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ON THE DEGREE OF C^l-DETERMINACY

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1. Introduction.

In this paper, we obtain estimates for the degree of $C^l-\mathscr{G}$-determinacy ($\mathscr{G}=\mathscr{R}$, $\mathscr{C}$ or $\mathscr{K}$) of C^∞ map-germs f: $(\mathrm{R}^n,0) \to (\mathrm{R}^p,0)$ that satisfy a convenient Lojasiewicz condition.

These estimates generalize a result of Takens [4], and refine in many cases, the results of D. Lefebvre and M. T. Pourprix [2].

When applied to homogeneous germs, our results imply the following: (3.14) Corollary: "Let f: $(\mathrm{R}^n,0) \to (\mathrm{R}^p,0)$ be a C^∞ map-germ of corank k, given by

$$x = (x_1,x_2,\ldots,x_n) \mapsto (x_1,\ldots,x_{p-k},f_1(x),\ldots f_k(x)),$$

with f_j homogeneous of degree r_j.

For all l, $1 \leqq l < \infty$ and $r=\max r_j$, we have

(a) If 0 is an isolated singular point of f, then f is $(r+l-1)-C^l-\mathscr{R}$-determined.
(b) If $f^{-1}(0)=\{0\}$, then f is $(r+l-1)-C^l$-determined.
(c) If 0 is an isolated singularity in $f^{-1}(0)$, then f is $(r+l-1)-C^l-\mathscr{K}$-determined.

Furthermore, with the hypothesis of (a), (b) or (c), it follows respectively that small deformations of order r are $C^0-\mathscr{G}$-trivial, $\mathscr{G}=\mathscr{R}, \mathscr{C}$ or $\mathscr{K}$.

The above estimates are sharp in the following sense: if $f(x)$ is $(r+l_0-2)-C^{l_0}-\mathscr{G}$-determined for some $2 \leqq l_0 < \infty$, then f is in fact $C^\infty-\mathscr{G}$-determined by its $(r+l_0-1)$-jet.

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2. Notation and basic definitions.

Let $C(n,p)$ be the space of smooth map-germs f: $(\mathrm{R}^n,0) \to (\mathrm{R}^p,0)$.

We denote by $J^k(n,p)$ the set of k-jets of elements of $C(n,p)$.

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(2.1) DEFINITION. For any group $\mathcal{G}$ acting on $C(n,p)$, we say that f is k-$\mathcal{G}$-determined if the $\mathcal{G}$-orbit of f contains all germs g such that $j^k g(0) = j^k f(0)$.

In this work, we are interested in the groups $C^l - \mathcal{G}$, $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ *and* $\mathcal{K}$, defined below.

(2.2) DEFINITION. (a) The group $\mathcal{R}$ is the group of germs of C^∞-diffeomorphisms $(\mathsf{R}^n, 0) \to (\mathsf{R}^n, 0)$. $\mathcal{R}$ acts on $C(n,p)$ by composition on the right;

(b) $\mathcal{C}$ is the group of germs of diffeomorphisms H of $(\mathsf{R}^n \times \mathsf{R}^p, 0)$ which (i) leave fixed the projection on R^n, and (ii) preserve the subspace $\mathsf{R}^n \times \{0\}$. We may also describe $\mathcal{C}$ as the group of germs of families of diffeomorphisms of $(\mathsf{R}^p, 0)$ into itself, parametetrized by $(\mathsf{R}^n, 0)$. Thus any H in $\mathcal{C}$ is of the form $H(x,y) = (x, h(x,y))$, where $h(x,0) = 0$. $\mathcal{C}$ acts on f in $(\mathsf{R}^n \times \mathsf{R}^p, 0)$, by the formula

$$(\mathrm{id}_{\mathsf{R}^n}, H.f) = H \circ (\mathrm{id}_{\mathsf{R}^n}, f), \qquad \text{where } H.f = h(x, f(x)),$$

and $\mathrm{id}_{\mathsf{R}^n}$ denotes the identity map on $(\mathsf{R}^n, 0)$.

(c) $\mathcal{K}$ denotes the group of invertible map-germs

$$H\colon (\mathsf{R}^n \times \mathsf{R}^p, 0) \to (\mathsf{R}^n \times \mathsf{R}^p, 0),$$

which preserve the subspace $(\mathsf{R}^n \times 0)$, and such that there exists a map germ $h\colon (\mathsf{R}^n, 0) \to (\mathsf{R}^n, 0)$, which makes the diagram below commutative:

$$\begin{array}{ccccc}
(\mathsf{R}^n, 0) & \xrightarrow{i} & (\mathsf{R}^n \times \mathsf{R}^p, 0) & \xrightarrow{\pi} & (\mathsf{R}^n, 0) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle h} \\
(\mathsf{R}^n, 0) & \xrightarrow{i} & (\mathsf{R}^n \times \mathsf{R}^p, 0) & \xrightarrow{\pi} & (\mathsf{R}^n, 0)
\end{array}$$

where i denotes the germ of inclusion $(\mathsf{R}^n, 0) \to (\mathsf{R}^n \times \mathsf{R}^p, 0)$ and π the germ of projection $(\mathsf{R}^n \times \mathsf{R}^p, 0) \to (\mathsf{R}^n, 0)$.

The action of $\mathcal{K}$ on f is defined by

$$(\mathrm{id}_{\mathsf{R}^n}, H.f) = H \circ (\mathrm{id}, f) \circ h^{-1}.$$

Clearly $\mathcal{C}$ is a subgroup of $\mathcal{K}$. The identification of $h \in \mathcal{R}$ with $(h, \mathrm{id}_{\mathsf{R}^p}) \in \mathcal{K}$ makes $\mathcal{R}$ a subgroup of $\mathcal{K}$. Furthermore $\mathcal{K} = \mathcal{R}.\mathcal{C}$ (semi-direct product).

(2.3) DEFINITION. $C^l - \mathcal{G}$, $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, $l \geqq 0$, are defined as before, taking diffeomorphisms of class C^l, $l \geqq 1$, or homeomorphisms, when $l = 0$.

Let $C(n)$ denote the ring of germs of smooth functions and $\mathfrak{m}_n$ its maximal ideal.

Following Wall [5], we denote by $I_{\mathcal{R}}(f) = J_f$, the ideal of $C(n)$ generated by the $p \times p$-minors of the Jacobean matrix of f, by $I_{\mathscr{C}}(f)=f^*(m_p)C(n)$, the ideal generated by the coordinate functions of f, and $I_{\mathscr{K}}(f(x))=I_{\mathscr{C}}(f(x)) + I_{\mathscr{R}}(f(x))$.

Now, write $N_{\mathscr{C}}(f(x))=|f(x)|^2$, $N_{\mathscr{R}}(f(x))=|df_x|^2_{\mathscr{R}}=\det\{(df_x)(df_x)^t\}$ $=$ sum of squares of $p \times p$-minors of df_x, and

$$N_{\mathscr{K}}(f(x)) = N_{\mathscr{C}}(f(x))+N_{\mathscr{R}}(f(x)) .$$

We say that $N_{\mathscr{G}}(f(x))$ satisfies a Lojasiewicz condition of order r (>0) if there exists a constant $c>0$ such that $N_{\mathscr{G}}(f(x))\geqq c|x|^r$; we denote such a condition (c_r).

The following proposition relates the existence of a Lojasiewicz condition for $\varphi=\sum_{i=1}^k \varphi_i^2$ with the condition that the ideal generated by the φ_i's is elliptic.

(2.4) PROPOSITION. *Let $I=\langle\varphi_1,\ldots,\varphi_k\rangle$ be a finitely generated ideal in $C(n)$. Then the following conditions are equivalent:*

(a) *I is elliptic (or, $I\supset \mathfrak{m}_n^\lambda$);*

(b) *there exists g in I such that $|g(x)|\geqq c|x|^\alpha$ for some $c>0$ and $\alpha>0$;*

(c) *there exists $c>0$ and $\alpha>0$ such that $\sum_{i=1}^k [\varphi_i(x)]^2\geqq c|x|^\alpha$.*

If φ_i are analytic, then the above conditions are equivalent to:

(d) *0 is an isolated point in $\varphi^{-1}(0)$, where $\varphi(x)=(\varphi_1(x),\ldots,\varphi_k(x))$.*

(See [5] for a proof and comments.)

To obtain good estimates for the degree of $C^l-\mathscr{G}$-determinacy it is necessary to impose a condition to control the growing of the derivative of $1/N_{\mathscr{G}}(f)$ such as:

$$\left|\frac{\operatorname{grad} N_{\mathscr{G}}(f)}{N_{\mathscr{G}}(f)}\right| \leqq \frac{C}{|x|^\lambda}, \quad \lambda \geqq 1 .$$

The control will be exercised via the condition (d_{r_0}), which we take to mean that r_0 is the largest integer such that $N_{\mathscr{G}}(f)\in \mathfrak{m}_n^{r_0}$.

The information contained in $I_{\mathscr{G}}(f)$ (hence in the tangent space to the $\mathscr{G}$-orbit of f) will be used in the construction of controlled vector fields, whose class of differentiability depends on the conditions (c_r) and (d_{r_0}) of the control function $N_{\mathscr{G}}(f)$.

3. Estimates for the degree of $\mathscr{C}^l$-$\mathscr{G}$-determinacy ($\mathscr{G}$-$\mathscr{R}$, $\mathscr{C}$ or $\mathscr{K}$).

Let f: $(\mathrm{R}^n,0) \to (\mathrm{R}^p,0)$ be a C^∞-map-germ, with corank k, in the form

(*) $$(x_1,x_2,\ldots,x_n) \mapsto (x_1,\ldots,x_{p-k},f_1(x),\ldots,f_k(x)) ,$$

and let $s=\max\{q \mid f_j \in \mathfrak{m}_n^q, j=1,\ldots,k\}$.

CASE 1. $\mathcal{G}=\mathcal{R}$.

(3.1) PROPOSITION. *If $N_{\mathcal{R}}(f)$ satisfies conditions (c_{2r}) and (d_{2r_0}), it follows that f is $N=[r+l(r-r_0+1)-(s-1)(k-1)]-C^l-\mathcal{R}$-determined.*

The proof follows easily from the following two lemmas.

(3.2) LEMMA. (See [5, Lemma 2.12], or [1, Lemma 4.7].) *Let θ_f be the set of germs of vector fields along f, that is*

$$\theta_f = \{\xi\colon (\mathrm{R}^n,0) \to T(\mathrm{R}^p) \mid \pi_{\mathrm{R}^p}\circ\xi=f\} .$$

Then, for $h \in \theta_f$ and

$$M = (\partial f_r/\partial x_{i_s})_{\substack{1\leqq r\leqq p\\ 1\leqq s\leqq p}} ,$$

a $p\times p$-minor of df_x, the following holds

$$(\det M).h = df\left[\sum_{s=1}^{p}\sum_{r=1}^{p}\left[\operatorname{cof}\left(\frac{\partial f_r}{\partial x_{i_s}}\right)h_r\right]\frac{\partial}{\partial x_{i_s}}\right]$$

where $\operatorname{cof}(\partial f_r/\partial x_{i_s})$ *denote the cofactor of $\partial f_r/\partial x_{i_s}$, and h_r are the component functions of h.*

(3.3) LEMMA. *Let $N(x)$ be defined by $N(x)=\sum_{j=1}^{L}(d_j(x))^2$.*
Suppose that $N(x)$ satisfies conditions (c_{2r}) and (d_{2r_0}).
Given any germ of a C^∞ function h, with $h(x)\in\mathfrak{m}_n^N$, $N=r+l(r-r_0+1)+1$, then $\varepsilon(x)=h(x)d_j(x)/N(x)$ is differentiable of class C^l, $l\geqq 1$, for any $j=1,\ldots,L$.

PROOF OF LEMMA (3.3). Since $|d(x)|=|d_j(x)|\leqq[N(x)]^{1/2}$, it follows that

$$|\varepsilon(x)| \leqq \frac{h(x)}{[N(x)]^{1/2}} \leqq c|x| ,$$

hence continuous.

We can now proceed by induction. Given

$$\varepsilon(x) = \frac{H(x)D(x)}{N^\alpha(x)} ,$$

where $H(x)\in\mathfrak{m}_n^N$, $N=[r+l(r-r_0+1)+1]+(\alpha-1)(r_0-1)$, $l\geqq\alpha-1$ and $D(x)=p^\alpha(d_j)$, a polynomial of degree α, in the variables d_j, $j=1,\ldots,L$.

Then

$$\frac{\partial \varepsilon(x)}{\partial x_j} = \frac{\dfrac{\partial H(x)}{\partial x_j} D(x) + H(x)\dfrac{\partial \dot{D}(x)}{\partial x_j}}{N^\alpha(x)} - \frac{\alpha H(x) D(x) \dfrac{\partial N}{\partial x_j}}{N^{\alpha+1}(x)},$$

which is a linear combination of terms in the form

$$\frac{\tilde{H}\tilde{D}}{N^{\alpha+1}(x)},$$

where $\tilde{H} \in \mathfrak{m}_n^N$, $N = r + l(r - r_0 + 1) + 1 + \alpha(r_0 - 1)$, $D(x) = p^{\alpha+1}(d_j)$, $l \geqq \alpha$.
It follows that

$$\left|\frac{\tilde{H}\tilde{D}}{N^{\alpha+1}(x)}\right| \leqq c|x|^k,$$

where $k \geqq r + \alpha(r - r_0 + 1) + 1 + \alpha(r_0 - 1) - (\alpha + 1)r = 1$, so that $\partial\varepsilon(x)/\partial x_j$ is continuous. The induction step, and hence the proof is complete.

PROOF OF PROPOSITION (3.1). Let g be such that the N-jets of g and f coincide at the origin, that is: $J^N g(0) = j^N f(0)$, and $F(x,t) = (f_t(x), t)$, where $f_t(x) = (1-t)f(x) + tg(x)$, $t \in [0,1]$. It is easy to see that

$$I_{\mathscr{R}}(f_t) = I_{\mathscr{R}}(f) + \mathfrak{m}_n^{r+1}, \quad \forall t \in [0,1].$$

Hence, $N_{\mathscr{R}}(f_t) + \varepsilon_t(x) = N_{\mathscr{R}}(f)$, where $\varepsilon_t \in \mathfrak{m}_n^{2r+2}$, $\forall t \in [0,1]$ and this implies $N_{\mathscr{R}}(f_t) \geqq c|x|^{2r}$ for all $t \in [0,1]$.
Now,

$$N_{\mathscr{R}}(f_t(x)) \frac{\partial f_t}{\partial t} = \sum_{J=1}^{L} \left[\partial f_t\left({}^*\hat{M}_t^J\left(\det M_t^J\right)\frac{\partial f_t}{\partial t}\right)\right],$$

where J enumerates all $p \times p$-minors of $(df_t)_x$ and

$${}^*\hat{M}_t^J \frac{\partial f_t}{\partial t} = \sum_{s=1}^{p} \sum_{r=1}^{p} \left[\operatorname{cof}\frac{\partial f_r}{\partial x_{i_s}} \cdot \frac{(\partial f_t)_r}{\partial x_{i_s}}\right]\frac{\partial}{\partial x_{i_s}},$$

as in Lemma 3.2. (The coefficients of $\partial/\partial x_j$ are zero for $j \neq i_s$.)
Defining

$$\varepsilon(t,x) = \sum_{J=1}^{L}\left[\frac{{}^*\hat{M}_t^J(\det M_t^J)\dfrac{\partial f_t}{\partial t}}{N_{\mathscr{R}}(f_t(x))}\right]\frac{\partial}{\partial x_J}$$

it follows from Lemma 3.3, that ε is differentiable of class C^l.

Now,

$$\frac{\partial f_t}{\partial t}(x,t) = (df_t)_x(x,t)(\varepsilon(x,t)),$$

and this implies the $C^l-\mathscr{R}$-triviality of the family $F(t,x)$ in a neighbourhood of $t=0$. Since the same argument is true in a neighbourhood of $t=\bar{t}$, $\forall\, \bar{t}\in[0,1]$, the proof is complete.

The estimates we obtain are, in many cases, more precise than the results in [2], as we can see in the following proposition and example.

(3.4) PROPOSITION. *Let* $f\colon (\mathsf{R}^n,0)\to(\mathsf{R}^p,0)$ *be as in* (*), *with corank* k. *Assume that each* f_j *is homogeneous of degree* r_j, $j=1,\dots,k$. *If* $I_{\mathscr{R}}(f)$ *is elliptic,* f *is* $(r+l-1)-C^l-\mathscr{R}$*-determined,* $r=\max_{1\leqq j\leqq k} r_j$.

PROOF. We may assume that $2\leqq r_1\leqq r_2\leqq\dots\leqq r_k$.
Each $p\times p$-minor of df_x is homogeneous of degree $\sum_{i=1}^{k}(r_i-1)=r=r_0$.
The elements of $\hat{M}=\operatorname{cof} M^t$ are $(p-1)\times(p-1)$-minors of df_x., which are homogeneous, and the degree depends on the omitted row. The smallest of these degrees is $\sum_{i=1}^{k-1}(r_i-1)$.
Hence, the degree of $C^l-\mathscr{R}$-determinacy of f is

$$r+l(r-r_0+1)-\sum_{i=1}^{k-1}(r_i-1) = r-1+l.$$

(3.5) REMARK. The above estimates are sharp in the following sense: if $f(x)$ is $(r+l_0-2)-C^{l_0}-\mathscr{R}$-determined, for some $2\leqq l_0<\infty$, then f is in fact $(r+l_0-1)-C^\infty-\mathscr{R}$-determined.
Let us assume f is $(r+l_0-2)-C^{l_0}-\mathscr{R}$-determined, for some l_0. Then, taking $(r+l_0-1)$-jets of $C^{l_0}-\mathscr{R}$-trivial families

$$f_t = f+t(g-f) = f\circ h_t,\quad t\in[0,1],$$

$$j^{r+l_0-2}g(0) = j^{r+l_0-2}f(0),\quad h_t\in C^{l_0}-\mathscr{R},\quad h_0 = \mathrm{id}_{\mathsf{R}^n},$$

we obtain

$$j^{r+l_0-1}(\partial f_t/\partial t)|_{t=0} = j^{r+l_0-1}(df(\partial h_t/\partial t|_{t=0})),$$

which in turn implies the $(r+l_0-1)-C^\infty-\mathscr{R}$-determinacy of f. (See [5] or [3] for more details.)

Finally, we recall that if f is $C^\infty-\mathscr{R}$-finitely determined and 0 is a singular point of f, then p must be equal to 1 ([5, Proposition 2.3]). Thus if $p>1$, $f(x)$ can not be $(r+l-2)-C^l-\mathscr{R}$-determined for all $l\geqq 1$.

(3.6) EXAMPLE. f: $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ defined by

$$\begin{cases} u(x,y) = x^r - y^r, \ r \text{ even} \\ v(x,y) = xy \,. \end{cases}$$

$N_{\mathscr{R}}(f) = r(x^r + y^r)$ and f is $(r+l-1)-C^l-\mathscr{R}$-determined for all $l \geqq 1$.

From Remark (3.5), it follows that $f(x)$ can not be $(r+l-2)-C^l-\mathscr{R}$-determined.

CASE 2. $\mathscr{G} = \mathscr{C}$.

We shall assume $I_{\mathscr{C}}(f)$ is elliptic. If f is as in (*), of corank k, we consider

$$N_{\mathscr{C}}^*(f) = (f_1)^2 + \ldots + (f_k)^2 + (x_1^s)^2 + \ldots + (x_{p-k}^s)^2 \,.$$

Clearly, $N_{\mathscr{C}}^*(f)$ satisfies a Lojasiewicz condition, that we shall denote by (c_{2r}^*).

In this case, $N_{\mathscr{C}}^*(f) \in \mathfrak{m}_n^{2s}$, that is $s = r_0$.

(3.7) PROPOSITION. *If $N_{\mathscr{C}}^*(f)$ satisfies conditions (c_{2r}^*) and (d_{2s}), it follows that f is $N = [r + l(r-s+1) - 1] - C^l - \mathscr{C}$-determined.*

It is not hard to show that f is $(N+1)-C^l-\mathscr{C}$-determined. The reduction to N depends on the next Lemma, in which we construct a conic bump function, with controlled derivatives.

(3.8) LEMMA. *Let $|y| \leqq c_1|x|$ and $|y| \leqq c_2|x|$ be cones in $\mathbb{R}^n \times \mathbb{R}^p$, with $c_1 < c_2$. There exists a function $p: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $p \in C^\infty$ in $\mathbb{R}^n \times \mathbb{R}^p - (0 \times 0)$,*

$$\begin{aligned} p(x,y) &= 1 && \text{if } |y| \leqq c_1|x|, \ (x,y) \neq (0,0)\,, \\ p(x,y) &= 0 && \text{if } |y| \geqq c_2|x|\,, \\ 0 \leqq p(x,y) &\leqq 1 && \text{if } c_1|x| \leqq |y| \leqq c_2|x|\,, \\ p(0,0) &= 0\,, \end{aligned}$$

such that

$$|D^s(p(x,y)y)| \leqq \frac{K_s}{|x|^{s-1}}, \quad K_s = \text{constant}, \ \forall s \geqq 1\,.$$

PROOF. For $n = p = 1$, let h: $\mathbb{R} \to \mathbb{R}$ be the usual C^∞ bump function,

$$\begin{aligned} h(\theta) &= 1 \quad , && \text{if } 0 \leqq \theta \leqq \theta_1; \\ h(\theta) &= 0 \quad , && \text{if } \theta \geqq \theta_2; \\ 0 \leqq h(\theta) &\leqq 1\,, && \text{if } \theta_1 < \theta < \theta_2\,. \end{aligned}$$

We define

$$p(x,y) = h(\theta), \quad \text{where } \theta = \operatorname{arctg} \frac{y}{|x|}\,.$$

Then $|p(x,y)y| \leqq (\operatorname{tg}\theta_2)|x|$. By successive derivations, we see easily that

$$|D^s p(x,y)y| \leqq \frac{K_s}{|x|^{s-1}}.$$

For $n \geqq 1$ and $p=1$, let $p(x,y)$ be defined by

$$p(x_1,\ldots,x_n,y) = h(\theta), \quad \text{if } |x| \neq 0$$

$$p(0,y) = 0 \quad , \quad \theta = \operatorname{arctg}\frac{y}{|x|}.$$

In the general case

$$p(x_1,\ldots,x_n,y_1,\ldots,y_p) = p_1(x,y)\,.p_2(x,y)\ldots p_p(x,y), \quad |x| \neq 0$$

where

$$p_i(x,y) = h(\theta_i), \quad \theta_i = \operatorname{arctg}\frac{y_i}{|x|}, \quad p(0,y) = 0.$$

Proof of Proposition (3.7). Let f: $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ be given by

$$(x_1,x_2,\ldots,x_n) \mapsto (x_1,\ldots,x_{p-k},f_1(x),f_2(x),\ldots,f_k(x)).$$

If $j^N g(0)=j^N f(0)$, where $N=[r+l(r-s+1)-1]$, and

$$F(x,t) = (f_t(x),t), \quad f_t(x) = f(x)+t(g(x)-f(x)), \quad t \in [0,1],$$

we have

$$N^*_{\mathscr{C}}(f_t)\frac{\partial f_t}{\partial t} = \sum_{j=1}^{k}\left[(f_t)_j\frac{\partial f_t}{\partial t}\right]F^*(y_{p-k+j})+\sum_{i=1}^{p-k}(x_t)_i^s\frac{\partial f_t}{\partial t}[F^*(y_i)]^s,$$

where $(x_t)_i$, $i=1,\ldots,p-k$ denote the first $p-k$ coordinate functions of $F(x,t)$, and $y=\{y_1,\ldots,y_p\}$ is the system of local coordinates at $(\mathbb{R}^p,0)$.

Let η: $(\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R},0) \to (\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R},0)$ be the vector field defined by

$$\eta(t,x,y) = \eta_1(t,x,y)+\eta_2(t,x,y),$$

where

$$\eta_1(t,x,y) = \frac{1}{N^*_{\mathscr{C}}(f_t)}\left[\sum_{j=1}^{k}(f_t)_j\frac{\partial f_t}{\partial t}y_{p-k+j}\frac{\partial}{\partial y_j}\right] \quad \text{and}$$

$$\eta_2(t,x,y) = \frac{1}{N^*_{\mathscr{C}}(f_t)}\left[\sum_{i=1}^{p-k}(x_t)_i^s\frac{\partial f_t}{\partial t}(x_t)_i^{s-1}y_i\frac{\partial}{\partial y_i}\right].$$

From Lemma (3.3), it follows that η_2 is of class C^l, while η_1 is only C^{l-1}. However, using the function $p(x,y)$ of Lemma (3.8), we may modify η_1 to obtain a C^l-vector field. We define

$$\tilde{\eta}_1(t,x,y) = p(t,x,y)\eta_1(t,x,y) .$$

Since $\tilde{\eta}_1$ coincides with η_1 in a conic neighbourhood of the graph of $F(t,x)$, equation $\partial f_t/\partial t = p \,.\eta_1 + \eta_2$ also holds.

The result follows as in Proposition (3.1), by integrating the vector fields.

(3.9) PROPOSITION. *Let* f: $\mathrm{R}^n,0 \to \mathrm{R}^p,0$ *be of corank* k:

$$(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_{p-k}, f_1(x),\ldots,f_k(x)) ,$$

where f_j *are homogeneous of degree* r_j.

If $I_{\mathscr{C}}(f)$ *is elliptic,* f *is* $(r+l-1)-C^l-\mathscr{C}$*-determined* $(l \geqq 1)$, $r = \max r_j$.

PROOF. Let $\mathscr{R} = r_1 \,.r_2 \,.\ \ldots\ .r_k$ and $\mathscr{R}_m = \prod_{i \neq m} r_i$. The convenient control function is given by

$$N_{\mathscr{C}}^{**}(f) = (f_1)^{2\mathscr{R}_1} + \ldots + (f_k)^{2\mathscr{R}_k} + (x_1)^{2\mathscr{R}} + \ldots + (x_{p-k})^{2\mathscr{R}} .$$

The rest of the proof is just as in Proposition 3.7.

(3.10) REMARK. Using similar arguments as in Remark 3.5, we conclude that the estimates of Proposition 3.9 are exact. Thus, if f is $(r+l_0-2)-C^{l_0}-\mathscr{C}$-determined for some $2 \leqq l_0 < \infty$, then f is in fact $(r+l_0-1)-C^{\infty}-\mathscr{C}$-determined.

CASE 3. $\mathscr{G} = \mathscr{K}$.

We are still considering f as in (*). Let

$$r_0 = \max\{q \mid d_i \in \mathrm{m}_n^q,\ i=1,\ldots,L\} ,$$

where $d_i = \det M_i$, $p \times p$-minor of df_x.

Let s be as before.

If $I_{\mathscr{K}}(f)$ is elliptic, there exist constants $\alpha > 0$, $r > 0$, such that:

$$N_{\mathscr{K}}^{*}(f) = (d_1^s)^2 + (d_2^s)^2 + \ldots + (d_L^s)^2 + (f_1^{r_0})^2 + \ldots + \\ + \ldots + (f_k^{r_0})^2 + (x_1^{sr_0})^2 + \ldots + (x_{p-k}^{sr_0})^2 \geqq \alpha|x|^{2r} .$$

Clearly, $r \geqq sr_0$ and $r_0 \geqq k(s-1)$.

With these assumptions, it is possible to obtain a result that envolves several variables, but gives good estimates.

(3.11) PROPOSITION. *Let*

$$N_1 = \frac{r}{s} + l(r/s - r_0 + 1) - (k-1)(s-1) \quad \textit{and} \quad N_2 = \frac{r}{r_0} + l(r/r_0 - s + 1) - 1 ,$$

then f is $N-C^l-\mathscr{K}$-determined, where N is the smallest integer greater than or equal to the $\max\{N_1, N_2\}$.

PROOF. For simplicity, we shall assume $p=k$.

For any g such that $j^N g(0)=j^N f(0)$, $N=\max\{N_1,N_2\}$, we consider the following unfolding of graph of f

$$F\colon (\mathsf{R}^n\times\mathsf{R},0) \to (\mathsf{R}^n\times\mathsf{R}^p\times\mathsf{R},0)$$
$$(x,t) \mapsto (x, f_t(x), t), \quad t\in[0,1],$$

where $f_t(x)=(1-t)f(x)+\operatorname{tg}(x)$, $t\in[0,1]$.

We aim to find C^l retractions h and k of $\mathrm{id}_{\mathsf{R}^n\times 0}$ and $\mathrm{id}_{\mathsf{R}^n\times\mathsf{R}^p\times 0}$, respectively, such that the following diagram commutes:

$$\begin{array}{ccccc}
(\mathsf{R}^n,0) & \xrightarrow{(\mathrm{id},f)} & ({}^n\times\mathsf{R}^p,0\times 0) & \xrightarrow{\pi_{\mathsf{R}^n}} & (\mathsf{R}^n,0)\\
\uparrow h & & \uparrow k & & \uparrow h\\
(\mathsf{R}^n\times\mathsf{R},0\times I) & \xrightarrow{F} & (\mathsf{R}^n\times\mathsf{R}^p\times\mathsf{R},0\times 0\times I) & \xrightarrow{\pi_{\mathsf{R}^n\times\mathsf{R}}} & (\mathsf{R}^n\times\mathsf{R},0\times I)
\end{array}$$

If we can do so, then

$h_1\colon(\mathsf{R}^n,0)\to(\mathsf{R}^n,0)$ defined by $h_1(x)=h(x,1)$ and
$k_1\colon(\mathsf{R}^n\times\mathsf{R}^p,0\times 0)\to(\mathsf{R}^n\times\mathsf{R}^p,0\times 0)$ defined by $k_1(x,y)$
$= k(x,y,1)$,

will give a $C^l-\mathscr{K}$-equivalence between f and g.

We shall construct h and k in a neighbourhood of $t=0$ as follows:
Since

$$N^*_{\mathscr{K}}(f_t)\frac{\partial f_t}{\partial t} = \big(N^*_{\mathscr{R}}(f_t)+N^*_{\mathscr{C}}(f_t)\big)\frac{\partial f_t}{\partial t},$$

$$N^*_{\mathscr{R}}(f_t) = \sum_{i=1}^{L}(d_i^s)^2 \quad\text{and}\quad N^*_{\mathscr{C}}(f_t) = \sum_{j=1}^{p}(f_t^{r_0})_j^2$$

we can proceed as in Propositions 3.1 and 3.7, to obtain the equation:

$$\frac{\partial f_t}{\partial t} = df_t\left[\sum_{i=1}^{L}\frac{d_i^{2s-1}*\hat{M}_i\dfrac{\partial f_t}{\partial t}}{N^*_{\mathscr{K}}(f_t)}\frac{\partial}{\partial x_i}\right]+\left[\sum_{j=1}^{p}\frac{(f_t)_j^{2r_0-1}\dfrac{\partial f_t}{\partial t}f_t^*(y_i)}{N^*_{\mathscr{K}}(f_t)}\right]. \tag{3.12}$$

To complete the proof, it remains to find germs of C^l vector fields

$\xi\colon(\mathsf{R}^n\times\mathsf{R},0)\to(\mathsf{R}^n\times\mathsf{R},0)$, $\pi_{\mathsf{R}}\circ\xi = \dfrac{\partial}{\partial t}$, $\pi_{\mathsf{R}^n}\circ\xi(0,t)=0$, and

$$\eta\colon (\mathrm{R}^n \times \mathrm{R}^p \times \mathrm{R}, 0) \to (\mathrm{R}^n \times \mathrm{R}^p \times \mathrm{R}, 0),$$

such that ξ is a lift for η over F, that is $dF(\xi)=\eta\circ F$.

So let,

$$\xi(x,t) = -\bar{\xi}(x,t)+\frac{\partial}{\partial t},$$

where

$$\bar{\xi}(x,t) = \frac{1}{N^*_{\mathscr{X}}(f_t)}\left[\sum_{i=1}^{L} d_i^{2s-1} * \hat{M}_i \frac{\partial f_t}{\partial t}\frac{\partial}{\partial x_i}\right] \quad \text{and}$$

$$\eta(x,y,t) = -\bar{\xi}(x,t)+\tilde{\eta}(x,y,t)+\frac{\partial}{\partial t},$$

where

$$\tilde{\eta}(x,y,t) = \frac{1}{N^*_{\mathscr{X}}(f_t)}\left[\sum_{j=1}^{p} (f_t)_j^{2r_0-1}\frac{\partial f_t}{\partial t} y_j \frac{\partial}{\partial y_j}\right].$$

Then $$dF(\xi) = \left(-\bar{\xi}, df_t(-\bar{\xi})+\frac{\partial f_t}{\partial t}, \frac{\partial}{\partial t}\right).$$

From equation (3.12), it follows that $dF(\xi)=\eta\circ F$.

To show ξ is of class C^l and η is of class C^{l-1}, it is enough to observe that

$$\left|\frac{\operatorname{grad} N^*_{\mathscr{X}}}{N^*_{\mathscr{X}}}\right| \leqq \frac{C}{|x|^{\lambda}},$$

where $\lambda \leqq \max\{\lambda_1,\lambda_2\}$, $\lambda_1 = r/s - r_0 + 1$ and $\lambda_2 = r/r_0 - s + 1$, and proceed by induction as in the proof of Lemma (3.3).

Using the function $p(x,y,t)$ of Lemma 3.8, we may now modify η to obtain a C^l-vector field. We define

$$\gamma = -\bar{\xi}+p\,.\tilde{\eta}+\frac{\partial}{\partial t}.$$

Since γ coincides with η in a conic neighbourhood of graph of f_t, the equation $dF(\xi)=\gamma\circ F$ also holds.

These vector fields are clearly integrable, hence determine C^l-diffeomorphisms H and K in $\mathrm{R}^n \times \mathrm{R}, 0$ and $\mathrm{R}^n \times \mathrm{R}^p \times \mathrm{R}, 0$, respectively.

The properties of ξ and γ imply that $\pi_{\mathrm{R}^n}\circ H = h$ and $\pi_{\mathrm{R}^n\times\mathrm{R}^p}\circ K = k$ are the desired retractions.

(3.13) PROPOSITION. *Let* $f\colon (\mathsf{R}^n,0) \to (\mathsf{R}^p,0)$ *be given by*

$$(x_1,\ldots,x_n) \to \bigl(x_1,\ldots,x_{p-k}, f_1(x),\ldots,f_k(x)\bigr)\,,$$

where f_j *are homogeneous of degree* r_j *and* $r=\max r_j$.

If $I_{\mathscr{K}}(f)$ *is elliptic (or equivalently, if* $N_{\mathscr{K}}(f)$ *satisfies a Lojasiewicz condition), then* f *is* $(r+l-1)-C^l-\mathscr{K}$*-determined* $(1\leqq l<\infty)$.

Furthermore, small deformations of f *of degree* r *are* $C^0-\mathscr{K}$*-trivial.*

The following corollary follows from Propositions (3.4), (3.9), (3.13) and from the Lojasiewicz Inequality for analytic functions (Proposition (2.4)).

(3.14) COROLLARY. *Given* f *as in* (3.13) *for all* l, $1\leqq l<\infty$ *and* $r=\max r_j$:

(a) *If* 0 *is an isolated singular point of* f, *then* f *is* $(r+l-1)-C^l-\mathscr{R}$*-determined.*

(b) *If* $f^{-1}(0)=\{0\}$, *then* f *is* $(r+l-1)-C^l-\mathscr{C}$*-determined.*

(c) *If* 0 *is an isolated singularity in* $f^{-1}(0)$, *then* f *is* $(r+l-1)-C^l-\mathscr{K}$*-determined.*

Moreover, with the hypothesis of (a), (b) *or* (c), *it follows, respectively, that small deformations of order* r *are* $C^0-\mathscr{G}$*-trivial,* $\mathscr{G}=\mathscr{R}, \mathscr{C}$ *or* $\mathscr{K}$.

(3.15) EXAMPLE. $f(x,y,z)=(ax^m+by^m+cz^m, xyz)$, $m\geqq 3$, $a\neq 0$, $b\neq 0$, $c\neq 0$. The usual procedure of computing the tangent space to the $\mathscr{K}$-orbit of f shows easily that f is $2(m-1)-\mathscr{K}$-determined.

So, f is $(m+l-1)-C^l-\mathscr{K}$-determined, for all $1\leqq l<m-1$.

This is a sharp result.

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