MATH. SCAND. 59 (1986), 41–44

AN UPPER BOUND ON THE DOMINATION NUMBER OF A GRAPH

DÁNUȚ MARCU

Introduction.

Graphs, considered here, are *finite* and *simple* (without loops or multiple edges), and [1], [2] are followed for terminology and notation.

Let $G = (V, E)$ be an *undirected graph* with V the set of *vertices* and E the set of *edges*. A graph is said to be *complete*, if every two vertices of the graph are joined by an edge. We shall denote by K_n the complete graph on n vertices. The *complement* G^c of G is the graph with vertex set V, two vertices being adjacent in G^c if and only if they are not adjacent in G. For any vertex v of G, the *neighbour set* of v is the set of all vertices adjacent to v; this set is denoted by $N(v)$. A vertex is said to be an isolated vertex, if its neighbour set is empty. A set of vertices in a graph is said to be a *dominating set*, if every vertex not in the set is adjacent to one or more vertices in the set. The *domination number* $\beta(G)$ of G is the size of the smallest dominating set. The well known upper bound for $\beta(G)$ is due to V. G. Vizing [1], [4] and is as follows:

$$\beta(G) \leqq n+1-\sqrt{1+2m},$$

where $n = |V|$ and $m = |E|$. But, if $\beta(G) > 2$, this bound can be attained only for graphs having at least an isolated vertex. In [3], we have suggested an upper bound for $\beta(G)$, which can be attained for graphs with no isolated vertices and having $\beta(G) > 2$. More exactly, we have proved that for a simple graph $G = (V, E)$ without isolated vertices and for which $\beta(G) > 2$, we have

$$\beta(G) \leqq \lceil (n+1-\delta)/2 \rceil,$$

where

$$\delta = \min_{v \in V} |N(v)|,$$

Received June 24, 1985.

and $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x. Our aim, in this note, is to suggest another upper bound for $\beta(G)$, when $\beta(G) \geqq 2$.

The main result.

In the sequel, we shall denote

$$\Delta = \max_{v \in V} |N(v)|,$$

and for any real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

LEMMA.

$$\beta(G^c) \leqq \lfloor \delta(\Delta - 1)/(n-1) \rfloor + 2.$$

PROOF. Obviously, if G contains at least an isolated vertex, then $\delta = 0$, $\beta(G^c) = 1$ and the lemma is proved. So, suppose that G does not contain isolated vertices.

Let $v \in V$ be, such that $|N(v)| = \delta$, and $W = V - (N(v) \cup \{v\})$. If W is empty, then, by the choice of v, we must have $|N(u)| \geqq \delta$ for each $u \in N(v)$, that is, $G = K_n$. Thus, $\delta = \Delta = n - 1$, and every vertex of G^c is isolated.

Consequently, $\beta(G^c) = n$, and the lemma is proved.

Let then $|W| \geqq 1$. Let $u \in N(v)$ and $D = N(v) \cap N(u)$.

Clearly, $D \cup \{v\} \cup \{u\}$ is a dominating set of G^c, that is,

$$\beta(G^c) \leqq 2 + |D|. \tag{1}$$

On the other hand, we have $W \cap N(u) = N(u) - (D \cup \{v\})$, that is,

$$|W \cap N(u)| \leqq \Delta - |D| - 1. \tag{2}$$

Hence, from (1) and (2), we obtain

$$|W \cap N(u)| \leqq \Delta + 1 - \beta(G^c), \quad \text{for each } u \in N(v). \tag{3}$$

Let $w \in W$ and $\tilde{D} = N(v) \cap N(w)$. Obviously, $\tilde{D} \cup \{v\} \cup \{w\}$ is a dominating set of G^c, that is,

$$\beta(G^c) - 2 \leqq |N(v) \cap N(w)| \quad \text{for each } w \in W. \tag{4}$$

From (4), it follows that

$$|W|\,[\beta(G^c) - 2] \leqq \sum_{w \in W} N(v) \cap N(w). \tag{5}$$

But,

$$\sum_{w \in W} |N(v) \cup N(w)| = \sum_{u \in N(v)} |W \cap N(u)|. \tag{6}$$

Hence, from (3), (5), and (6), we obtain

$$|W|\,[\beta(G^c) - 2] \leqq [\Delta + 1 - \beta(G^c)]\,|N(v)|$$

or

$$(n - \delta - 1)\,[\beta(G^c) - 2] \leqq \delta[\Delta + 1 - \beta(G^c)]$$

implying

$$\beta(G^c) \leqq \delta(\Delta - 1)/(n - 1) + 2,$$

that is,

$$\beta(G^c) \leqq \lfloor \delta(\Delta - 1)/(n - 1) \rfloor + 2.$$

THEOREM.

$$\beta(G) \leqq \lfloor (n - \Delta - 1)(n - \delta - 2)/(n - 1) \rfloor + 2.$$

PROOF. The theorem follows from the lemma, since $(G^c)^c = G$.

EXAMPLE. Let us consider the graph $G = K_1 + K_{1,4}$.
It is easy to see that $\beta(G) = 2$. For this graph, our upper bound gives the correct value, whereas Vizing's bound is larger.

ACKNOWLEDGEMENT. I wish to thank the referee for his helpful comments, kindness and interest concerning this paper.

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FACULTY OF MATHEMATICS
UNIVERSITY OF BUCHAREST
ACADEMIEI 14
70109-BUCHAREST
ROMANIA