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AUTOMORPHISM GROUPS OF COMPACT KLEIN SURFACES WITH ONE BOUNDARY COMPONENT

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Abstract.

We obtain in this paper the minimum genus of a compact Klein surface with one boundary component, which has a given group of automorphisms, and we give upper bounds depending of the genus for the order of the groups of automorphisms.

1. Introduction.

In this paper we study the automorphism groups of compact Klein surfaces with one boundary component [see 1].

In section 2 we give some results on groups of automorphisms of Klein surfaces. In section 3 we characterize the existing epimorphism from a non-Euclidean crystallographic (N.E.C.) group onto a cyclic group having as kernel the group of a surface. In section 4 we obtain the minimum genus of a compact Klein surface with one boundary component, which has a given group of automorphisms and we get Klein surfaces reaching the minimum genus. In section 5 we give upper bounds depending of the genus for the order of the groups of automorphisms and we get groups of automorphisms reaching the bounds.

2. Compact Klein surfaces and N.E.C. groups.

By an N.E.C. group [see 10], we shall mean a discrete subgroup Γ of the group of isometries of the non-Euclidean plane $D = \mathbf{C}^{+}$ with compact quotient space D/Γ, including isometries that reverse orientation, reflections and glide-reflections.

N.E.C. groups are classified according to their signature (see § 3 of [5]). The signature of an N.E.C. group Γ has the form

$$(*) \qquad (g, \pm, [m_1, \ldots, m_\tau], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

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If Γ has this signature, then D/Γ is a compact surface of genus g with k holes; it is orientable if the sign is "+", and non-orientable if the sign is "−". The integers $m_1,\ldots,m_\tau$, are called the periods and represent the branching over interior points of D/Γ in the natural projection from D onto D/Γ. The brackets $(n_{i1},\ldots,n_{is_i})$ are the period-cycles and the integers $n_{i1},\ldots,n_{is_i}$ represent the branching around the ith hole.

The group Γ with signature (*) has the presentation given by generators x_i $(i=1,\ldots,\tau)$, e_i $(i=1,\ldots,k)$, c_{ij} $(i=1,\ldots,k,\ j=0,\ldots,s_i)$, a_j, b_j (if sign "+"), d_j (if sign "−") $(j=1,\ldots,g)$, and relations

$$x_i^{m_i}=1\ (i=1,\ldots,\tau),\quad c_{is_i}=e_i^{-1}c_{i0}e_i\ \ (i=1,\ldots,k),$$

$$c_{i,j-1}^2=c_{ij}^2=(c_{i,j-1}c_{ij})^{n_{ij}}=1\ \ (i=1,\ldots,k, j=0,\ldots,s_i),$$

$$x_1\ldots x_\tau e_1\ldots e_k a_1 b_1 a_1^{-1} b_1^{-1}\ldots a_g b_g a_g^{-1} b_g^{-1}=1\ \text{(if sign "+")},$$

$$x_1\ldots x_\tau e_1\ldots e_k d_1^2\ldots d_g^2=1\ \text{(if sign "−")}.$$

From now on, we will denote by x_i, e_i, c_{ij}, a_j, b_j, d_j the above generators associated with an N. E. C. group.

(2.1). Definition. We shall say that an N.E.C. group Γ is the group of a surface with one boundary component if it has signature $(g,\pm,[-],\{(-)\})$, "+" for orientable, and "−" for non-orientable surfaces.

From now on, we will abreviate by K.S. a compact Klein surface with one boundary component.

(2.2). Theorem (Preston [8]). *Let X be a* K.S. *of algebraic genus greater or equal than 2. Then X can be represented in the form D/Γ, where Γ is the group of a surface with one boundary component.*

(2.3). Theorem (May [6]). *Let Γ be the group of a surface with one boundary component; then G is a group of automorphisms of the Klein surface D/Γ if and only if $G\simeq\Gamma'/\Gamma$, where Γ' is an* N.E.C. *group such that $\Gamma\lhd\Gamma'$.*

If Γ is an N.E.C. group with signature (*), then Γ has associated a fundamental region of the non-euclidean plane (see § 6 of [10]) whose area is

$$2\pi\Big(\alpha g+k-2+\sum_{i=1}^{\tau}(1-1/m_i)+\tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1-1/n_{ij})\Big),$$

α being 2 for sign "+" and 1 for sign "−" (see Theorem 1 of [9]), that we denote $|\Gamma|$. Moreover, if $G\simeq\Gamma'/\Gamma$, then $\operatorname{order}(G)=|\Gamma|/|\Gamma'|$.

(2.4). Theorem. *Let G be an automorphism group of a* K.S. *Then G is cyclic if the order of G is odd, and G is cyclic or dihedral if its order is even.*

PROOF. If G is a group of automorphisms of a K.S. and C is the boundary component, then each $g \in G$ must map C onto itself. Thus G induces a finite group of automorphisms of the circle, which must be cyclic or dihedral. This group is an isomorphic copy of G, since, by standard methods of analytic continuation, any element of G which fixes the whole boundary component must be the identity.

As it will be seen in section 4, given a dihedral or cyclic group, there is a K.S. which has that group as an automorphism group.

3. Homomorphism of a surface.

(3.1). DEFINITION. A homomorphism θ of an N.E.C. group Γ in a finite group G is an orientable surface homomorphism (o. s. h.), if θ is an epimorphism and $\ker\theta$ is the group of an orientable surface with one boundary component.

(3.2). DEFINITION. A homomorphism θ of an N.E.C. group Γ in a finite group G is a non-orientable surface homomorphism (n.s.h.) if θ is an epimorphism and $\ker\theta$ is the group of a non-orientable surface with one boundary component.

It follows immediately from section 2 that G is an automorphism group of a K.S. if and only if there are an N.E.C. group Γ and a homomorphism $\theta\colon \Gamma \to G$ which is an o. s. h. or an n. s. h.

(3.3). PROPOSITION. *Let G be a cyclic group of order greater than 2, and Γ an* N.E.C. *group. The existence of either an* o.s.h. *or an* n.s.h. *from Γ in G implies that the signature of Γ has only one period-cycle which is empty if the order of G is odd. If the order is even, then all period-cycles are empty.*

PROOF. If the order of G is odd, from (2.3) of [2] the signature of Γ has only one empty period-cycle.

If the order of G is even and there is either an o.s.h. or an n.s.h. $\theta\colon \Gamma \to G = \mathsf{Z}/(N)$, we will show that if we assume the existence of some non-empty period-cycle, we will get a contradiction. Let $(n_1, n_2, \ldots, n_s)$ be a non-empty period-cycle belonging to Γ, let $c_0, c_1, \ldots, c_s$ be the generating reflections of the period-cycle and let e be the element relating c_0 with c_s. Then the following relations hold in Γ:

$$e^{-1} c_0 e c_s = 1,$$
$$c_0^2 = c_1^2 = c_2^2 = \ldots = c_s^2 = 1,$$
$$(c_0 c_1)^{n_1} = \ldots = (c_{s-1} c_s)^{n_s} = 1.$$

There are two cases:

Case 1. Suppose that the reflections $c_0, c_1, \ldots, c_s$ do not belong to $\ker\theta$. Then, as N is even and θ is a homomorphism, necessarily

$$\theta(c_0) = \theta(c_1) = \ldots = \theta(c_s) = \overline{N/2};$$

so $c_0 c_1, c_1 c_2, \ldots, c_{s-1} c_s$ belong to $\ker\theta$, and as

$$(c_0 c_1)^{n_1} = \ldots = (c_{s-1} c_s)^{n_s} = 1,$$

by (2.1) of [3], there will be periods in the signature of $\ker\theta$, and $\ker\theta$ is not the group of a surface with one boundary component. Therefore θ is neither o. s. h. nor n. s. h.

Case 2. Suppose that at least one of the reflections $c_0, c_1, \ldots, c_s$ belongs to $\ker\theta$. Let $c_j \in \ker\theta$.

We can distinguish several subcases:

i) Let $j \neq 0 \neq s \neq j$. Then we have that c_{j-1} and $c_{j+1} \notin \ker\theta$, since if they belonged to $\ker\theta$, by (2.2) of [3] there would exist a period-cycle with the element n_j or n_{j+1} in $\ker\theta$.

On the other hand, $n_j = 2$, since if n_j is odd, we should have that $c_{j-1} \in \ker\theta$, as $(c_{j-1} c_j)^{n_j} = 1$, and if $n_j \neq 2$ and even, the value $n_j/2$ should appear among the period-cycles of $\ker\theta$ associated to the reflections $c_j, c_{j-1} c_j c_{j-1}$. In the same way, we prove that $n_{j+1} = 2$.

Let q be the least number for which $(c_{j-1} c_j)^q \in \ker\theta$. We consider

$$\Gamma/\ker\theta = \{\beta_1 \ker\theta, \beta_2 \ker\theta, \ldots, \beta_{2q} \ker\theta, \ldots, \beta_N \ker\theta\},$$

where

$$\beta_1 = c_{j+1}, \ \beta_2 = c_{j-1} c_{j+1}, \ \beta_3 = c_{j-1} c_{j+1} c_{j-1}, \ldots, \ \beta_{2q} = (c_{j-1} c_{j+1})^q.$$

From §4 of [5] and §6 of [10], we get that given Γ there is a fundamental region for Γ which is a polygon $P = \gamma_{j-1} \gamma_j \gamma_{j+1} A$, where A denotes the other sides of the polygon.

Moreover, c_k is the element of Γ fixing the side γ_k (where $k = j-1$, j, $j+1$). So a fundamental region for $\ker\theta$ is

$$P' = \beta_1 P \cup \ldots \cup \beta_{2q} P \cup \ldots \cup \beta_N P.$$

Now, it is not difficult to show that the sides of P'

$$\beta_1(\gamma_j), \beta_2(\gamma_j), \ldots, \beta_{2q}(\gamma_j)$$

generate a hole in the surface obtained if we consider the identification produced in the fundamental region P' by the equivalence relation given by $\ker\theta$.

Then in $P'/\ker\theta$ there will be $N/2q$ holes. As we can only have one hole in $P'/\ker\theta$, we get $N = 2q$ and $\Gamma/\ker\theta \simeq D_{N/2}$, and this is impossible since $\Gamma/\ker\theta \simeq Z/(N)$.

ii) If $j = 0$ or s, and $s \neq 0$, then, as in the above subcase, we have that c_{s-1} and c_1 are not in $\ker\theta$, and $n_1 = n_s = 2$.

Let k be the least natural number for which $e^k \in \ker\theta$ and let Γ be the least natural number for which $(e^{k-1} c_s e^{-k+1} c_1)^r \in \ker\theta$ (let $e^0 = 1$). Then we can consider

$$\Gamma/\ker\theta = \{\beta_1 \ker\theta,\ \beta_2 \ker\theta, \ldots, \beta_{2r} \ker\theta, \ldots, \beta_N \ker\theta\},$$

where

$$\beta_1 = h_2,\ \beta_2 = h_2 h_1,\ \beta_3 = h_2 h_1 h_2, \ldots,\ \beta_{2r} = (h_2 h_1)^r, \quad \text{and}$$

$$h_1 = e^{k-1} c_{s-1} e^{-k+1},\ h_2 = e^{k-1} c_{s-1} e^{-k+1} c_1 e^{k-1} c_{s-1} e^{-k+1}.$$

A fundamental region for $\ker\theta$ is

$$P' = \beta_1 P \cup \ldots \cup \beta_{2r} P \cup \ldots \cup \beta_N P,$$

where P is a fundamental region for Γ, which is a polygon

$$\varepsilon\gamma_0 \gamma_1 C \gamma_{s-1} \gamma_s \varepsilon' D$$

where C and D are the other sides of the polygon, and e is the element of Γ transforming ε' into ε. So, after a lengthy and straightforward calculation, we have that the sides of P'

$$\beta_1(\gamma_0),\ \beta_1(\gamma_s),\ \beta_2(\gamma_0),\ \beta_2(\gamma_s), \ldots,\ \beta_{2r}(\gamma_0),\ \beta_{2r}(\gamma_s)$$

generate a hole in the surface obtained if we consider the identification produced in P' by $\ker\theta$. As we can only have one hole in $P'/\ker\theta$, we have $N = 2r$, and $\Gamma/\ker\theta \simeq D_{N/2}$, impossible since $\Gamma/\ker\theta \simeq Z/(N)$. The remaining possibility after i) and ii) is $s = j = 0$. Then the period-cycle of Γ we are considering is empty.

Finally, from 1) and 2), it results that the period-cycles of Γ have to be empty.

(3.4). **Proposition.** *Let $G \simeq Z/(N)$.*

a) *If N is odd and Γ is an* N.E.C. *group of signature* $(g,+,[m_1,\dots,m_\tau],\{(-)\})$, *then there is no* n.s.h. *from Γ onto G.*

b) *If Γ is an* N.E.C. *group of signature* $(g,-,[m_1,\dots,m_\tau],\{(-)_{i=1,\dots,k}\})$, *then there is no* o.s.h. *from Γ onto G.*

Proof. If N is odd there is a homomorphism $\theta:\Gamma \to G$, θ being either an o.s.h. or an n.s.h. So $G \simeq \Gamma/\ker\theta$; as $\ker\theta \lhd \Gamma$, by (2.4) of [2], $\ker\theta$ and Γ will have the same sign in their signatures, and hence in case a), there is no n.s.h. and in case b) there is no o.s.h.

We will show that b) also holds if N is even. In fact, if N is even, there is an $i \in \{1,\dots,k\}$ such that $\theta(e_i)=\overline{q_i}$, $q_i < N$ and q_i prime with N, since if there is no $i \in \{1,\dots,k\}$ with $\theta(e_i)$ verifying the above conditions, then by (2.3) of [3] there would appear more than one period-cycle in the signature of $\ker\theta$.

Given this e_i, there is an r_i such that $\theta(e_i^{r_i})=\overline{1}$.

Let $\theta(d_1)=\overline{s}$. If $\overline{s}=\overline{N}$, then the glide-reflection d_1 belongs to $\ker\theta$, so the signature of $\ker\theta$ will have the sign "$-$". If $\overline{s}\neq\overline{N}$, then $\theta(e_i^{r_i(N-s)}d_1) = \overline{N}$, therefore $e_i^{r_i(N-s)}d_1$ is a glide-reflection belonging to $\ker\theta$, so the signature of $\ker\theta$ will have the sign "$-$".

In both cases there is no o.s.h. from Γ onto G.

(3.5). **Proposition.** *Let $G \simeq Z/(N)$, and let Γ be an* N.E.C. *group of signature* $(g,+,[m_1,\dots,m_\tau],\{(-)_{i=1,\dots,k}\})$. *Then there exists an* o.s.h. *$\theta:\Gamma\to G$ if and only if*

i) $k=1$.

ii) l.c.m. $(m_1,\dots,m_\tau)=N$.

iii) *If N is even, the number of periods divisible by the maximum power of* 2 *dividing N is odd.*

Proof. First we will show the conditions are necessary.

Assume that there is an o.s.h. $\theta:\Gamma\to G$. Then:

i) If N is odd, by (3.3) $k=1$. We will see that the case N even and $k>1$ is impossible.

As θ is an o.s.h., $\ker\theta$ has exactly one period-cycle in its signature, and by (2.3) of [3] there is only one reflection of the generators of Γ in $\ker\theta$, and the remaining reflections of the generators of Γ are such that their squares belong to $\ker\theta$.

Let $c_j \in \ker\theta$. Then necessarily $\theta(c_j)=\overline{N}$ and $\theta(c_i)=\overline{N/2}$ for all $i\in\{1,\dots,k\}$, $i\neq j$.

Let e_j be the element of the generators of Γ associated with the period-cycle of c_j. Then $\theta(e_j) = \overline{q_j}$, $q_j < N$ and prime with N, since otherwise, by (2.3) of [3], there would be more than one period-cycle in the signature of $\ker\theta$. Therefore there is an r_j such that $\theta(e_j^{r_j}) = \overline{1}$.

Let us consider the elements $e_j^{r_j N/2} c_i$ with $i \neq j$. These elements are glide-reflections belonging to $\ker\theta$, because $\theta(e_j^{r_j N/2} c_i) = \overline{N}$. Therefore the signature of $\ker\theta$ will have the sign "$-$", and hence θ is not an o.s.h.

ii) As θ is an o.s.h., $\ker\theta$ has no periods in its signature, so $\theta(x_i)$ must have order m_i modulo $\ker\theta$.

By i), $k = 1$, and by (2.3) of [3], $\theta(e_1)$ must have order N modulo $\ker\theta$, since otherwise there would be more than one period-cycle in the signature of $\ker\theta$. As $x_1 x_2 \dots x_\tau e_1 = 1$, $\theta(x_1 x_2 \dots x_\tau e_1) = \overline{N}$. So $\theta(e_1)$ divides l.c.m. $(m_1, \dots, m_\tau)$; and as $m_i | N$ for all $i = 1, \dots, \tau$, l.c.m. $(m_1, \dots, m_\tau) = N$.

iii) $\ker\theta$ is the group of an orientable surface and $\Gamma/\ker\theta \simeq \mathsf{Z}/(N)$; so from the formulas of the areas of the fundamental regions of Γ and $\ker\theta$ (see section 2), we have that $|\ker\theta|/|\Gamma| = N$, and so

$$\frac{2g'-1}{N} = 2g - 1 + \sum_{i=1}^{\tau} (1 - 1/m_i)$$

(where g' is the genus of $\ker\theta$) and

$$2g' = 1 + 2gN - N + \sum_{i=1}^{\tau} (N - N/m_i).$$

Therefore iii) holds.

The conditions are sufficient: Suppose that Γ has i), ii), and iii). We define $\theta : \Gamma \to \mathsf{Z}/(N)$ in the following way: $\theta(c_1) = \overline{N}$, $\theta(a_j) = \overline{N}$, $\theta(b_j) = \overline{N}$, for $j = 1, \dots, g$. By b) of Theorem 4 of [4] we are able to give $\theta(x_i)$, $i = 1, \dots, \tau$ and $\theta(e_1)$ values such that $\theta(x_1 x_2 \dots x_\tau e_1) = \overline{N}$, $\theta(x_i)$ has order m_i for $i = 1, \dots, \tau$, and $\theta(e_1)$ has order N.

By [2] and [3], $\ker\theta$ is the group of an orientable surface with one boundary component, and so θ is an o.s.h.

(3.6). **Proposition.** *Let $G \simeq \mathsf{Z}/(N)$, and let Γ be an* N.E.C. *group with signature $(g, -, [m_1, \dots, m_\tau], \{(-)_{i=1,\dots,k}\})$. Then there exists an* n.s.h. *$\theta : \Gamma \to G$ if and only if*

i) $m_i | N$, $i = 1, \dots, \tau$.
ii) *If Γ has sign "+" then N is even and $k > 1$.*
iii) *$k = 1$ if N is odd and Γ has sign "$-$".*

iv) *If N is even, Γ has sign "$-$", and $k=1$, then the number of periods divisible by the maximum power of 2 dividing N is odd, and there exists an $i \in 1,\ldots,\tau$ such that $m_i \nmid N/2$.*

PROOF. Let us suppose that there exists an n.s.h. $\theta: \Gamma \to G$. Then:

i) As θ is an n.s.h., $\ker\theta$ has no periods in its signature, and $\theta(x_i)$ must have order m_i modulo $\ker\theta$; so $m_i | N$ for all $i=1,\ldots,\tau$.

ii) If Γ has sign "$+$", then by (3.4) N cannot be odd. If N is even and $k=1$, by (2.2) of [3], the unique reflection among the generators of Γ must belong to $\ker\theta$, since $\ker\theta$ has to have a period-cycle. So $\ker\theta$ has no glide-reflections among its generators, and so $\ker\theta$ has sign "$+$" in its signature. In this case θ cannot be an n.s.h.

iii) Immediate from (3.3).

iv) If N is even, Γ has sign "$-$" and $k=1$, then one of the relations appearing among the generators of Γ is $d_1^2 d_2^2 \ldots d_g^2 x_1 \ldots x_\tau e_1 = 1$, and $\theta(d_1^2 d_2^2 \ldots d_g^2 x_1 \ldots x_\tau e_1) = \bar{N}$. As $k=1$, by (2.2) of [3] the unique reflection c_1 among the generators of Γ has to belong to $\ker\theta$, and the element e_1 of the generators of Γ associated with c_1 must have order N modulo $\ker\theta$, since otherwise there would be more than one period-cycle in the signature of $\ker\theta$. So $\theta(e_1)$ has order N.

Now $\theta(d_1^2 d_2^2 \ldots d_g^2 x_1 \ldots x_\tau e_1) = \bar{N}$ and $\theta(e_1)$ has order N; then the number of periods divisible by the maximum power of 2 dividing N is odd.

Consider $\theta(d_i^2) = \overline{2q_i}$, $i=1,\ldots,g$. The order of $\theta(d_i^2)$ in $Z/(N)$ is an integer p_i such that $p_i | N/2$, and as $\theta(e_i)$ has order N, there is an m_i such that $m_i \nmid N/2$, since $N | \text{l.c.m.}(m_1,\ldots,m_\tau,p_1,\ldots,p_g)$ and $\theta(d_1^2 d_2^2 \ldots d_g^2 x_1 \ldots x_\tau e_1) = \bar{N}$.

If Γ verifies i)–iv) we define the homomorphism $\theta: \Gamma \to Z/(N)$ in the following way: If Γ has sign "$+$",

$$\theta(e_1) = \bar{1}, \qquad \theta(e_2) = \overline{-\left(1 + \sum_{i=1}^{\tau} N/m_i\right)}, \qquad \theta(e_i) = \bar{N} \quad \text{for } i > 2,$$

$$\theta(x_i) = \overline{N/m_i}, \qquad \theta(a_i) = \bar{N}, \qquad \theta(b_i) = \bar{N},$$

$$\theta(c_1) = \bar{N}, \qquad \theta(c_i) = \overline{N/2} \qquad \text{for } i > 1.$$

If Γ has sign "$-$" and $k>1$, we define $\theta(e_i)$, $\theta(x_i)$, and $\theta(c_i)$ in the same way as above, and $\theta(d_i) = \bar{N}$, $i=1,\ldots,g$.

If Γ has sign "$-$" and $k=1$, we define $\theta(c_1)=\bar{N}$, $\theta(d_i)=\bar{N}$ for $i=2,\dots,g$. From b) of Theorem 4 of [4], we are able to give values to $\theta(x_i)$, $i=1,\dots,\tau$, $\theta(d_1)$ and $\theta(e_1)$, such that $\theta(d_1^2 x_1 \dots x_\tau e_1)=\bar{N}$, $\theta(x_i)$ having order m_i, $\theta(e_1)$ having order N, and $\theta(d_1^2)$ having order $N/2$. By [3], $\ker\theta$ is the group of a non-orientable surface with one boundary component, and therefore θ is an n.s.h.

4. Minimum genus.

(4.1). THEOREM. *Let* $G \simeq Z/(N)$, $N=q_1^{\beta_1}q_2^{\beta_2}\dots q_r^{\beta_r}$, $q_1<q_2<\dots<q_r$ *primes. Then the minimum genus* g' *of a* K.S. X *with an automorphism group isomorphic to* G *depends of the prime factors of* N, *and its value is*:

$$g'=\frac{N}{2}-\frac{N}{2q_1}-\frac{q_1-1}{2} \quad \text{if } \beta_1=1,\quad \beta_2\neq 0,\quad X \text{ orientable},$$

$$g'=\frac{q_1-1}{2} \quad \text{if } N=q_1,\quad q_1\neq 2,\quad X \text{ orientable},$$

$$g'=1 \quad \text{if } N=2,\quad X \text{ orientable},$$

$$g'=\frac{N}{2}-\frac{N}{2q_1} \quad \text{if } \beta_1\neq 1,\quad X \text{ orientable},$$

$$g'=N-\frac{N}{q_1}+1 \quad \text{if } X \text{ is non-orientable}.$$

PROOF. If X is orientable, $\beta_1=1$ and $\beta_2\neq 0$, we consider an N.E.C. group Γ' of signature $(0,+,[q_1,N/q_1],\{(-)\})$; this group fulfils the condition of (3.5); so there exists an orientable K.S. $X=D/\Gamma$ of genus g' with an automorphism group Γ'/Γ isomorphic to G. Then, by section 2,

$$\frac{|\Gamma|}{|\Gamma'|}=\text{order}\,(G)=N\,,$$

and so

$$\frac{2g'-1}{N}=1-\frac{1}{q_1}-\frac{q_1}{N}$$

implies

$$g'=\frac{N}{2}-\frac{N}{2q_1}-\frac{q_1-1}{2}.$$

Now, let us see that g' is the minimum genus. If we take any other N.E.C. group Γ with the condition of (3.5), then

$$\frac{2g'-1}{N} = 2g-1+\sum_{i=1}^{\tau}(1-1/m_i)$$

implies

$$g' = \tfrac{1}{2}+(2g-1)\frac{N}{2}+\frac{N}{2}\sum_{i=1}^{\tau}(1-1/m_i),$$

and the only possible cases with genus less than the above take place when $g=0$ and $\sum_{i=1}^{\tau}(1-1/m_i)<2$, so $\tau \leqq 3$, but as Γ is an N.E.C. group $\tau \leqq 2$ and so we have that only the following cases can hold: $\tau = 2$ and $\tau = 3$. If $\tau = 2$ or $\tau = 3$ ever, the minimum genus

$$g' \geqq \frac{N}{2}-\frac{N}{2q_1}-\frac{q_1-1}{2}$$

(this is immediate from Lemmas 1 and 2 of [4]).

If X is orientable, $N = q_1$ and $q_1 \neq 2$, we consider an N.E.C. group of signature $(0,+,[q_1,q_1],\{(-)\})$; this group fulfils the conditions of (3.5); therefore there exists an automorphism group isomorphic to G and such that

$$\frac{2g'-1}{q_1} = 1-\frac{2}{q_1},$$

which implies

$$g' = \frac{q_1-1}{2}.$$

If X is orientable, $N = 2$, we consider an N.E.C. group of signature $(0,+,[2,2,2],\{(-)\})$; this group fulfils the conditions of (3.5); therefore there exists an automorphism group isomorphic to G and such that

$$\frac{2g'-1}{2} = -1+\frac{3}{2},$$

which implies

$$g' = 1.$$

If X is orientable, $\beta_1 \neq 1$, we consider an N.E.C. group of signature $(0,+,[N,2],\{(-)\})$; this group fulfils the conditions of (3.5) and so there exists an automorphism group isomorphic to G and such that

$$\frac{2g'-1}{N} = 1 - \frac{1}{N} - \frac{1}{q_1},$$

which implies

$$g' = \frac{N}{2} - \frac{N}{2q_1}.$$

If X is non-orientable and $q_1 \neq 2$, we consider an N.E.C. group of signature $(1,-,[q_1],\{(-)\})$; this group fulfils the conditions of (3.6) and therefore there exists a non-orientable K.S. X of genus g' with an automorphism group isomorphic to G and such that

$$\frac{g'-1}{N} = 1 - \frac{1}{q_1},$$

which implies

$$g' = N - \frac{N}{q_1} + 1.$$

If X is non-orientable and $q_1 = 2$, we consider an N.E.C. group of signature $(0,+,[2],\{(-)(-)\})$; this group fulfils the conditions of (3.6) and therefore there exists a non-orientable K.S. X of genus g' with an automorphism group isomorphic to G and such that $(g'-1)/N = \frac{1}{2}$; so $g' = N/2+1$.

In any case, operating in the same way as before, we get that those are the minimum genus.

(4.2). THEOREM. *Let* $G \simeq D_N$, $N = q_1^{\beta_1} q_2^{\beta_2} \dots q_r^{\beta_r}$, $q_1 < q_2 < \dots < q_r$ *primes. Then the minimum genus of a* K.S. X *with an automorphism group isomorphic to G is the same as the one of the above theorem* (4.1).

PROOF. If D_N is an automorphism group of a K.S. X, then $Z/(N)$ is an automorphism group of X; so the minimum genus of a K.S. with an automorphism group isomorphic to D_N is greater or equal than the minimum genus of a K.S. with an automorphism group isomorphic to $Z/(N)$. Now we will see that the minimum genus is the same in every case.

To prove it, we will find a series of N.E.C. groups such that for each group of the series there is either an o.s.h. or an n.s.h. in the conditions that we wish.

If X is orientable, $\beta_1 = 1$ and $\beta_2 \neq 0$, we consider an N.E.C. group Γ of signature $(0,+,[-],\{(2,q_1,N/q_1,2)\})$, and we define

$$\theta: \Gamma \to G = \langle x,y \mid x^2 = y^2 = (xy)^N = 1\rangle$$

in the following way:

$$\theta(c_1) = 1, \quad \theta(c_2) = x, \quad \theta(c_3) = y(xy)^{(N/q_1)-1},$$
$$\theta(c_4) = y(xy)^{(N/q_1)-1}(xy)^{q_1}, \quad \theta(c_5) = 1.$$

Hence $\ker\theta$ is a normal N.E.C. subgroup of Γ. From a fundamental region of Γ, we get one for $\ker\theta$, and from the last one we have that $\ker\theta$ is the group of an orientable K.S. Moreover, $\Gamma/\ker\theta \simeq D_N$, where $\Gamma/\ker\theta$ is an automorphism group of the orientable K.S. $D/\ker\theta$, and the genus g' of $D/\ker\theta$ is given by

$$\frac{2g'-1}{2N} = \frac{1}{2} - \frac{1}{2q_1} - \frac{q_1}{2N}; \quad \text{so} \quad g' = \frac{N}{2} - \frac{N}{q_1} - \frac{q_1-1}{2}.$$

Following the same method as in the above case, we get the following results:

If X is orientable, $N = q_1$ and $q_1 \neq 2$, we consider an N.E.C. group of signature $(0,+,[q_1],\{(2,2)\})$, and we define $\theta: \Gamma \to G$ in the following way:

$$\theta(x_1) = yx,\ \theta(e_1) = xy,\ \theta(c_1) = 1,\ \theta(c_2) = x,\ \theta(c_3) = 1.$$

In this case $D/\ker\theta$ is an orientable K.S. of genus g' given by

$$\frac{2g'-1}{2q_1} = \frac{1}{2} - \frac{1}{q_1}, \quad \text{and so} \quad g' = \frac{q_1-1}{2}.$$

If X is orientable and $N = 2$, we consider an N.E.C. group Γ of signature $(0,+,[-],\{(2,2,2,2,2)\})$, and we define $\theta: \Gamma \to G$ in the following way:

$$\theta(c_1) = y,\ \theta(c_2) = 1,\ \theta(c_3) = x,\ \theta(c_4) = y,\ \theta(c_5) = x,\ \theta(c_6) = y.$$

In this case $D/\ker\theta$ is an orientable K.S. of genus g' given by

$$\frac{2g'-1}{4} = \frac{1}{4} \quad \text{and so} \quad g' = 1.$$

If X is orientable, $\beta_1 \neq 1$, we consider an N.E.C. group Γ of signature $(0,+,[-],\{(2,q_1,N,2)\})$, and we define $\theta\colon \Gamma \to G$ in the following way:

$$\theta(c_1)=1,\ \theta(c_2)=x,\ \theta(c_3)=y(xy)^{(N/q_1)-1},$$
$$\theta(c_4)=y(xy)^{(N/q_1)-2},\ \theta(c_5)=1.$$

In this case $D/\ker\theta$ is an orientable K.S. of genus g' given by

$$\frac{2g'-1}{2N}=\frac{1}{2}-\frac{1}{2q_1}-\frac{1}{2N},\quad \text{and so}\quad g'=\frac{N}{2}-\frac{N}{q_1}.$$

If X is non-orientable and $q_1=2$, we consider an N.E.C. group Γ of signature $(0,+,[-],\{(2,2,2,2,2)\})$, and we define $\theta\colon \Gamma\to G$ in the following way:

$$\theta(c_1)=y,\quad \theta(c_2)=1,\quad \theta(c_3)=x,\quad \theta(c_4)=x(yx)^{N/2},$$
$$\theta(c_5)=(xy)^{N/2},\quad \theta(c_6)=y.$$

In this case $D/\ker\theta$ is a non-orientable K.S. of genus g' given by

$$\frac{g'-1}{2N}=\frac{1}{4},\quad \text{and so}\quad g'=\frac{N+2}{2}.$$

If X is non-orientable and $q_1\neq 2$, we consider an N.E.C. group Γ of signature $(0,+,[2],\{(2,2,q_1)\})$, and we define $\theta\colon\Gamma\to G$ in the following way:

$$\theta(x_1)=x(yx)^{((N/q_1)-1)/2},\ \theta(c_1)=y,\ \theta(c_2)=1,\ \theta(c_3)=x,$$
$$\theta(c_4)=x(yx)^{N/q_1},\ \theta(e_1)=x(yx)^{((N/q_1)-1)/2}.$$

In this case $D/\ker\theta$ is a non-orientable K.S. of genus g' given by

$$\frac{g'-1}{2N}=\frac{1}{2}-\frac{1}{2q_1},\quad \text{and so}\quad g'=N-\frac{N}{q_1}+1.$$

5. Upper bounds for the order of the groups of automorphisms.

Given an orientable K.S. X of genus g' and with an automorphism group isomorphic to $\mathbb{Z}/(N)$, we have by (4.1) that $g'\geqq (N-2)/4$, and so $4g'+2\geqq N$.

Moreover, for each $g'\geqq 1$ in (4.1) we have found an orientable K.S. with an automorphism group isomorphic to $\mathbb{Z}/(4g'+2)$. This upper bound had been found in terms of algebraic genus by May in Theorem 1 of [7].

Given a non-orientable K.S. X of genus g' and with an automorphism group isomorphic to $\mathsf{Z}/(N)$, we have by (4.1) that $g' \geqq N/2+1$, and so $2g'-2 \geqq N$.

In fact, for each $g' \geqq 2$, in (4.1) we have found a non-orientable K.S. with an automorphism group isomorphic to $\mathsf{Z}/(2g'-2)$.

Given an orientable K.S. X of genus g' and with an automorphism group isomorphic to D_N, we have by (4.2) that $g' \geqq (N-2)/4$ and $4g'+2 \geqq N$. Besides, for each $g' > 1$, we obtain in (4.2) an orientable K.S. with an automorphism group isomorphic to $D_{4g'+2}$.

Finally, given a non-orientable K.S. X of genus g' and with an automorphism group isomorphic to D_N, we have by (4.2) that $g' \geqq N/2+1$, and so $2g'-2 \geqq N$. For each $g' \geqq 2$, we have found in (4.2) a non-orientable K.S. with automorphism group isomorphic to $D_{2g'-2}$.

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