(k)} - 1$, for weak pleasantness of A_k. In particular, this is the situation for $k = 4$, since then $A_{k-1} = A_3$ must be pleasant by (1.5). It is shown in [8] that A_4 is weakly pleasant but *not* pleasant if and only if the following conditions are satisfied (cf. (2.2)):

$$a_4 = aa_3 + b, \quad 1 \leqq b \leqq r_2 < a_2 \leqq a + b$$

$$a + b - r_2 + 1 \leqq f_2 \leqq \langle (r_2 + 1)/b \rangle (a + b - 1) - a_2 - r_2 + 2.$$

As usual, $\langle x \rangle$ denotes the smallest integer $\geqq x$.

An important contribution to the establishment of the final inequality was made by Hans-Georg Beuter.

The first such bases A_4 are given by

h_0	4		5		6			
a_2	2	3	3	3	2	3	4	4
a_3	7	8	11	11	9	11	11	15
a_4	15	17	23	24	28	34	34	33

4. Non-pleasant partial bases $\{1, a_2, a_i\}$.

It was first observed by Beuter that the result (1.5) under certain circumstances can be generalized to the case when a partial basis $\{1, a_2, a_i\}$ is non-pleasant for some i with $3 \leqq i \leqq k$. We put $a_i = Qa_2 - S$, $0 \leqq S < a_2$, where now $Q \leqq S$ by (1.3). Assume that a_i really *appears* in the regular representation (2.4) (always for $i = k$, otherwise if $f_i > 1$). It is then simple to see that $n_{\tilde{h}_0}(A_k) > g_{\tilde{h}_0}(A_k)$, by transforming the following three terms of $g_{\tilde{h}_0}(A_k) + 1$:

$$(a_2 - 1) + (f_2 - 1)a_2 + a_i = (a_2 - 1 - S) + (f_2 - 1 + Q)a_2.$$

Since $Q \leqq S$, the coefficient sum is *reduced* with at least 1, meaning that $g_{\tilde{h}_0}(A_k) + 1$ has a (non-regular) $\tilde{h}_0$-representation by A_k.

This argument fails if $i < k$ and $f_i = 1$. All the same, we can prove the following generalization of (1.5):

THEOREM 3. *If the partial basis* $\{1, a_2, a_i\}$ *is non-pleasant for some i with* $3 \leqq i \leqq k$, *then*

(4.1) $$n_h(A_k) > g_h(A_k), \qquad h \geqq h_0^{(i)}$$

(4.2) $$n_h(A_k) \geqq g_h(A_k) + a_i - 1, \quad h \geqq \tilde{h}_0^{(i)}.$$

COROLLARY.

$$k \geqq 4,\ A_k \text{ weakly pleasant} \Rightarrow \{1, a_2, a_i\} \text{ pleasant},\quad 3 \leqq i \leqq k.$$

This is the promised strengthening of Zöllner's result (1.6).

To prove Theorem 3, we use the following

LEMMA. *Every integer n such that*

$$a_{p-1} \leqq n < a_p,\quad 1 < p \leqq k,$$

can be written in the form

(4.3) $$n = a_p - t_{p-1}a_{p-1} - t_{p-2}a_{p-2} - \dots - t_2a_2 - t_1,$$

where the integer coefficients t_j *satisfy the conditions*

(i) $-1 \leqq t_j \leqq f_j - 1,\ 1 < j \leqq p-1;\ 0 \leqq t_1 \leqq a_2 - 1.$
(ii) $t_l = -1 \Rightarrow t_{l+1} = t_{l+2} = \dots = t_{L-1} = 0,\ t_L > 0,\ L \leqq p-1.$
(iii) *If* $m = \min\{\mu > 1 \mid t_\mu = -1\}$, *then*

$$t_{m-1}a_{m-1} + t_{m-2}a_{m-2} + \dots + t_2a_2 + t_1 > 0.$$

Here the integers f_j are taken from the algorithm (2.2). The statements (ii) and (iii) may of course be empty.

For $1 \leqq n < a_2$, hence $n = a_2 - (a_2 - n)$, the above conditions are satisfied. We may therefore use *induction* on n, assuming that the Lemma holds for all $n < N < a_k$. If then $a_{p-1} \leqq N < a_p$, we first write

(4.4) $$N = a_p - e_{p-1}a_{p-1} - e_{p-2}a_{p-2} - \dots - e_2a_2 - e_1,$$

where $a_p - N = \sum e_j a_j$ is the *regular* representation by A_{p-1}. Clearly $0 \leqq e_1 \leqq a_2 - 1$, and further $0 \leqq e_j \leqq f_j$ for $j > 1$, since $\sum e_j a_j$ is regular and $(f_j + 1)a_j > a_{j+1}$ by (2.2).

If already $e_j \leqq f_j - 1$ for all $j > 1$, we are finished. Otherwise, there is a *maximal* index $\tau \leqq p-1$ such that $e_\tau = f_\tau$. If $\tau = p-1$, we get $N \leqq a_p - f_{p-1}a_{p-1} = r_{p-1} - r_{p-2} < a_{p-1}$, so $\tau < p-1$. If $e_j = f_j - 1$ for all $j > \tau$, we get the same contradiction

$$N \leqq a_p - (f_{p-1} - 1)a_{p-1} - \dots - (f_{\tau+1} - 1)a_{\tau+1} - f_\tau a_\tau = r_{p-1} - r_{\tau-1} < a_{p-1}.$$

There is consequently a *minimal* index T with $\tau < T < p$ such that $e_T \leqq f_T - 2$. We write

(4.5) $$N = a_p - e_{p-1}a_{p-1} - \dots - e_{T+1}a_{T+1} - (e_T + 1)a_T + N'.$$

Since $\sum e_j a_j$ is regular, a comparison with (4.4) shows that $N' > 0$. On the other hand,

$$N' = a_T - (f_{T-1} - 1)a_{T-1} - \dots - (f_{\tau+1} - 1)a_{\tau+1} - f_\tau a_\tau - e_{\tau-1}a_{\tau-1} - \dots - e_1$$
$$= r_{T-1} - r_{\tau-1} - e_{\tau-1}a_{\tau-1} - \dots - e_1 < a_{T-1} \leqq a_{p-2} < N.$$

We can now use the induction hypothesis on N'. If $a_{w-1} \leqq N' < a_w$, then

$$N' = a_w - t'_{w-1}a_{w-1} - t'_{w-2}a_{w-2} - \dots - t'_2a_2 - t'_1,$$

with t_j' satisfying (i)–(iii). Substitution of N' into (4.5) shows that we now have an expression (4.3) for N if we put

$$\begin{aligned} 0 \leqq t_j &= e_j \leqq f_j - 1, && j = T+1, \ldots, p-1 \\ 0 < t_T &= e_T + 1 \leqq f_T - 1 && \\ 0 = t_j &\leqq f_j - 1, && j = w+1, \ldots, T-1 \quad (\text{if } w < T-1) \\ t_w &= -1 && \\ t_j &= t_j', && j = 1, \ldots, w-1. \end{aligned}$$

This completes the proof of the Lemma.

We can now prove Theorem 3. Here (4.1) is trivial if $\tilde{h}_0^{(i)} > h_0^{(i)}$ (replace k by i in (3.4) and use (3.5)). It therefore suffices to prove (4.2) with $h = \tilde{h}_0^{(i)} = H$ (say). There is then a largest κ, $i \leqq \kappa \leqq k$, such that $H = \tilde{h}_0^{(\kappa)}$, with

$$g_H(A_k) = g_H(A_\kappa) = (a_2 - 2) + (f_2 - 1)a_2 + \ldots + + (f_{i-1} - 1)a_{i-1} + 1 \cdot a_\kappa. \tag{4.6}$$

We must show that for $1 \leqq \Delta \leqq a_i - 1$, there exists a (not necessarily regular) representation of $g_H(A_\kappa) + \Delta$ by A_κ with at most H addends. For this purpose, we apply (4.3) to $n = a_\kappa - a_i + \Delta$. Clearly $0 < n < a_\kappa$, and so $1 < p \leqq \kappa$. Substitution of $a_\kappa = n + a_i - \Delta$ into (4.6) gives

$$g_H(A_\kappa) + \Delta = (a_2 - 2 - t_1) + (f_2 - 1 - t_2)a_2 + \ldots + + (f_{\kappa-1} - 1 - t_{\kappa-1})a_{\kappa-1} + a_i + a_p \tag{4.7}$$

(where $f_j = 1$ for $j > i$, and $t_j = 0$ for $j \geqq p$). This representation has the coefficient sum $H + 1 - \sum_1^{p-1} t_j \leqq H$, since it is easily seen that $\sum t_j > 0$: By (ii), every $t_l = -1$ is compensated by a $t_L > 0$, and by (iii), there exists at least one $t_j > 0$ for $j < m$.

It then remains to examine whether the representation (4.7) is "legal", hence has non-negative coefficients for the elements a_j. By (i), this holds for all $j > 1$, but we will get a constant term -1 if $t_1 = a_2 - 1$. Only in this case do we need to use the fact that $\{1, a_2, a_i\}$ is *non-pleasant*. As before, we put $a_i = Qa_2 - S$, $0 \leqq S < a_2$, and transform the following three terms of (4.7):

$$-1 + (f_2 - 1 - t_2)a_2 + a_i = (a_2 - 1 - S) + (f_2 - 2 - t_2 + Q)a_2.$$

The resulting representation of $g_H(A_\kappa)+\Delta$ is legal and has a coefficient sum $H+Q-S-\sum_2^{p-1} t_j \leqq H$ by (ii) and $Q \leqq S$, which completes the proof of Theorem 3.

In fact, it is easily seen that $Q \leqq S$ can be replaced by the *weaker* condition

(4.8) $$a_i = \sum_2^{i-1} b_j a_j - S',\ \ b_j \geqq 0,\ \ b_2 > 0,\ \ 0 \leqq S' < a_2;\ \ \sum_2^{i-1} b_j \leqq S'.$$

In connection with Theorem 3, we finally note that the bound (4.2) is *sharp*. The simplest example is given by

$$k = i = 3,\ \ A_3 = \{1,3,4\},\ \ h_0 = \tilde{h}_0 = 2,\ \ n_2(A_3) = 8,\ \ g_2(A_3) = 5.$$

We now turn to a similar result which involves the *whole* basis A_k, even if $i < k$.

THEOREM 4. *Let the partial basis $\{1, a_2, a_i\}$ be non-pleasant for some i with $3 \leqq i \leqq k$, and put*

$$a_i = Qa_2 - S,\ \ 0 \leqq S < a_2\ \ (\textit{hence } Q \leqq S),$$
$$\delta = \min\{S-Q,\ a_2-1-S\}.$$

Then

(4.9) $$n_h(A_k) \geqq g_h(A_k) + a_i - 1 + \delta(a_k - 1),\ \ h \geqq \tilde{h}_0^{(k)}.$$

PROOF. By (3.5), it suffices to prove this for $h = \tilde{h}_0^{(k)} = H$. If $\delta = 0$, the result follows from (4.2). If $\delta > 0$, choose d with $1 \leqq d \leqq \delta$ and N with

$$g_H(A_k) + a_i + (d-1)(a_k-1) \leqq N \leqq g_H(A_k) + a_i - 1 + d(a_k - 1).$$

We must show that N has a (not necessarily regular) representation by A_k with at most H addends. We can write

$$N = g_H(A_k) + a_i - 1 + (d-1)(a_k-1) + n,\ \ 0 < n < a_k.$$

To this n, we can thus apply (4.3), and get

$$N = (a_2 - 2 - t_1 - d) + (f_2 - 1 - t_2)a_2 + \ldots + (f_{k-1} - 1 - t_{k-1})a_{k-1} + \\ + da_k + a_i + a_p$$

(where $t_j = 0$ for $j \geqq p$). This representation has the coefficient sum $H + 1 - \sum_1^{p-1} t_j \leqq H$, and is not legal only if $a_2 - 2 - t_1 - d \leqq -1$. Then N may be written as

$$N = (2a_2 - 2 - t_1 - d - S) + (f_2 - 2 - t_2 + Q)a_2 + \ldots + \\ + (f_{k-1} - 1 - t_{k-1})a_{k-1} + da_k + a_p.$$

This is a legal representation since $t_1 \leqq a_2 - 1$ and $d + S \leqq \delta + S \leqq a_2 - 1$. Further $2a_2 - 2 - t_1 - d - S \leqq a_2 - 1 - S$, so the coefficient sum is at most (cf. (2.3))

$$a_2 - 1 - S + Q + \textstyle\sum_2^{k-1} f_j - \sum_2^{p-1} t_j + d - k + 2 \\ = H - S + Q + d - \textstyle\sum_2^{p-1} t_j \leqq H - S + Q + d \leqq H,$$

since $d \leqq \delta \leqq S - Q$.

This completes the proof of Theorem 4. Again, the condition $Q \leqq S$ (making $\delta \geqq 0$) may be replaced by (4.8). At the same time, we must then replace δ by

$$\delta' = \min\{S' - \textstyle\sum_2^{i-1} b_j,\ a_2 - 1 - S'\}.$$

Theorem 4 has an interesting application to *extremal bases* $A_3 = A_3^*$, which for given h have the largest possible h-range $n_h(3) = n_h(A_3^*)$. These bases were determined by Hofmeister [2], who in particular found that

$$n_h(3) \sim \frac{4}{81} h^3 \tag{4.10}$$

(asymptotically, as $h \to \infty$).

Hofmeister also determined the extremal h_0-bases A_3 (that is, bases which are admissible for a given $h = h_0$, but not for $h = h_0 - 1$), and the corresponding extremal h_0-ranges $l_{h_0}(3)$, where

$$l_{h_0}(3) \sim \frac{\sqrt{3}}{36} h_0^3. \tag{4.11}$$

Quite surprisingly, the "simple" result (4.9), with $i = k = 3$, yields both results (4.10–4.11) as (asymptotically) lower bounds for $n_h(3)$ and $l_{h_0}(3)$. Details are found in [6].

REFERENCES

1. G. Hofmeister, *Über eine Menge von Abschnittsbasen*, J. Reine Angew. Math. 213 (1963), 43–57.
2. G. Hofmeister, *Asymptotische Abschätzungen für dreielementige Extremalbasen in natürlichen Zahlen*, J. Reine Angew. Math. 232 (1968), 77–101.
3. G. Hofmeister, *Vorlesungen über endliche additive Zahlentheorie*, Lecture notes, Math. Inst., Joh. Gutenberg-Univ., Mainz, 1976.

4. G. Hofmeister, *Lineare diophantische Probleme*, Lecture notes, Math. Inst., Joh. Gutenberg-Univ., Mainz, 1978.
5. G. Hofmeister, *Zum Reichweitenproblem*, Mainzer Seminarberichte in additiver Zahlentheorie 1 (1983), 30–52.
6. C. Kirfel and E. S. Selmer, *Regular h-ranges, weakly pleasant h-bases and extremal bases* A_3, Inst. report No. 39, Dept. of Pure Math., Univ. of Bergen, 1985.
7. E. S. Selmer, *On the postage stamp problem with three stamp denominations*, Math. Scand. 47 (1980), 29–71.
8. E. S. Selmer, *The local postage stamp problem*, Inst. reports No. 42, 44, Dept. of Pure Math., Univ. of Bergen, 1986.
9. J. Zöllner, *Über Mengen natürlicher Zahlen für die jede euklidische Darstellung eine minimale Koeffizientensumme besitzt*, Diplomarbeit, Math. Inst., Joh. Gutenberg-Univ., Mainz, 1974.
10. J. Zöllner, *Über angenehme Mengen*, Mainzer Seminarberichte in additiver Zahlentheorie 1 (1983), 53–71.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BERGEN
N-5000 BERGEN
NORWAY