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# GENERAL TAUBERIAN THEOREMS IN $\mathsf{R}^d$ CONNECTED WITH A THEOREM OF KORENBLUM

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## 1. Introduction.

This paper deals with so called Tauberian remainder problems. To explain the nature of these problems, consider first the following well known theorem.

WIENER'S THEOREM. *Let* $\Phi\colon \mathsf{R} \to \mathsf{R}$ *be bounded, let* $F \in L^1(\mathsf{R})$ *and suppose* $\hat{F}(\xi) \neq 0,\ \xi \in \mathsf{R}$. *If* $\Phi$ *satisfies the Tauberian condition*

$$\lim_{h\to 0+}\ \lim_{x\to\infty}\ \inf_{\substack{u \geqq x\\ 0\leqq y\leqq h}} \big(\Phi(u+y)-\Phi(u)\big) = 0 \tag{1.1}$$

*then*

$$\Phi * F(x) = o(1), \quad x \to \infty \tag{1.2}$$

*implies*

$$\Phi(x) = o(1), \quad x \to \infty. \tag{1.3}$$

In a Tauberian remainder theorem we introduce stronger conditions on $\hat{F}$, mainly consisting of restrictions on the order of magnitude of $1/\hat{F}$ and its derivatives, and we also use stronger Tauberian conditions than in Wiener's theorem. Knowing the order of magnitude of $\Phi * F(x)$ as $x \to \infty$, say $\Phi * F(x) = o\big(m(x)\big)$, $x \to \infty$, where $m \searrow$, we then obtain a more refined estimate of $\Phi(x)$ as $x \to \infty$ than merely $\Phi(x) = o(1)$, $x \to \infty$. The Tauberian condition mostly used in this connection is that $\Phi(x) + Ax \nearrow$, $x \geqq x_0$, for some constant $A$. In the papers [5] and [6] some Tauberian remainder theorems were proved which, in many special cases, yielded sharp results for such a Tauberian condition.

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In the present paper corresponding problems are studied for stronger Tauberian conditions, conditions which are strong enough to yield the same result as the monotonicity of $\Phi$. For instance, in Theorem $3_0$ below, conditions on $m$ and $\hat{F}$ are introduced such that if $\Phi$ is a bounded function satisfying the Tauberian condition

$$\lim_{h\to 0+}\lim_{x\to\infty}\inf_{\substack{u\geqq x\\ 0\leqq y\leqq h}}\frac{\Phi(u+y)-\Phi(u)}{m(u)} = 0 \tag{1.4}$$

then

$$\Phi * F(x) = o(m(x)), \quad x\to\infty\,, \tag{1.5}$$

implies

$$\Phi(x) = o(m(x)), \quad x\to\infty\,, \tag{1.6}$$

which expresses the analogy of Wiener's theorem.

Theorems 1 and 2 below contain some preliminary Tauberian results which might be interesting in themselves. The main result of the present paper is given in Theorem 3 and the above mentioned Theorem $3_0$ is a special case of this theorem. From Theorem 3 we deduce Theorem 4 which may be considered a general Tauberian theorem corresponding to Korenblum's Tauberian theorem for Laplace transforms in [3]. Theorem $4_0$ is a special case of Theorem 4. It is a generalization to $\mathsf{R}^d$ of Korenblum's theorem with the "non decreasing" condition relaxed and it may be applied even to some functions which tend to zero as $x$ tends to infinity.

The methods used in the earlier papers [5] and [6] do not apply directly to the strong Tauberian conditions considered here. Therefore, though the main idea is the same, the methods are modified on some essential points in order to suit the new purpose. The methods are also extended to $\mathsf{R}^d$, $d\geqq 1$ and to the case when $|\Phi * F(x)|$ may tend to infinity as $x$ tends to infinity.

Due to these extensions the theorems are burdened with some supplementary conditions. However, in order to include Koremblum's theorem, an extension to unbounded $\Phi$ is necessary.

A generalization in another direction of Korenblum's theorem to $\mathsf{R}^d$ is presented by Stadtmüller and Trautner in [8] and [9].

Tauberian theorems in $\mathsf{R}^d$, $d\geqq 1$, for Tauberian conditions corresponding to the condition $\Phi(x)+Ax\nearrow$, $x\geqq x_0$, were considered long ago by Frennemo and applied to the $d$-dimensional Laplace transform (see [1] and [2]).

**2. Definitions and notations.**

All functions are supposed to be measurable. Let $x = (x_1, \ldots, x_d) \in \mathsf{R}^d$, $d \geqq 1$, $\mathit{0} = (0, \ldots, 0) \in \mathsf{R}^d$, $\mathit{1} = (1, \ldots, 1) \in \mathsf{R}^d$. Let $y \in \mathsf{R}^d$, $xy = \sum_{j=1}^d x_j y_j$, $|x| = \sum_{j=1}^d |x_j|$, $\|x\|_\infty = \max_{j=1,2\ldots d} |x_j|$, and $\max(x, y) = (\max(x_1, y_1), \ldots, \max(x_d, y_d)) \in \mathsf{R}^d$.

The notation $x \leqq y$ means $x_j \leqq y_j$, $j = 1, 2, \ldots, d$, and $x \to \infty$ means $x_j \to \infty$, $j = 1, 2, \ldots, d$. The notations $\min(x, y)$, $x < y$ and $x \to \mathit{0}$ are defined in an analogous way. Let

$$\mathsf{R}^d_+ = \{x \mid x \in \mathsf{R}^d,\ x \geqq \mathit{0}\}, \quad \mathsf{R}^d_{++} = \{x \mid x \in \mathsf{R}^d,\ x > \mathit{0}\}.$$

Let $f\colon \mathsf{R}^d \to \mathsf{R}^1$. The notation $f \nearrow$ means that $f$ is componentwise nondecreasing, i.e.

$$f(x_1, \ldots, x_j + h, \ldots, x_d) \geqq f(x_1, \ldots, x_d), \quad h > 0,\ j = 1, 2, \ldots, d,$$

and $f \searrow$ means that $f$ is componentwise nonincreasing. I use standard notations for convolution and for the $L^p$-norm. Thus

$$f_1 * f_2(x) = \int_{\mathsf{R}^d} f_1(x-y) f_2(y)\,dy, \quad \|f\|_p = \left(\int_{\mathsf{R}^d} |f(x)|^p\,dx\right)^{1/p}.$$

Let $I$ be a subset of $\{1, 2, \ldots, d\}$ and let $J$ be the complementary subset. Thus

$$I \cup J = \{1, 2, \ldots, d\}, \quad I \cap J = \varnothing. \tag{2.1}$$

Let $x$ and $\xi$ belong to $\mathsf{R}^d$ (or $\mathsf{C}^d$) and let $x^I_\xi \in \mathsf{R}^d$ (or $\mathsf{C}^d$) be defined by

$$(x^I_\xi)_j = \xi_j \ \text{ for } j \in I \ \text{ and } \ (x^I_\xi)_j = x_j \ \text{ for } j \in J. \tag{2.2}$$

Introduce the class $\mathscr{R}$ of functions $k\colon \mathsf{R}^d \to \mathsf{R}^1_{++}$ as follows

DEFINITION. $k \in \mathscr{R}$ if

$$k(x+y) \leqq k(x) k(y), \quad x, y \in \mathsf{R}^d \tag{2.3}$$

and

$$\begin{cases} k(t_1 x_1, \ldots, t_d x_d) \geqq k(x_1, \ldots, x_d), & x \in \mathsf{R}^d,\ t \in \mathsf{R}^d,\ t \geqq \mathit{1} \\ k(x) \geqq k(x^I_{\mathit{0}}) \geqq 1, & I \subset \{1, 2, \ldots, d\},\ x \in \mathsf{R}^d. \end{cases} \tag{2.4}$$

For functions $k \in \mathscr{R}$ we introduce the class $R[k]$ of functions $m\colon D_m \to \mathsf{R}^1_{++}$, $D_m \subset \mathsf{R}^d$, as follows

DEFINITION. $m \in R[k]$ if

$$m(\eta) \leqq m(\xi)k(\xi - \eta), \quad \xi, \eta \in D_m. \tag{2.5}$$

We also need the subclass $\mathscr{W}$ of $\mathscr{R}$ of functions $\kappa : \mathsf{R}^d \to \mathsf{R}^1_{++}$ defined as follows

DEFINITION. $\kappa \in \mathscr{W}$ if $\kappa \in \mathscr{R}$,

$$\kappa(x) = \prod_1^d \kappa_j(x_j) \quad \text{and} \quad \kappa_j(0) = 1, \ j = 1, 2, \ldots, d. \tag{2.6}$$

Note that if $m_1 \in R[k]$ and $m_2 \in R[k]$ then $\max(m_1, m_2) \in R[k]$, and if

$$\kappa_j(x_j) = 1, \ x_j \leqq 0, \ \kappa_j \nearrow, \ \frac{D^+ \kappa_j(x_j)}{\kappa_j(x_j)} \searrow, \ x_j \geqq 0, \ j = 1, 2, \ldots, d, \tag{2.7}$$

then $\kappa(x) = \prod_{j=1}^d \kappa_j(x_j)$ belongs to $\mathscr{W}$.

Note also that if $\kappa \in \mathscr{W}$ then $\kappa(-x)$ and $\kappa(|x|)$ belong to $\mathscr{W}$,

$$\frac{1}{\kappa} \in R[\kappa], \quad \max_{j=1,2,\ldots,d} \frac{1}{\kappa_j(x_j)} \in R[\kappa],$$

and $\kappa \in R[\kappa(-x)]$.

Let $a \in \mathsf{R}^d_+$, $b \in \mathsf{R}^d_+$ and

$$q(x) = q_{a,b}(x) = e^{a \max(x,0) + b \max(-x,0)}. \tag{2.8}$$

Then $q \in \mathscr{W}$ and

$$q_j(x_j) = e^{a_j x_j}, \ x_j \geqq 0, \ q_j(x_j) = e^{-b_j x_j}, \ x_j < 0, \ j = 1, 2, \ldots, d.$$

## 3. Some preliminary lemmas.

First, three lemmas concerning the class $R[k]$ will be proved.

LEMMA 1. *Let $q = q_{a,b}$ be defined by* (2.8) *and let $w \in \mathscr{R}$. Let $m(x)$, $x \geqq X$, belong to $R[qw]$. Let $\alpha \in \mathsf{R}^d$, $0 \leqq \alpha \leqq a$, and*

$$m^*(x) = e^{\alpha \max(X - x, 0)} m(\max(x, X)), \quad x \in \mathsf{R}^d.$$

*Then $m^* \in R[qw]$.*

PROOF. Letting $x^* = \max(x, X)$ and $x^{(0)} = \max(X-x, 0)$, we have $m^*(\eta) = e^{\alpha\eta^{(0)}} m(\eta^*)$. By assumption $m(\eta^*) \leqq q(\xi^*-\eta^*)w(\xi^*-\eta^*)m(\xi^*)$. Therefore (2.5) is satisfied with $m$ replaced by $m^*$ and $k = qw$ if we can prove

$$e^{\alpha(\eta^{(0)}-\xi^{(0)})} q(\xi^*-\eta^*)w(\xi^*-\eta^*) \leqq q(\xi-\eta)w(\xi-\eta), \quad \xi \in \mathrm{R}^d, \quad \eta \in \mathrm{R}^d. \tag{3.1}$$

Using the definition of $\mathscr{R}$ we find that (3.1) is satisfied if, for $\xi \in \mathrm{R}^d$, $\eta \in \mathrm{R}^d$ and $j = 1, 2, \ldots, d$,

$$e^{\alpha_j(\eta_j^{(0)}-\xi_j^{(0)})} q_j(\xi_j^*-\eta_j^*) \leqq q_j(\xi_j-\eta_j) \tag{3.2}$$

and

$$\xi_j^*-\eta_j^* = 0 \quad \text{or} \quad \xi_j-\eta_j = t_j(\xi_j^*-\eta_j^*) \quad \text{for some } t \geqq 1. \tag{3.3}$$

Choose $\xi$ and $\eta$ and denote the left hand side of (3.2) by $\varphi_j$. Consider for instance the case $\eta_j \leqq X_j \leqq \xi_j$. Then $\xi_j-\eta_j \geqq \xi_j-X_j = \xi_j^*-\eta_j^* \geqq 0$ and (3.3) is satisfied. Furthermore,

$$\eta_j^{(0)} - \xi_j^{(0)} = X_j - \eta_j \geqq 0$$

and

$$\varphi_j = e^{\alpha_j(X_j-\eta_j)+a_j(\xi_j-X_j)} \leqq e^{a_j(\xi_j-\eta_j)} = q_j(\xi_j-\eta_j).$$

Thus (3.2) is satisfied. The remaining cases may be treated similarly. Thus (3.1) holds true and Lemma 1 is proved.

We will now prove the following fundamental Lemma 2. For $d = 1$ the result of Lemma 2 is easily obtained and for $d = 1$ and $\Phi$ bounded it is used earlier (see [5, p. 87–88]). The notation $\xi_Y^I$ is introduced in (2.2).

LEMMA 2. *Let $b$ and $Y$ be constants in $\mathrm{R}_+^d$. Let $\Phi\colon \mathrm{R}^d \to \mathrm{R}^1$ and suppose $\Phi(x)e^{-bx}$, $x \geqq Y$, bounded. Let $k \in \mathscr{R}$, $k(x)e^{bx} \leqq c$, $x \leqq 0$, and let $m(x)$, $x \geqq Y$, belong to $R[k]$.*

*Then there exists a majorant $\mu(x)$, $x \geqq Y$, of $\Phi(x)$, $x \geqq Y$, such that $\mu/m \nearrow$,*

$$\mu \in R[k(x)k(\max(-x, 0))e^{b\max(-x,0)}] \tag{3.4}$$

*and $\mu$ has the following property: There exist $I$ and $J$ satisfying* (2.1) *such that*

(I) *For some constant $C$ and for every $x \geqq Y$ there exists $\xi$, $\xi \geqq x$, satisfying*

$$\frac{\mu(x)}{m(x)} \leqq C\frac{\mu(\xi_Y^I)}{m(\xi_Y^I)}. \tag{3.5}$$

(II) *Either* $J = \emptyset$ *or there exists a sequence* $(x^{(n)})_1^\infty$ *in* $\{x \mid x \geqq Y\}$ *such that* $x_j^{(n)} \nearrow \infty,\ n \to \infty,\ j \in J$ *and*

$$\overline{\lim_{n\to\infty}} \frac{|\Phi(x^{(n)})|}{\mu(x^{(n)})} \geqq \frac{1}{2c}. \tag{3.6}$$

PROOF. If $\Phi(x) = 0,\ x \geqq Y$, we may choose $\mu = m$ and there is nothing to prove. Suppose $\sup_{y \geqq Y} |\Phi(y)| > 0$. Let

$$\varphi(x) = e^{bx} \sup_{y \geqq x} |\Phi(y)| e^{-by},\quad x \geqq Y,$$

and

$$\mu(x) = m(x) \sup_{Y \leqq y \leqq x} \frac{\varphi(y)}{m(y)},\quad x \geqq Y.$$

Then $\mu > 0$, $\mu \geqq |\Phi|$ and $\mu/m \nearrow$. We will prove that for $\xi \geqq Y$, $\eta \geqq Y$

$$\mu(\eta) \leqq \mu(\xi) k(\xi - \eta) k(\max(\eta - \xi, 0)) e^{b\max(\eta - \xi, 0)}. \tag{3.7}$$

Choose $\xi$ and $\eta$. Introducing $I$ and $J$ by (2.1) and letting

$$E_I = \{y \in \mathbf{R}^d \mid Y \leqq y \leqq \eta,\ y_j \leqq \xi_j \text{ for } j \in I \text{ and } \xi_j < y_j \text{ for } j \in J\}, \tag{3.8}$$

then, for some $I$,

$$\sup_{Y \leqq y \leqq \eta} \frac{\varphi(y)}{m(y)} = \sup_{y \in E_I} \frac{\varphi(y)}{m(y)}.$$

Now

$$0 \leqq y - y_\xi^J = \max(y - \xi, 0),\quad y \in E_I. \tag{3.9}$$

Using (3.9) and $\varphi(y)e^{-by} \searrow$ we find

$$\varphi(y) \leqq e^{b\max(\eta - \xi, 0)} \varphi(y_\xi^J),\quad y \in E_I.$$

Hence

$$\sup_{y \in E_I} \frac{\varphi(y)}{m(y)} \leqq e^{b\max(\eta - \xi, 0)} \sup_{y \in E_I} \frac{\varphi(y_\xi^J)}{m(y)}.$$

Furthermore,

$$\sup_{Y \leqq y \leqq \xi} \frac{\varphi(y)}{m(y)} \geqq \sup_{Y \leqq y \leqq \xi} \frac{\varphi(y_\xi^J)}{m(y_\xi^J)} = \sup_{y \in E_I} \frac{\varphi(y_\xi^J)}{m(y_\xi^J)}.$$

Thus we have proved

$$\frac{\mu(\eta)}{\mu(\xi)} \leqq \frac{m(\eta)}{m(\xi)} e^{b\max(\eta-\xi,0)} \sup_{y\in E_I} \frac{m(y_\xi^J)}{m(y)}.$$

If we further use $m \in R[k]$ and (3.9) we find

$$\frac{\mu(\eta)}{\mu(\xi)} \leqq k(\xi-\eta)e^{b\max(\eta-\xi,0)}k(\max(\eta-\xi).0),$$

which proves (3.7). Thus (3.4) is satisfied.

Let $\mathscr{P}$ denote the set of all subsets of $\{1,2,\dots,d\}$. Let $Q$ denote the set of subsets $U$ of $\mathscr{P}$ with the property that $I \in U$ implies that all subsets of $I$ belong to $U$.

The last property of $\mu$ will follow by induction when we have proved the following

PROPOSITION. *Let $U \in Q$, $U \neq Q$. If* (I) *holds true and* (II) *is false for every $I \in U$, then there exists $U_1 \in Q$, $U \subset U_1$, $U \neq U_1$ such that* (I) *holds true for every $I \in U_1$.*

PROOF. Let $V = Q \setminus U$. Then $\varnothing \notin V$. It is sufficient to prove that there exists $I_0 \in V$, such that (I) holds true for $I = I_0$. Let $U = \{I_1,\dots,I_n\}$ and introduce $J_\nu$ as usual, i.e. $I_\nu \cup J_\nu = \{1,2,\dots,d\}$, $I_\nu \cap J_\nu = \varnothing$, $\nu = 1,2,\dots,n$. Since (II) is false for $J = J_\nu$, $\nu = 1,2,\dots,n$, by assumption, there exists $A \geqq \|Y\|_\infty$, such that, letting

$$E = \bigcup_{\nu=1}^{n} \left\{x \mid \min_{j\in J_\nu} x_j > A\right\} \tag{3.10}$$

we have

$$|\Phi(x)| < 2^{-3/4}c^{-1}\mu(x), \quad x \in E. \tag{3.11}$$

Choose

$$y \in \complement E = \bigcap_{\nu=1}^{n} \left\{x \mid \min_{j\in J_\nu} x_j \leqq A\right\}.$$

Then there exist $j_\nu \in J_\nu$ such that $y_{j_\nu} \leqq A$, $\nu = 1,2,\dots,n$. Let $I = I_\nu = \{j_1,\dots,j_n\}$. Then $I \in V$ and $\max_{j\in I} y_j \leqq A$. Letting

$$a_I = \left\{x \mid \max_{j\in I} x_j \leqq A\right\} \tag{3.12}$$

we thus have

$$\complement E \subset \bigcup_{I \in V} a_I. \tag{3.13}$$

Let us now prove

$\alpha$) *For every* $x \geqq Y$ *there exists* $\xi$, $\xi \geqq x$, *such that*

$$\frac{\mu(\xi)}{m(\xi)} = \sup_{\substack{Y \leqq y \leqq \xi \\ y \in \complement E}} \frac{\varphi(y)}{m(y)}. \tag{3.14}$$

First, we prove the following auxiliary result

$\alpha_1$) *If*

$$\frac{\mu(\zeta)}{m(\zeta)} = \sup_{\substack{Y \leqq y \leqq \zeta \\ y \in E}} \frac{\varphi(y)}{m(y)} \tag{3.15}$$

*then there exists* $\eta$, $\eta > \zeta$, *such that*

$$\mu(\zeta)e^{-b\zeta} \leqq 2^{-1/2}\mu(\eta)e^{-b\eta}. \tag{3.16}$$

Using (3.15) and the definition of $\varphi$ we find $v$ and $\eta$, $v \in E$, $Y \leqq v \leqq \zeta$, $\eta \geqq v$, such that

$$\frac{\mu(\zeta)}{m(\zeta)} \leqq 2^{1/8}\frac{\varphi(v)}{m(v)} \leqq 2^{1/4}\frac{|\Phi(\eta)|}{m(v)}e^{-b(\eta - v)}.$$

Now $\eta \in E$ since $v \in E$ and $\eta \geqq v$. Using (3.11) with $x = \eta$ we find

$$\frac{\mu(\zeta)}{m(\zeta)} \leqq \frac{2^{-1/2}}{c}\frac{\mu(\eta)}{m(v)}e^{-b(\eta - v)}. \tag{3.17}$$

The inequalities $m(\eta) \leqq cm(v)e^{b(\eta - v)}$ and $m(\zeta) \leqq cm(v)e^{b(\zeta - v)}$ follow from $m \in R[k]$ and $k(x) \leqq ce^{-bx}$, $x \leqq 0$. Combining the first of these inequalities with (3.17) and using $\mu/m \nearrow$ we find $\eta > \zeta$. Combining the second inequality with (3.17), we have proved (3.16). Thus $\alpha_1$) is proved.

We will now give an indirect proof of $\alpha$). Choose $x$ and suppose (3.14) false for every $\xi \geqq x$. Then (3.15) holds true for every $\zeta \geqq x$. Let $x = x^{(1)}$. Repeated applications of $\alpha_1$) with $\zeta = x^{(n)}$, $\eta = x^{(n+1)}$, $n = 1, 2, \ldots$, yield

$$\mu(x)e^{-bx} \leqq 2^{-n/2}\mu(x^{(n+1)})e^{-bx^{(n+1)}}, \quad n = 1, 2, \ldots, \tag{3.18}$$

where $x^{(n)} < x^{(n+1)}$, $n = 1, 2, \ldots$.

Using the definition of $\mu$, $m \in R[k]$ and $k(x) \leqq ce^{-bx}$, $x \leqq 0$, it is easy to verify that

$$\sup_{y \geqq Y} \mu(y)e^{-by} \leqq c \sup_{y \geqq Y} |\Phi(y)| e^{-by} < \infty.$$

Thus (3.18) is impossible, (3.14) holds true and $\alpha$) is proved.

Remembering that $\mu/m \nearrow$ the following result follows. There exists $\xi^0 = \xi^0(x)$ such that

$$\frac{\mu(x)}{m(x)} \leqq \sup_{\substack{Y \leqq y \leqq \xi^0 \\ y \in \complement E}} \frac{\varphi(y)}{m(y)}, \quad \xi^0 = \xi^0(x) \geqq x, \quad x \geqq Y. \tag{3.19}$$

Let, for $I \in V$, $a_I$ be defined by (3.12), choose $\xi^{(0)} = \xi^{(0)}(x)$ according to (3.19) and let

$$W_I = \left\{ x \,\middle|\, \frac{\mu(x)}{m(x)} \leqq \sup_{\substack{Y \leqq y \leqq \xi^{(0)} \\ y \in a_I}} \frac{\varphi(y)}{m(y)} \right\}.$$

Then

$$\bigcup_{I \in V} W_I = \{x \mid x \geqq Y\}$$

according to (3.19) and (3.13). Therefore at least one of the sets $W_I$, say $W_{I_0}$, has the property that

$$W_{I_0} \cap \{x \mid x \geqq X\} \neq \emptyset \quad \text{for every } X.$$

Introduce $\xi = \xi(x)$ as follows. Let $\xi(x) = \xi^0(x)$, $x \in W_{I_0}$. If $x \notin W_{I_0}$ choose $x_0$, $x_0 > x$, $x_0 \in W_{I_0}$ and let $\xi(x) = \xi^0(x_0)$. Then

$$\frac{\mu(x)}{m(x)} \leqq \sup_{\substack{Y \leqq y \leqq \xi \\ y \in a_{I_0}}} \frac{\varphi(y)}{m(y)}, \quad \xi = \xi(x) \geqq x, \quad x \geqq Y. \tag{3.20}$$

Using $\varphi(y)e^{-by} \searrow$, the definition of $a_{I_0}$ and $m \in R[k]$ we find that the right hand side of (3.20) is majorized by

$$C \sup_{Y \leqq y \leqq \xi} \frac{\varphi(y_Y^{I_0})}{m(y_Y^{I_0})} = \frac{\mu(\xi_Y^{I_0})}{m(\xi_Y^{I_0})},$$

where $C = k(AI - Y)e^{b(AI - Y)}$. Thus, we have proved that for every $x$, $x \geqq Y$, there exists $\xi$, $\xi \geqq x$, satisfying

$$\frac{\mu(x)}{m(x)} \leqq C \frac{\mu(\xi_Y^{I_0})}{m(\xi_Y^{I_0})},$$

i.e. (I) holds true for $I = I_0$. Since $I_0 \in V$ we have proved the proposition.

We are now ready to prove the last property of $\mu$. Suppose, to prove the contrary, that (I) and (II) are not satisfied for any $I$. Since (I) is trivially satisfied for $I = \varnothing$, $U = \{\varnothing\}$, using the Proposition repeatedly we find that (I) is satisfied for every $I \in Q$. Thus (I) is satisfied for $I = \{1,2,\dots,d\}$. Since (II) is trivially satisfied for $J = \varnothing$ we have obtained a contradiction.

This concludes the proof of Lemma 2.

The following simple variant of Lemma 2 will be used first in the proof of Theorem 3.

LEMMA 3. *Let $\varphi : \mathsf{R}^d \to \mathsf{R}^1$ be bounded, $\varphi \searrow$ and $\varphi(x) \to 0$, $x \to \infty$. Let $\kappa$ satisfy*

(3.21) $\kappa \in \mathscr{W}$, $\kappa \nearrow$, *and* $\kappa_j(x_j) \to \infty$, $x_j \to \infty$, $j = 1,2,\dots,d$.

*Then there exists a majorant $\mu$ of $\varphi$ such that $\mu \in R[\kappa]$ and $\mu(x) \to 0$, $x \to \infty$.*

PROOF. If $\varphi \equiv 0$ choose $\mu = 1/\kappa$. If $\varphi \not\equiv 0$ let

$$\mu(x) = \frac{1}{\kappa(x)} \sup_{y \leqq x} \varphi(y)\kappa(y), \quad x \in \mathsf{R}^d.$$

Then $\mu \geqq \varphi$ and $\kappa\mu \nearrow$. To prove that $\mu \in R[\kappa]$ choose $\xi$ and $\eta$ and let $E_I$ be defined by (3.8) with the restriction $y \geqq Y$ omitted. Then, for some $I$,

$$\sup_{y \leqq \eta} \varphi(y)\kappa(y) = \sup_{y \in E_I} \varphi(y)\kappa(y).$$

Using $\varphi \searrow$ and $\kappa \nearrow$ we find

$$\sup_{y \in E_I} \varphi(y)\kappa(y) \leqq \Big(\prod_{j \in J} \kappa_j(\eta_j)\Big) \sup_{y \leqq \xi} \Big(\varphi(y_\xi^J) \prod_{j \in I} \kappa_j(y_j)\Big).$$

Furthermore,

$$\sup_{y \leqq \xi} \varphi(y)\kappa(y) \geqq \sup_{y \leqq \xi} \varphi(y_\xi^J)\kappa(y_\xi^J) = \Big(\prod_{j \in J} \kappa_j(\xi_j)\Big) \sup_{y \leqq \xi} \Big(\varphi(y_\xi^J) \prod_{j \in I} \kappa_j(y_j)\Big).$$

Hence

$$\frac{\mu(\eta)}{\mu(\xi)} \leqq \frac{\kappa(\xi)}{\kappa(\eta)} \prod_{j \in J} \frac{\kappa_j(\eta_j)}{\kappa_j(\xi_j)} = \prod_{j \in I} \frac{\kappa_j(\xi_j)}{\kappa_j(\eta_j)}.$$

Using (2.3) we find

$$\mu(\eta) \leqq \mu(\xi) \prod_{j \in I} \kappa_j(\xi_j - \eta_j).$$

Now $\kappa_j(x_j) = 1,\ x_j \leqq 0,\ j = 1,2,\ldots,d$, and therefore the right hand side equals $\mu(\xi)\kappa(\xi - \eta)$. Thus $\mu \in R[\kappa]$.

It remains to prove that $\mu(x) \to 0,\ x \to \infty$. Choose $\varepsilon > 0$ and choose $X^{(0)}$ such that $\varphi(y) \leqq \varepsilon,\ y \geqq X^{(0)}$. Using $\kappa \nearrow$ we find

$$\frac{1}{\kappa(x)} \sup_{X^{(0)} \leqq y \leqq x} \varphi(y)\kappa(y) \leqq \varepsilon, \quad x \geqq X^{(0)}. \tag{3.22}$$

Let

$$a_{j,x} = \{y \in \mathsf{R}^d \mid y \leqq x,\ y_j \leqq X_j^{(0)}\},\ j = 1,2,\ldots,d,\ x \in \mathsf{R}^d.$$

Then

$$\frac{1}{\kappa(x)} \sup_{y \in a_{j,x}} \varphi(y)\kappa(y) \leqq \|\varphi\|_\infty \frac{\kappa_j(X_j^{(0)})}{\kappa_j(x_j)};\ j = 1,2,\ldots,d.$$

Let $A_x = \bigcup_{j=1}^d a_{j,x}$. Since $\kappa_j(x_j) \to \infty,\ x_j \to \infty,\ j = 1,2,\ldots,d$, there exists $X,\ X > X^{(0)}$ such that

$$\frac{1}{\kappa(x)} \sup_{y \in A_x} \varphi(y)\kappa(y) \leqq \varepsilon, \quad x \geqq X. \tag{3.23}$$

Now

$$A_x \cup \{y \mid X^{(0)} \leqq y \leqq x\} = \{y \mid y \leqq x\}$$

and (3.22) and (3.23) yield

$$\mu(x) = \frac{1}{\kappa(x)} \sup_{y \leqq x} \varphi(y)\kappa(y) \leqq \varepsilon, \quad x \geqq X.$$

Thus $\mu(x) \to 0,\ x \to \infty$, and the lemma is proved.

Let $\Phi\colon \mathsf{R}^d \to \mathsf{R}^1,\ m\colon \mathsf{R}^d \to \mathsf{R}^1_{++}$ and $t\colon \mathsf{R}^d \to \mathsf{R}^1_{++},\ t \searrow$. In Theorems 1 and 2 we introduce a Tauberian condition of the following type

$$\Phi(x+y) - \Phi(x) \geqq -m(x),\ \boldsymbol{0} \leqq y \leqq t(x)\boldsymbol{1},\ x \geqq x^{(0)}. \tag{3.24}$$

In the following lemma we will derive corresponding results for any $y \geqq \boldsymbol{0}$ provided that $m \in R[k]$.

LEMMA 4. *Let $t \searrow$, $m \in R[k]$ and let* (3.24) *be satisfied. Then*

$$\Phi(x+y) - \Phi(x) \geqq -\left(\frac{\|y\|_\infty}{t(x+y)} + 1\right)k(-y)m(x),\ y \geqq \boldsymbol{0},\ x \geqq x^0. \tag{3.25}$$

PROOF. Choose $x$ and $y$, $x \geqq x^{(0)}$, $y \geqq 0$. If $\|y\|_\infty \leqq t(x)$ there is nothing to prove. If $\|y\|_\infty > t(x)$, let $y = \eta$, $x = \xi = \xi^{(1)}$ and define recursively

$$\xi^{(n+1)} = \xi^{(n)} + \frac{t(\xi^{(n)})\eta}{\|\eta\|_\infty}, \quad n = 1, 2, \ldots .$$

Choose the integer $N$ such that

$$\sum_{n=1}^{N-1} t(\xi^{(n)}) < \|\eta\|_\infty \leqq \sum_{n=1}^{N} t(\xi^{(n)}).$$

Then

$$N < \frac{\|\eta\|_\infty}{t(\xi+\eta)} + 1. \tag{3.26}$$

Applying (3.24) with $x = \xi^{(n)}$, $y = t(\xi^{(n)})\eta/\|\eta\|_\infty$, $n = 1, 2, \ldots, N-1$, and $x = \xi^{(N)}$, $y = \xi + \eta - \xi^{(N)}$ and adding the inequalities thus obtained we get

$$\Phi(\xi+\eta) - \Phi(\xi) \geqq -\sum_{n=1}^{N} m(\xi^{(n)}).$$

Using $m \in R[k]$ we find

$$m(\xi^{(n)}) \leqq m(\xi)k(\xi - \xi^{(n)}) \leqq m(\xi)k(-\eta), \quad n = 1, 2, \ldots, N.$$

Therefore

$$\Phi(\xi+\eta) - \Phi(\xi) \geqq -Nk(-\eta)m(\xi).$$

Using the estimate (3.26) of $N$ we have proved (3.25).

## 4. A Tauberian theorem for positive kernels.

Introduce the class $\mathscr{E}$ of functions $t: \mathrm{R}^d \to \mathrm{R}^1_{++}$ as follows:

DEFINITION. $t \in \mathscr{E}$ if $t \searrow$ and

$$\lim_{x\to\infty} \frac{t(x + at(x))}{t(x)} = 1 \quad \text{for every } a \in \mathrm{R}^d.$$

The definition may appear odd. However, such a class has been used earlier in Tauberian theorems (see [7, p. 167]).

We will prove the following theorem which is a modification of a result in $\mathrm{R}^1$ proved earlier ([5, Lemma 6, p. 84]).

THEOREM 1. *Let* $K: \mathsf{R}^d \to \mathsf{R}^1_+$, $K \not\equiv 0$, $e^{\varepsilon|u|}K(u) \in L^1(\mathsf{R}^d)$ *for some* $\varepsilon > 0$. *Let* $m \in R[ce^{\lambda|x|}]$ *for some* $c \geqq 1$ *and some* $\lambda > 0$. *Let* $\Phi: \mathsf{R}^d \to \mathsf{R}^1$ *and suppose for some* $s$, $\mathit{0} \leqq s \leqq \lambda\mathit{1}$

$$(4.1) \quad \frac{\Phi(x)e^{-s\max(X-x,\mathit{0})}}{m(\max(x,X))} \text{ bounded in } \mathsf{R}^d \setminus \{x \mid x \geqq X\} \text{ for every } X \geqq X^{(0)}.$$

*Let*

$$(4.2) \quad t \in \mathscr{E},\ t(x) \to 0,\ x \to \infty,\ \text{and}\ t(x) \geqq |x|e^{-\lambda|x|},\ x \geqq X^{(0)},$$

*and*

$$(4.3) \qquad \Phi(x+y) - \Phi(x) \geqq -m(x),\ \mathit{0} \leqq y \leqq t(x)\mathit{1},\ x \geqq X^{(0)}.$$

*Suppose that for some* $\omega > 0$ *and for every* $\tau: \mathsf{R}^d \to \mathsf{R}^1_+$ *satisfying* $\tau(x)/t(x) \to \omega$, $x \to \infty$

$$(4.4) \qquad \overline{\lim_{x\to\infty}} \frac{1}{m(x)} \left| \int_{\mathsf{R}^d} \Phi(x - u\tau(x))K(u)du \right| \leqq 1.$$

*Then*

$$(4.5) \qquad \overline{\lim_{x\to\infty}} \frac{|\Phi(x)|}{m(x)} \leqq C(1+\omega),$$

*where* $C = C(c, \varepsilon, K)$.

PROOF. It is easily seen, by using Lemma 4 and the properties of the class $\mathscr{E}$, that it suffices to prove the theorem for $\omega = 1$. We may also suppose $X^{(0)} \geqq \mathit{0}$ and $\int_{\mathsf{R}^d} K(u)du = 1$. This is obviously no restriction.

Let, for $a \in \mathsf{R}^d_{++}$, $E_a = \{x \mid x \in \mathsf{R}^d,\ -a < x < a\}$. Let $\delta = 10^{-1}$ and choose $a$, $a = a(c, \varepsilon, K)$, such that

$$(4.6) \qquad \int_{\mathsf{R}^d \setminus E_a} e^{\varepsilon|u|} K(u)du \leqq \frac{\delta}{c^3}.$$

Then

$$(4.7) \qquad \int_{E_a} K(u)du > 1 - \delta.$$

Introduce the notations

$$\tau_n(x) = t\big(x + (-1)^n t(x)a\big),$$

and

$$I_n(x) = \int_{\mathbf{R}^d} \Phi\big(x - u\tau_n(x)\big)K(u)\,du, \quad n = 1,2.$$

We will first prove that there exists $x^{(1)}$ such that

$$|I_n(x + (-1)^{n-1} t(x)a)| \leqq 2c\,m(x), \quad x \geqq x^{(1)}, \quad n = 1,2. \tag{4.8}$$

Observing that $t \in \mathscr{E}$ implies that $\tau_n(x)/t(x) \to 1$, $x \to \infty$, $n = 1,2,$ and using the assumption (4.4) we find $x^{(0)} \geqq X^{(0)}$ such that

$$|I_n(x)| \leqq \sqrt{2}\,m(x), \quad x \geqq x^{(0)}, \quad n = 1,2. \tag{4.9}$$

Choose $x^{(1)} \geqq x^{(0)}$ such that

$$e^{4\lambda t(x)|a|} \leqq 2, \qquad x \geqq x^{(1)}, \tag{4.10}$$

$$\frac{t(x)}{t\big(x + 2at(x)\big)} \leqq 2, \quad x \geqq x^{(1)}, \tag{4.11}$$

$$4\lambda t(x) \leqq \varepsilon, \qquad x \geqq x^{(1)}. \tag{4.12}$$

Using $m \in R[ce^{\lambda|x|}]$ and (4.10) we obtain

$$m\big(x + (-1)^{n-1} t(x)a\big) \leqq c2^{1/4} m(x), \quad x \geqq x^{(1)}.$$

Thus (4.8) follows from (4.9).

Let $C$ denote a positive constant depending on $c$, $\varepsilon$ and $K$, not necessarily the same one each time it occurs. Let

$$J_n(x) = \int_{\mathbf{R}^d \setminus E_a} \Phi\big(x + (-1)^{n-1} t(x)a - t(x)u\big)K(u)\,du, \quad n = 1,2. \tag{4.13}$$

We will now prove

$$\Phi(x)\int_{E_a} K(u)\,du \leqq Cm(x) - J_1(x), \quad x \geqq x^{(1)}. \tag{4.14}$$

Applying Lemma 4 with $k(x) = ce^{\lambda|x|}$ and using $t \searrow$, (4.10) and (4.11) we obtain from the Tauberian condition (4.3)

$$\Phi(x+y) - \Phi(x) \geqq -Cm(x), \quad 0 \leqq y \leqq 2t(x)a, \quad x \geqq x^{(1)}. \tag{4.15}$$

Choose $x \geqq x^{(1)}$, and let $t = t(x)$. Letting $y = ta - tu$ in (4.15), multiplying with $K(u)$ and integrating over $E_a$ we get

$$\Phi(x) \int_{E_a} K(u)du \leqq Cm(x) + \int_{E_a} \Phi(x+ta-tu)K(u)du.$$

The last term is $I_1(x+ta) - J_1(x)$. Using (4.8) with $n = 1$ we have proved (4.14).

We will now prove that $\Phi(x)e^{-2\lambda|x|}$ is bounded. Applying Lemma 4 with $x = x^{(1)}$ and using (4.3), (4.2), and (4.11) we find that $\Phi(x)e^{-2\lambda|x|}$, $x \geqq x^{(1)}$, is bounded from below. Since $m(\max(x, x^{(1)})) \in R[ce^{\lambda|x|}]$ according to Lemma 1, $m(\max(x, x^{(1)}))e^{-\lambda|x|}$ is bounded. The assumption (4.1) thus yields that $\Phi(x)e^{-2\lambda|x|}$ is bounded in $\mathrm{R}^d \setminus \{x \mid x \geqq x^{(1)}\}$. Hence $\Phi(x)e^{-2\lambda|x|}$ is bounded from below in $\mathrm{R}^d$. Thus, for some $A$, $\Phi(x) \geqq -Ae^{2\lambda|x|}$, $x \in \mathrm{R}^d$. Inserting in (4.13) and using (4.10) and (4.12) we find

$$J_1(x) \geqq -2^{1/2}Ae^{2\lambda|x|} \int_{\mathrm{R}^d \setminus E_a} e^{\varepsilon|u|}K(u)du, \quad x \geqq x^{(1)}. \tag{4.16}$$

Combining (4.16) and (4.14) and using (4.6) and (4.7) we obtain that $\Phi(x)e^{-2\lambda|x|}$ is bounded from above for $x \geqq x^{(1)}$. Thus we have proved that $\Phi(x)e^{-2\lambda|x|}$, $x \in \mathrm{R}^d$, is bounded.

Let $Y = x^{(1)} + 2t(x^{(1)})a$. We will now prove

$$\Phi(x) \int_{E_a} K(u)du \geqq -Cm(x) - J_2(x), \quad x \geqq Y. \tag{4.17}$$

Replacing $x$ by $x - y$ in (4.15) and using $t \searrow$, $m \in R[ce^{\lambda|x|}]$ and (4.10) we find

$$\Phi(x) \geqq \Phi(x-y) - Cm(x), \quad 0 \leqq y \leqq 2t(x)a, \quad x \geqq Y. \tag{4.18}$$

Choosing $x \geqq Y$, letting $t = t(x)$ and $y = ta + tu$ in (4.18), multiplying with $K(u)$ and integrating over $E_a$ we get

$$\Phi(x) \int_{E_a} K(u)du \geqq -Cm(x) + \int_{E_a} \Phi(x-ta-tu)K(u)du.$$

The last term is $I_2(x-ta)-J_2(x)$. Using (4.8) with $n=2$ we have proved (4.17).

Combining (4.7), (4.14), and (4.17) we obtain

$$-Cm(x)-J_2(x) \leqq (1-\delta)\Phi(x) \leqq Cm(x)-J_1(x), \quad x \geqq Y. \tag{4.19}$$

We will use this inequality to estimate $\overline{\lim}_{x\to\infty} |\Phi(x)|/m(x)$. Applying Lemma 2 with $k(x)=ce^{\lambda|x|}$, $b=2\lambda 1$ we find a majorant $\mu(x)$, $x \geqq Y$, of $\Phi$ such that $\mu \in R[v]$, where $v(x)=c^2e^{4\lambda|x|}$. Let for $x \in \mathrm{R}^d$

$$\mu_1(x)=\mu(\max(x,Y))$$

and

$$m^*(x)=e^{\lambda|\max(Y-x,0)|}m(\max(x,Y)).$$

Then $\mu_1$ and $m^*$ belong to $R[v]$ according to Lemma 1. From the assumption (4.1) it follows that $|\Phi(x)| \leqq Am^*(x)$, $x \in \mathrm{R}^d \setminus \{x \mid x \geqq Y\}$ for some $A$. Letting $\mu^*=\max(\mu_1, Am^*)$ we find $\mu^* \in R[v]$ and

$$|\Phi(x)| \leqq \mu^*(x), \quad x \in \mathrm{R}^d. \tag{4.20}$$

Proceeding as in the proof of (4.16) and using (4.20), (4.6), (4.10), and (4.12) we obtain

$$|J_n(x)| \leqq \frac{2\delta\mu^*(x)}{c}, \quad x \geqq Y, \ n=1,2. \tag{4.21}$$

Combining (4.19) and (4.21) we have proved

$$(1-\delta)|\Phi(x)| \leqq Cm(x)+\frac{2\delta\mu^*(x)}{c}, \quad x \geqq Y. \tag{4.22}$$

So far we have not used the last property in Lemma 2 of the function $\mu$. We will now use this property and the inequality (4.22) to prove that $\mu/m$ is bounded.

Suppose, to prove the contrary, that $\mu(x)/m(x)$, $x \geqq Y$ is unbounded. Then (I) and (II) of Lemma 2 hold true for some $J \neq \varnothing$ and some sequence $(x^{(n)})_1^\infty$, $x^n \geqq Y$, $n=1,2\ldots$. Using (3.5) we find that $\mu(x_Y^J)/m(x_Y^J)$, $x \geqq Y$, is unbounded. Since $\mu/m \nearrow$ and $x_j^{(n)} \nearrow \infty$, $n\to\infty$, $j \in J$, this implies that $\mu(x^{(n)})/m(x^{(n)}) \to \infty$, $n \to \infty$. Therefore $\mu^*(x^{(n)}) = \mu(x^{(n)})$, $n \geqq n_0$. By dividing both sides of (4.22) by $\mu(x)$, choosing

$x = x^{(n)}$, $n = 1, 2, \ldots$, taking the upper limit as $n \to \infty$ and using (3.6) and (4.21), we obtain

$$\frac{1-5\delta}{2c} \leqq C \varlimsup_{n\to\infty} \frac{m(x^{(n)})}{\mu(x^{(n)})}.$$

Remembering that $\delta = 10^{-1}$ we have proved

$$\varliminf_{n\to\infty} \frac{\mu(x^{(n)})}{m(x^{(n)})} \leqq C. \tag{4.23}$$

Let

$$\psi(n) = \frac{\mu\big((x^{(n)})_Y^I\big)}{m\big((x^{(n)})_Y^I\big)}.$$

Then $\psi(n)\nearrow$, $\psi(n) \leqq \mu(x^{(n)})/m(x^{(n)})$, and (4.23) yields $\psi(n) \leqq C$, $n = 1, 2, \ldots$. Since $\mu/m \nearrow$ and $x_j^{(n)} \nearrow \infty$, $n \to \infty$, $j \in J$, this implies

$$\frac{\mu(x_Y^I)}{m(x_Y^I)} \leqq C, \quad x \geqq Y.$$

We have obtained a contradiction and conclude that $\mu/m$ and hence $\Phi/m$ are bounded in $\{x \mid x \geqq Y\}$.

If

$$\varlimsup_{x\to\infty} \frac{|\Phi(x)|}{m(x)} = B < \infty$$

then it is easy to verify that

$$\varlimsup_{x\to\infty} \frac{|J_n(x)|}{m(x)} \leqq \frac{B\delta}{c}, \quad n = 1, 2.$$

The result $B \leqq C$ then follows directly from (4.19). Thus Theorem 1 is proved.

## 5. Tauberian theorems with conditions on the Fourier transform of the kernel.

First, we introduce some notations. A function $\psi: \mathsf{R}^d \to \mathsf{R}_+^1$, $\psi \not\equiv 0$, is called *almost increasing* if there exists a positive constant $B$ such that

$$\psi(\eta) \leqq B\psi(\xi), \quad \eta \leqq \xi, \tag{5.1}$$

for all $\xi,\eta$ where $\psi$ is defined. Let $\alpha \in Z^d_+$, $\gamma \in Z^d_+$, $x \in R^d$ and

$$x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}, \quad \alpha! = \prod_{j=1}^d \alpha_j!, \quad \binom{\alpha}{\gamma} = \frac{\alpha!}{\gamma!(\alpha-\gamma)!}, \quad \gamma \leqq \alpha.$$

We will now introduce log convex sequences in a way similar to the log convex sequences introduced in [5, Section 2.1].

Let $P_j(n)$, $n \in Z_+$, $j = 1,2,\ldots,d$, satisfy $P_j(0) = 1$, $\log P_j(n)$ convex in $n$,

$$\text{(5.2)} \qquad \frac{P_j(n)}{n!} \text{ almost increasing and } n^{-1}P_j(n)^{1/n} \to \infty, \quad n \to \infty.$$

For $\alpha \in Z^d_+$ let

$$P(\alpha) = \prod_{j=1}^d P_j(\alpha_j).$$

Let

$$\text{(5.3)} \qquad p(x) = \sup_{\alpha \in Z^d_+} \frac{|x^\alpha|}{P(\alpha)}, \quad x \in R^d,$$

and

$$\text{(5.4)} \qquad h_P(x) = \sum_{\alpha \in Z^d_+} \frac{|x^\alpha|}{P(\alpha)}, \quad x \in R^d.$$

Then

$$\text{(5.5)} \qquad h_P(\theta_1 x) \leqq C(P,\theta_1,\theta_2)p(\theta_2 x), \quad \theta_1 < \theta_2, \quad x \in R^d.$$

Let

$$\delta_j(0) = 0, \quad \delta_j(n) = \frac{P_j(n)}{P_j(n-1)}, \quad n = 1,2,\ldots, \quad j = 1,2,\ldots,d,$$

and let, for $\theta > 0$, $\alpha \in Z^d_+$,

$$\text{(5.6)} \qquad \Omega_{\alpha,\theta} = \{x \mid \delta_j(\alpha_j) \leqq \theta|x_j| \leqq \delta_j(\alpha_j+1), \ j = 1,2,\ldots,d\}.$$

Then

$$\text{(5.7)} \qquad p(\theta x) = \frac{\theta^{|\alpha|}|x^\alpha|}{P(\alpha)}, \quad x \in \Omega_{\alpha,\theta}.$$

For the above results see [5, p. 68]. The condition $n^{-1}P_j(n)^{1/n} \to \infty$, $n \to \infty$, $j = 1,2,\ldots,d$, was not introduced in [5]. Note that this condition implies that for every $\varepsilon > 0$, $h_P(x)e^{-\varepsilon|x|}$ is bounded in $\mathbf{R}^d$.

Let $a \in \mathbf{R}^d_+$, $b \in \mathbf{R}^d_+$ be given. Let $I = \{j \mid a_j + b_j > 0\}$. By renumbering the variables, let $I = \{1,2,\ldots,k\}$ and $J = \{k+1,k+2,\ldots,d\}$. Let

$$\begin{aligned}
\zeta \ &= \xi + i\eta \in \mathbf{C}^d,\\
\zeta' &= \xi' + i\eta' = (\zeta_1,\zeta_2,\ldots,\zeta_k),\\
\zeta'' &= \xi'' + i\eta'' = (\zeta_{k+1},\zeta_{k+2},\ldots,\zeta_d),\\
\zeta \ &= (\zeta',\zeta'')
\end{aligned}$$

and introduce the notations

$$\begin{aligned}
B'(a,b) &= \{\eta' \in \mathbf{R}^k \mid -b_j < \eta'_j < a_j,\ j = 1,2,\ldots,k\}\\
B(a,b) &= \{\eta \in \mathbf{R}^d \mid \eta' \in B'(a,b),\ \eta'' = 0\}\\
T'(a,b) &= \{\zeta' \in \mathbf{C}^k \mid \eta' \in B'(a,b),\ \xi' \in \mathbf{R}^k\}\\
T(a,b) &= \{\zeta \in \mathbf{C}^d \mid \eta \in B(a,b),\ \xi \in \mathbf{R}^d\}.
\end{aligned}$$

If $a+b>0$, then $B'(a,b) = B(a,b)$ and $T'(a,b) = T(a,b)$. If $a+b=0$, then $T'(a,b) = \varnothing$, otherwise $T'(a,b)$ is a tube in $\mathbf{C}^k$ with base $B'(a,b)$.

Let $F\colon \mathbf{R}^d \to \mathbf{C}$ belong to $L^1$ and

$$\hat{F}(\xi) = \int_{\mathbf{R}^d} e^{-i\xi x} F(x)dx, \quad \xi \in \mathbf{R}^d.$$

Let $W\colon \mathbf{R}^1_+ \to \mathbf{R}^1_+$, $W \nearrow$ and introduce the class $\mathscr{A} = \mathscr{A}(W,P,a,b)$ of functions $F$ as follows.

Definition. $F \in \mathscr{A}(W,P,a,b)$ if $\hat{F}(\xi) \neq 0$, $\xi \in \mathbf{R}^d$, $1/\hat{F}(\xi) = f(\xi)$, $\xi \in \mathbf{R}^d$, and

a) If $a+b \neq 0$ then for almost all $\xi'' \in \mathbf{R}^{d-k}$ there exists a function $\varphi(\zeta') = f(\zeta',\xi'')$ analytic in $T'(a,b)$ and

$$\lim_{\substack{\eta' \to 0\\ \eta' \in B'(a,b)}} f(\xi' + i\eta', \xi'') = f(\xi) \ \text{ a.e. in } \mathbf{R}^d.$$

b) There exists a constant $C$ such that

$$\left\| \frac{D^\alpha f(\xi + i\beta)}{W(\|\xi\|_\infty)} \right\|_2 \leq CP(\alpha), \quad \alpha \in \mathbf{Z}^d_+, \ \beta \in B(a,b). \tag{5.8}$$

We use functions $W$ of the type

$$W(r) = \exp(e^{rv(r)}), \quad r \geqq r_0, \tag{5.9}$$

where

$$v \text{ decreasing and } v(r) \to 0, \quad r \to \infty, \tag{5.10}$$

and we let $v^{-1}$ denote the inverse function of $v$.

Theorem 2 below may be considered a variant of a theorem proved earlier ([5, Theorem 1], cf. [6]). This theorem was proved by using certain non-quasianalytic sequences and therefore it was necessary to impose a condition on the function $t$, introduced in the Tauberian condition (4.3), of the following type: As $x \to \infty$, $t(x)$ tends to zero as fast as

$$(\log|x|)^{-1}(\log\log|x|)^{-1} \cdots (\log \cdots \log|x|)^{-1-\delta}$$

for some $\delta > 0$. In Theorem 2 below we let an argument of Paley–Wiener type replace the non-quasianalytic sequences used in [5]. In doing so, we gain that the function $t$ may tend to zero arbitrarily slowly as $x$ tends to infinity. On the other hand, the condition (5.15), imposed in Theorem 2 below, is stronger than the following condition, which was sufficient in Theorem 1 in [5]: For some $\omega > 0$

$$\rho(x) W\left(\frac{1}{\omega t(x)}\right) \leqq m(x). \tag{5.11}$$

The functions $q_{a,b}$ and $h_P$ are defined by (2.8) and (5.4) respectively.

THEOREM 2. (1) *Let* $a \in \mathsf{R}^d_+$, $b \in \mathsf{R}^d_+$ *and* $F \in \mathscr{A}(W, P, a, b)$. *Let* $w \in \mathscr{R}$ *and let for some* $\theta$, $0 < \theta < 1$,

$$\frac{w(x)}{h_P(\theta x)} \in L^2(\mathsf{R}^d). \tag{5.12}$$

*Let* $m \in R[wq_{a,b}]$ *and let* $\Phi: \mathsf{R}^d \to \mathsf{R}^1$ *satisfy*

$$\frac{|\Phi| * |F|(x)}{w(-x)q_{a,b}(-x)} \text{ bounded.} \tag{5.13}$$

(2) *Let* $\Phi$ *and* $m$ *satisfy* (4.1) *for some* $s \geqq 0$, *let* $t$ *satisfy* (4.2) *for some* $\lambda > 0$ *and let* (4.3) *hold true. Let* $\rho \in R[wq_{a,b}]$ *and*

$$|\Phi * F(x)| \leqq \rho(x), \quad x \in \mathsf{R}^d. \tag{5.14}$$

*Suppose that for some* $\omega > 0$

$$\rho(x)W\big(v^{-1}(\omega t(x))\big) \leqq m(x), \quad x \geqq X^{(0)}. \tag{5.15}$$

*Then*

$$\Phi(x) = O\big(m(x)\big), \quad x \to \infty.$$

REMARK. If, in Theorem 2, $a = b = \mathbf{0}$ and (5.2) is replaced by $n!/P_j(n)$ sub-multiplicative and

$$\sum_{n=0}^{\infty} P_j(n)^{-1/n} < \infty, \quad j = 1, 2, \ldots, d,$$

then the conditions (5.9) and (5.10) may be omitted and (5.15) may be replaced by the weaker condition (5.11).

PROOF OF THEOREM 2. Let

$$\hat{G}(\xi) = \prod_{j=1}^{d} \exp\big(-(e^{\xi_j} + e^{-\xi_j})\big), \quad \xi \in \mathsf{R}^d,$$

and let $\hat{K} = \hat{G} * \hat{G}$. Then $\hat{K}$ is the Fourier transform of a function $K$, $K \geqq 0$. Furthermore, $\hat{K}$ is analytic in the tube $T(\frac{1}{2}\pi\mathbf{1}, \frac{1}{2}\pi\mathbf{1}) \subset \mathsf{C}^d$ and

$$|\hat{K}(\xi + i\eta)| \leqq \frac{2^d}{\prod_{j=1}^{d} \cos\eta_j} \exp\left(-\sum_{j=1}^{d} e^{|\xi_j|/2}\cos\eta_j\right), \quad \eta \in B(\tfrac{1}{2}\pi\mathbf{1}, \tfrac{1}{2}\pi\mathbf{1}). \tag{5.16}$$

Hence $\hat{K}(\xi + i\beta) \in L^2(\mathsf{R}^d)$, $\beta \in B(\frac{1}{2}\pi\mathbf{1}, \frac{1}{2}\pi\mathbf{1})$.

Thus $\hat{K} \in H^2\big(T(\frac{1}{2}\pi\mathbf{1}, \frac{1}{2}\pi\mathbf{1})\big)$, $\hat{K}(\xi + i\beta)$ is the Fourier transform of $e^{\beta u}K(u)$, $\beta \in B(\frac{1}{2}\pi\mathbf{1}, \frac{1}{2}\pi\mathbf{1})$ and Parseval's relation yields

$$e^{\beta u}K(u) \in L^2(\mathsf{R}^d), \quad -\tfrac{1}{2}\pi\mathbf{1} < \beta < \tfrac{1}{2}\pi\mathbf{1}.$$

It follows that

$$e^{\varepsilon|u|}K(u) \in L^1(\mathsf{R}^d), \quad \varepsilon < \tfrac{1}{2}\pi. \tag{5.17}$$

The inequality (5.16) further implies that for $\zeta = \xi + i\eta$

$$|\hat{K}(\zeta)| \leqq 2^{3d/2} \exp\left(-\frac{1}{\sqrt{2}} e^{\|\xi\|_\infty/2}\right), \quad \zeta \in T(\tfrac{1}{4}\pi\mathbf{1}, \tfrac{1}{4}\pi\mathbf{1}). \tag{5.18}$$

Using Cauchy's inequality and (5.18) we obtain

$$(5.19)\qquad |D^\alpha \hat{K}(\zeta)| \leqq 2^{3d/2}\alpha!\,(\tfrac{1}{8}\pi)^{|\alpha|}\exp\left(-\frac{1}{\sqrt{2}}e^{\|\xi\|_\infty/2}\right),\quad \zeta\in T(\tfrac{1}{8}\pi\mathbf{1},\tfrac{1}{8}\pi\mathbf{1}).$$

Let $\lambda_1 = 2+\max(\|a\|_\infty,\|b\|_\infty)$. Choose $r_1 \geqq r_0$ such that $r_1 \geqq (2^5\lambda_1)/\pi$, $r_1 v(r_1)\geqq 1$ and let $\tau_1 = 4/r_1$. Let, for $0<\tau\leqq\tau_1$, $K_\tau(x) = \tau^{-d}K(x\tau^{-1})$ and $\hat{U}_\tau(\zeta) = \hat{K}_\tau(\zeta)f(\zeta)$, $\zeta\in T(a,b)$.

We will first prove that there exists a constant $C$ such that, for $0<\tau\leqq\tau_1$ and $\beta\in B(a,b)$

$$(5.20)\qquad \|D^\alpha \hat{U}_\tau(\xi+i\beta)\|_2 \leqq CP(\alpha)W(v^{-1}(\tau/4)).$$

Let $C$ denote a positive constant independent of $\tau$ and $\beta$, not necessarily the same one each time it occurs. Using $\hat{K}_\tau(\zeta) = \hat{K}(\tau\zeta)$ and Leibniz' formula we find

$$(5.21)\qquad D^\alpha U_\tau(\zeta) = \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}D^\gamma \hat{K}(\tau\zeta)D^{\alpha-\gamma}f(\zeta).$$

If $0<\tau\leqq\tau_1$ and $\zeta\in T(a,b)$ then $\tau\zeta\in T(\tfrac{1}{8}\pi\mathbf{1},\tfrac{1}{8}\pi\mathbf{1})$ and (5.19) yields

$$(5.22)\qquad |D^\gamma \hat{K}(\tau\zeta)| \leqq 2^{3d/2}\gamma!\,2^{-|\gamma|}\exp\left(-\frac{1}{\sqrt{2}}e^{\tau\|\xi\|_\infty/2}\right),\quad \zeta\in T(a,b).$$

Let

$$\psi(\tau,r) = W(r)\exp\left(-\frac{1}{\sqrt{2}}e^{\tau r/2}\right)$$

and $\chi(\tau) = \sup_{r\geqq 0}\psi(\tau,r)$. The inequalities (5.21) and (5.22) yield

$$|D^\alpha \hat{U}_\tau(\xi+i\beta)| \leqq \chi(\tau)2^{3d/2}\alpha!\sum_{\gamma\leqq\alpha}\frac{2^{-|\gamma|}|D^{\alpha-\gamma}f(\xi+i\beta)|}{(\alpha-\gamma)!\,W(\|\xi\|_\infty)}.$$

Using Minkowsky's inequality and the assumption (5.8) we get

$$\|D^\alpha \hat{U}_\tau(\xi+i\beta)\|_2 \leqq C\chi(\tau)\alpha!\sum_{\gamma\leqq\alpha}2^{-|\gamma|}\frac{P(\alpha-\gamma)}{(\alpha-\gamma)!}.$$

Remembering that $P(\alpha)/\alpha!$ is almost increasing we find, after a summation over $\gamma$

(5.23) $$\|D^\alpha \hat{U}_\tau(\xi+i\beta)\|_2 \leqq CP(\alpha)\chi(\tau).$$

Now, $\psi(\tau,r) \leqq W(v^{-1}(\tau/4))$, $0 \leqq r \leqq v^{-1}(\tau/4)$. If $r \geqq v^{-1}(\tau/4)$, then $v(r) \leqq \tau/4 \leqq 1/r_1 \leqq v(r_1)$ and hence $r \geqq r_1 \geqq r_0$. Putting $\varphi(r) = \varphi_\tau(r) = \log\psi(\tau,r)$ we thus have

$$\varphi(r) = e^{rv(r)} - \frac{1}{\sqrt{2}}e^{\tau r/2}, \quad r \geqq v^{-1}(\tau/4).$$

Since $v \searrow$ we find

$$D^+\varphi(r) \leqq v(r)e^{rv(r)} - \frac{\tau}{2\sqrt{2}}e^{\tau r/2} < 0, \quad r \geqq v^{-1}(\tau/4).$$

Hence $\psi(\tau,r) \leqq W(v^{-1}(\tau/4))$, $r \geqq v^{-1}(\tau/4)$. We have proved that $\chi(\tau) \leqq W(v^{-1}(\tau/4))$. Thus (5.20) follows from (5.23).

Let $\hat{U}_\tau(\xi) = \hat{K}_\tau(\xi)f(\xi)$, $\xi \in \mathrm{R}^d$. If $a+b=0$, then (5.20) applied with $\alpha=0$, yields that $\hat{U}_\tau \in L^2(\mathrm{R}^d)$. If $a+b \neq 0$ then the assumption $F \in \mathscr{A}(W,P,a,b)$ further implies

(5.24) $$\lim_{\substack{\eta'\to 0\\ \eta'\in B'(a,b)}} \hat{U}_\tau(\xi'+i\eta',\xi'') = \hat{U}_\tau(\xi) \text{ a.e. in } \mathrm{R}^d,$$

and (5.20) and Fatou's lemma yield that $\hat{U}_\tau \in L^2(\mathrm{R}^d)$. Thus $\hat{U}_\tau$ is the Fourier transform in the $L^2$-sense of a function $U_\tau \in L^2(\mathrm{R}^d)$. If $a+b \neq 0$, then $g(\zeta') = g_{\xi''}(\zeta') = \hat{U}_\tau(\zeta',\xi'')$ is analytic in $T'(a,b)$ for almost all $\xi'' \in \mathrm{R}^{d-k}$. Using (5.20) and Fubini's theorem we find that $g_{\xi''}(\zeta') \in H^2(T'(a,b))$ for almost all $\xi'' \in \mathrm{R}^{d-k}$. From the theory of $H^2$-spaces in tubes (see for instance [8, Chapter III]) and (5.24) it follows that, for $\beta \in B(a,b)$, $\hat{U}_\tau(\xi+i\beta)$ is the Fourier transform of $e^{\beta x}U_\tau(x)$.

Similarly, if $\beta \in B(a,b)$ and $P(\alpha) < \infty$, then $D^\alpha\hat{U}_\tau(\xi+i\beta)$ is the Fourier transform of $(-i)^{|\alpha|}x^\alpha e^{\beta x}U_\tau(x)$. Using Parseval's relation and (5.20) we thus find

$$\|x^\alpha e^{\beta x}U_\tau(x)\|_2 \leqq CP(\alpha)W(v^{-1}(\tau/4)), \quad \beta \in B(a,b),$$

and it follows that

(5.25) $$\|q_{a,b}(x)x^\alpha U_\tau(x)\|_2 \leqq CP(\alpha)W(v^{-1}(\tau/4)).$$

Let us now prove that for $0 < \tau \leqq \tau_1$

$$\|q_{a,b} w U_\tau\|_1 \leqq CW(v^{-1}(\tau/4)). \tag{5.26}$$

Let $q = q_{a,b}$ and choose $\theta_0$, $\theta < \theta_0 < 1$. From (5.5) and the assumption (5.12) it follows that

$$\frac{w(x)}{p(\theta_0 x)} \in L^2(\mathsf{R}^d). \tag{5.27}$$

Introduce $\Omega_\alpha = \Omega_{\alpha,\theta_0}$ according to (5.6). Using Schwarz' inequality, (5.27) and (5.7) we get

$$\int_{\Omega_\alpha} q(x)w(x)|U_\tau(x)|\,dx \leqq C\theta_0^{|\alpha|}\,\|q(x)x^\alpha U_\tau(x)\|_2\,P(\alpha)^{-1}.$$

Combining with (5.25) we obtain

$$\int_{\Omega_\alpha} q(x)w(x)|U_\tau(x)|\,dx \leqq C\theta_0^{|\alpha|}\,W(v^{-1}(\tau/4)). \tag{5.28}$$

Now $\bigcup_{\alpha\in Z_+^d}\Omega_\alpha = \mathsf{R}^d$ and therefore (5.26) follows from (5.28) after a summation over $\alpha$.

Let $\Psi = \Phi * F$. By assumption $|\Psi| \leqq \rho$ and $\rho \in R[qw]$. Hence

$$\left|\int_{\mathsf{R}^d} \Psi(x-u)U_\tau(u)\,du\right| \leqq \int_{\mathsf{R}^d}\rho(x-u)|U_\tau(u)|\,du \leqq \rho(x)\|qwU_\tau\|_1, \quad x\in\mathsf{R}^d.$$

Using (5.26) we have proved that for $0<\tau\leqq\tau_1$

$$|\Psi * U_\tau(x)| \leqq C\rho(x)W(v^{-1}(\tau/4)), \quad x \in \mathsf{R}^d. \tag{5.29}$$

Using (5.26) and the assumptions $w \in \mathscr{R}$ and (5.13) it follows that the integral $\Phi * F * U_\tau$ is absolutely convergent and may be inverted. Thus

$$\Psi * U_\tau = \Phi * (F * U_\tau) = \Phi * K_\tau.$$

Furthermore

$$\Phi * K_\tau(x) = \int_{\mathsf{R}^d}\Phi(x-u\tau)K(u)du.$$

The last two identities and (5.29) yield that for $0 < \tau \leqq \tau_1$

$$\left| \int_{\mathbb{R}^d} \Phi(x - u\tau)K(u)du \right| \leqq C\rho(x)W(v^{-1}(\tau/4)), \quad x \in \mathbb{R}^d. \tag{5.30}$$

Let $\tau : \mathbb{R}^d \to \mathbb{R}^1$ satisfy

$$\frac{\tau(x)}{t(x)} \to 5\omega, \quad x \to \infty. \tag{5.31}$$

Then $4\omega t(x) \leqq \tau(x) \leqq \tau_1$, $x \geqq x^{(1)}$ and $v^{-1}(\tau(x)/4) \leqq v^{-1}(\omega t(x))$, $x \geqq x^{(1)}$. Choosing $x \geqq x^{(1)}$ and applying (5.30) with $\tau = \tau(x)$ we have proved

$$\left| \int_{\mathbb{R}^d} \Phi(x - u\tau(x))K(u)\,du \right| \leqq C\rho(x)W\big(v^{-1}(\omega t(x))\big), \quad x \geqq x^{(1)}. \tag{5.32}$$

From (5.32) and the assumption (5.15) it now follows that there exists $C \geqq 1$, such that for every $\tau$ satisfying (5.31)

$$\varlimsup_{x \to \infty} \frac{1}{m(x)} \left| \int_{\mathbb{R}^d} \Phi(x - u\tau(x))K(u)du \right| \leqq C.$$

We will apply Theorem 1 with $m$ replaced by $Cm$ and $\omega$ replaced by $5\omega$. The conditions on $K$ are satisfied according to (5.17). Since $h_P(x)e^{-\varepsilon|x|}$ is bounded for every $\varepsilon > 0$, the assumptions $w \in \mathscr{R}$ and (5.12) imply that $w(x) \leqq ce^{|x|}$, $x \in \mathbb{R}^d$, for some $c \geqq 1$. Letting

$$\lambda_0 = \max(\lambda, \lambda_1, \|s\|_\infty)$$

we thus have $w(x)q(x) \leqq ce^{\lambda_0|x|}$ and $m \in R[ce^{\lambda_0|x|}]$. Applying Theorem 1 with $\lambda$ replaced by $\lambda_0$, we obtain $\Phi(x) = O(m(x))$, $x \to \infty$. This completes the proof of Theorem 2.

The result of the Remark follows from Theorem 1 in [5] if $d = 1$ and $\Phi$ is bounded. The general case may be treated by the same methods.

The Tauberian condition of Theorems 1 and 2 may be varied. By considering $-\Phi$ instead of $\Phi$ it follows that the condition (4.3) may be replaced by

$$\Phi(x + y) - \Phi(x) \leqq m(x), \quad 0 \leqq y \leqq t(x), \quad x \geqq X^{(0)}. \tag{5.33}$$

Also, the theorems hold true for complex valued functions $\Phi$ as well, provided that $\mathrm{Re}\,\Phi$ and $\mathrm{Im}\,\Phi$ satisfy either (4.3) or (5.33).

Note also that the condition $F \in \mathscr{A}(W, P, a, b)$ in Theorem 2 is symmetric in the sense that it yields an estimate of $|\Phi(x)|$ as $x \to -\infty$ as well, under appropriate changes of the conditions (4.1), (4.3), (4.4), and (5.15). Using the notation

$$\tilde{\varphi}(x) = \varphi(-x)$$

the variant when $x \to -\infty$ is stated in parenthesis in Theorem 3 below.

Introduce the following notations:

If $\Phi$ is real valued, let

$$\tag{5.34} \tau(\Phi, m) = \lim_{h \to 0+} \lim_{x \to \infty} \inf_{\substack{u \geqq x \\ 0 \leqq y \leqq hl}} \frac{\Phi(u+y) - \Phi(u)}{m(u)}$$

and

$$\chi(\Phi,m) = \max\bigl(\tau(\Phi,m),\ \ \tau(-\Phi,m)\bigr).$$

If $\Phi$ is complex valued, let

$$\chi(\Phi, m) = \min\bigl(\chi(\operatorname{Re}\Phi, m),\ \ \chi(\operatorname{Im}\Phi, m)\bigr).$$

Then Theorem 3 may be stated as follows.

THEOREM 3. (1) *Let Condition* (1) *of Theorem* 2 *hold true, but for the fact that $\Phi$ might be complex valued. Let $(\Phi * F)/m$ be bounded and let $\Phi$ and $m$ (respectively $\tilde{\Phi}$ and $\tilde{m}$) satisfy* (4.1) *for some $s \geqq 0$ and some $X^{(0)}$.*

(2) *Let*

$$\tag{5.35} \chi(\Phi, m) = 0 \quad (\textit{respectively } \chi(\tilde{\Phi}, \tilde{m}) = 0)$$

*and*

$$\tag{5.36} \Phi * F(x) = o\bigl(m(x)\bigr), \quad x \to \infty \quad (\textit{respectively } x \to -\infty).$$

*Then*

$$\Phi(x) = o\bigl(m(x)\bigr), \quad x \to \infty \quad (\textit{respectively } x \to -\infty).$$

COROLLARY. *Let Condition* (1) *of Theorem* 3 *hold true. If $\chi(\Phi, m)$ (respectively $\chi(\tilde{\Phi}, \tilde{m})$) is bounded then*

$$\Phi(x) = O\bigl(m(x)\bigr), \quad x \to \infty \quad (\textit{respectively } x \to -\infty).$$

REMARK. It might be interesting to investigate the necessity of the analyticity condition imposed on $1/\hat{F}$ in Theorem 3. An analogous problem in $\mathsf{R}^1$ has been treated earlier ([4, Theorem 6, p. 347]). The methods used in that paper yield the following result.

Let $d = 1$, $a+b>0$ and let $F$ satisfy the supplementary condition

$$x^{\frac{1}{2}+\delta}F(x) \in L^1(\mathsf{R}^1) \quad \text{for some } \delta > 0.$$

Suppose further that $F$ has the following property:

If $a > 0$ then for every $\alpha < a$ there exists a function $\Phi \not\equiv 0$, bounded on $(-\infty, X)$ for every $X$, and satisfying the conditions of Theorem 3 with $m(x) = e^{-\alpha x}$, $x > 0$. If $b > 0$ then for every $\beta < b$ there exists a function $\Phi \not\equiv 0$, bounded on $(X, \infty)$ for every $X$, and satisfying the conditions of Theorem 3 with $m(x) = e^{\beta x}$, $x < 0$.

Then the condition $1/\hat{F}$ analytic in the strip $T(a,b)$ is necessary in Theorem 3.

PROOF OF THEOREM 3. First, we make two observations. It is no restriction to suppose

$$\frac{rD^+W(r)}{W(r)} \geqq 1, \quad r \geqq r_0. \tag{5.37}$$

Secondly, note that the assumption $m(x)/h_P(\theta x) \in L^2(\mathsf{R}^d)$ implies that there exists a function $\kappa$ satisfying the condition (3.21) of Lemma 3 such that $\kappa_j(x_j) \leqq 1 + x_j$, $x_j \geqq 0$, $j = 1, 2, \ldots, d$, and

$$\frac{w(x)\kappa(x)}{h_P(\theta x)} \in L^2(\mathsf{R}^d). \tag{5.38}$$

This is easily seen by constructing a function $\kappa(x) = \prod_{j=1}^{d} \kappa_j(x_j)$ satisfying (5.38) and (2.7) such that $\kappa_j(x_j) \to \infty$ as $x_j \to \infty$ and $\kappa_j(x_j) \leqq 1 + x_j$, $x_j \geqq 0$, $j = 1, 2, \ldots, d$.

Let

$$\psi(x) = \sup_{y \geqq x} \frac{|\Phi * F(y)|}{m(y)}, \quad x \in \mathsf{R}^d.$$

Then $\psi$ is bounded, $\psi \searrow$ and $\psi(x) \to 0$, $x \to \infty$. According to Lemma 3, there exists a majorant $\mu^*$ of $\psi$ such that $\mu^* \in R[\kappa]$ and $\mu^*(x) \to 0$, $x \to \infty$. Let $\rho = m\mu^*$. Then $\rho \in R[q_{a,b} w\kappa]$ and

$$|\Phi * F(x)| \leqq \rho(x), \quad x \in \mathsf{R}^d. \tag{5.39}$$

Let $\mu_1 = \sqrt{\mu^*}$ and

$$t(x) = v[W^{-1}(1/\mu_1(x))]. \tag{5.40}$$

Then $t \searrow$, $t(x) \to 0$, $x \to \infty$. Using (5.37), $rv(r) \nearrow$ and $\mu_1 \in R[\kappa]$ it is easy to verify that $t \in \mathscr{E}$ and $t(x)^{-1}\kappa(x)^{-1}$ is bounded. Thus (4.2) is satisfied. Furthermore, $W[v^{-1}(t(x))] = 1/\mu_1(x)$ and hence

$$\rho(x) W[v^{-1}(t(x))] = m(x)\mu_1(x). \tag{5.41}$$

Let us prove the theorem under the assumption that $\Phi$ is real valued and $\chi(\Phi, m) = \tau(\Phi, m)$. Consider the function

$$\sigma(x) = - \inf_{\substack{u \geqq x \\ 0 \leqq y \leqq t(x)1}} \frac{\Phi(u+y) - \Phi(u)}{m(u)}.$$

By definition, $\sigma \searrow$ and $\sigma \geqq 0$. Since $\tau(\Phi, m) = 0$ by assumption, $\sigma(x) \to 0$, $x \to \infty$, and we choose $x^{(0)}$ such that $\sigma^*(x) = \sigma(\max(x, x^{(0)}))$ is bounded in $\mathsf{R}^d$. Then $\sigma^* \searrow$, $\sigma^*(x) \to 0$, $x \to \infty$, and

$$\Phi(x+y) - \Phi(x) \geqq -m(x)\sigma^*(x), \quad 0 \leqq y \leqq t(x)1, \quad x \geqq x^{(0)}. \tag{5.42}$$

According to Lemma 3 there exists a majorant $\mu_2$ of $\sigma^*$ such that $\mu_2 \in R[\kappa]$ and $\mu_2(x) \to 0$, $x \to \infty$.

Let

$$\mu_3(x) = \max_{j=1,2,\ldots,d} \frac{1}{\kappa_j(x_j)}.$$

Then $\mu_3 \in R[\kappa]$, $\mu_3(x) \to 0$, $x \to \infty$, and for every $X$

$$\frac{1}{\mu_3(\max(x, X))} \text{ is bounded in } \mathsf{R}^d \setminus \{x \mid x \geqq X\}. \tag{5.43}$$

Now, let

$$\mu = \max(\mu_1, \mu_2, \mu_3).$$

Then $\mu \in R[\kappa]$ and $\mu(x) \to 0$, $x \to \infty$. Let $m_1 = m\mu$ and $w_1 = w\kappa$. Then $m_1 \in R[w_1 q_{a,b}]$. We will apply Theorem 2 with $m$ replaced by $m_1$ and $w$ replaced by $w_1$. From (5.41) and $\mu_1 \leqq \mu$ it follows that (5.15) is satisfied with $m$ replaced by $m_1$ and $\omega = 1$. Since $\sigma^* \leqq \mu_2 \leqq \mu$, (5.42) yields that (4.3) holds true with $m$ replaced by $m_1$. Since $\mu_3 \leqq \mu$ we find from (5.43) that the assumption that (4.1) holds true for $m$ implies that (4.1) holds true with $m$ replaced by $m_1$.

Thus we may apply Theorem 2 and find $\Phi(x) = O(m_1(x))$, $x \to \infty$. Since $m_1(x) = o(m(x))$, $x \to \infty$, this yields $\Phi(x) = o(m(x))$, $x \to \infty$. If $\chi(\Phi, m) = \tau(-\Phi, m)$ or $\Phi$ is complex valued or $x \to -\infty$ the result follows in the same way. Thus Theorem 3 is proved.

The Corollary is easily derived from the Theorem and the details are omitted.

Let $d = 1$. If $m \searrow$ and $\Phi$ is bounded then the conditions (5.13) and (4.1) in Theorem 3 are trivially satisfied. If $P(1) < \infty$, then $h_P(x)^{-1}(1 + |x|)$ is bounded and it follows that (5.12) holds true for $w \equiv c$. Thus we obtain the following special case of Theorem 3, mentioned in the introduction.

THEOREM $3_0$. *Let $d = 1$, let $F \in \mathscr{A}(\mathscr{W}, P, a, 0)$ for some $a > 0$ and some $P$, $P(1) < \infty$, and let $m \searrow$, $m \in R[cq_{a,0}]$ for some $c \geqq 1$. If $\Phi$ is a bounded function satisfying* (1.4) *and* (1.5), *then* (1.6) *holds true.*

In the Tauberian theorems considered above, the behaviour of $\Phi * K(x)$ determines, in a certain sense, the behaviour of $\Phi(x)$ as $x \to \infty$. Another problem of Tauberian character, a special case of which was considered by Korenblum in [3], may be expressed as follows.

When does $\dfrac{\Phi_1 * K(x)}{\Phi_2 * K(x)} \to 1$ as $x \to \infty$ imply that $\dfrac{\Phi_1(x)}{\Phi_2(x)} \to 1$ as $x \to \infty$?

Theorem 4 below is of this type. It yields Korenblum's Tauberian theorem for Laplace transforms ([3, Theorem 2, p. 176]) for an appropriate choice of the kernel $K$. Theorem 4 is derived from Theorem 3, and Korenblum's theorem may, from this point of view, be considered a remainder Tauberian theorem.

The notion "almost increasing" is defined by (5.1), $\tau(\Phi, m)$ is defined by (5.34) and $x_\xi^I$ is defined by (2.2).

THEOREM 4. *Let $K: \mathsf{R}^d \to \mathsf{R}^1_+$. Suppose that, for some $a \in \mathsf{R}^d_+$ and for every $b \in \mathsf{R}^d_+$, there exist $P$, $P(1) < \infty$, and $W$ such that $q_{a,b}K \in L^1(\mathsf{R}^d)$ and $K \in \mathscr{A}(W, P, a, b)$.*

*Let $\Phi_n: \mathsf{R}^d \to \mathsf{R}^1_+$ and suppose $e^{ax}\Phi_n(x)$ almost increasing, $n = 1, 2$. Suppose further*

$$\tau(-\Phi_1, \Phi_1) = 0, \tag{5.44}$$

$$\tau(\Phi_2, \Phi_2) = 0, \tag{5.45}$$

*and*

$$\frac{\Phi_2 * K(x)}{\Phi_1 * K(x)} \to 1, \quad x \to \infty. \tag{5.46}$$

*Then*

$$\frac{\Phi_2(x)}{\Phi_1(x)} \to 1, \quad x \to \infty. \tag{5.47}$$

PROOF. The condition $e^{ax}\Phi_n(x)$ almost increasing yields that there exists $X^{(1)} \geqq \mathit{0}$ such that $\Phi_n(x) > 0,\ x \geqq X^{(1)},\ n = 1,2$ and constants $B_n \geqq 1$ such that

$$\Phi_n(x) \leqq B_n e^{a(y-x)} \Phi_n(y), \quad x \leqq y, \ n = 1,2. \tag{5.48}$$

From (5.44) it follows that there exists $X^{(2)} \geqq X^{(1)}$ and $h_0 > 0$ such that

$$\Phi_1(x+y) \leqq 2\Phi_1(x), \quad \mathit{0} \leqq y \leqq h_0 \mathit{1}, \ x \geqq X^{(2)}. \tag{5.49}$$

Let $b_0 = \log 2/h_0$, $b = b_0 \mathit{1}$, $c = 2B_1$, and $q = q_{a,b}$. We will prove

$$\Phi_1(x), \quad x \geqq X^{(2)}, \text{ belongs to } R[cq]. \tag{5.50}$$

Let us first prove

$$\Phi_1(\eta) \leqq \Phi_1(\xi) 2e^{b(\eta-\xi)}, \quad X^{(2)} \leqq \xi \leqq \eta. \tag{5.51}$$

Choose $\xi$ and $\eta$, $X^{(2)} \leqq \xi \leqq \eta$. Let $\xi = \xi^{(1)}$, $y^{(0)} = h_0(\eta - \xi)/\|\eta - \xi\|_\infty$ and $\xi^{n+1} = \xi^{(n)} + y^{(0)}$, $n = 1,2,\ldots$. Choosing the integer $N$ so that

$$N < \frac{\|\eta - \xi\|_\infty}{h_0} \leqq N+1,$$

applying (5.49) with $x = \xi^{(n)}$, $y = y^{(0)}$, $n = 1,2,\ldots,N-1$, and $x = \xi^{(N)}$, $y = \eta - \xi^{(N)}$ and multiplying the inequalities thus obtained, we get

$$\Phi_1(\eta) \leqq \Phi_1(\xi) 2^{N+1}.$$

Using the above estimate of $N$ and the definition of $b_0$ this proves (5.51).

Now, choose points $\xi \geqq X^{(2)}$, $\eta \geqq X^{(2)}$, and let $I = \{j \mid \eta_j < \xi_j\}$. Then

$$\eta^I_\xi - \eta = \max(\xi - \eta, \mathit{0}) \geqq \mathit{0} \quad \text{and} \quad \eta^I_\xi - \xi = \max(\eta - \xi, \mathit{0}) \geqq \mathit{0}.$$

Using (5.48) and (5.51) we get

$$\begin{aligned}\Phi_1(\eta) &\leqq B_1 e^{a(\eta^I_\xi - \eta)} \Phi_1(\eta^I_\xi) \leqq 2B_1 e^{a(\eta^I_\xi - \eta)} e^{b(\eta^I_\xi - \xi)} \Phi_1(\xi) \\ &= 2B_1 q(\xi - \eta) \Phi_1(\xi).\end{aligned}$$

Thus (5.50) is proved.

Let $\Psi_n = \Phi_n * K$, $n=1,2$. Then $\Psi_n \geqq 0$, $n=1,2$, and (5.46) yields that there exists $X^{(3)} \geqq X^{(2)}$ such that

$$\Psi_2(x) \leqq 2\Psi_1(x), \quad x \geqq X^{(3)}. \tag{5.52}$$

Let

$$m(x) = e^{a\max(X^{(3)}-x,0)}\Phi_1\big(\max(x, X^{(3)})\big). \tag{5.53}$$

Then $m \in R[cq]$ according to Lemma 1 and (5.50), and $m(x) = \Phi_1(x)$, $x \geqq X^{(3)}$. Since $e^{ax}\Phi_1(x)$ is almost increasing, $\Phi_1/m$ is bounded.

We will now prove that $\Phi_2/m$ is bounded. Choose $\alpha \in \mathsf{R}^d_+$ such that

$$\int_{u \leqq \alpha} e^{au} K(u)du = \omega > 0.$$

The inequality (5.48) with $n = 2$ yields

$$e^{au}\Phi_2(x) \leqq B_2 e^{a\alpha}\Phi_2(x+\alpha-u), \quad u \leqq \alpha.$$

Multiplying this inequality by $K(u)$ and integrating over $u \leqq \alpha$ we obtain

$$\begin{aligned} \omega\Phi_2(x) &\leqq B_2 e^{a\alpha}\int_{u\leqq\alpha}\Phi_2(x+\alpha-u)K(u)\,du \\ &\leqq B_2 e^{a\alpha}\Psi_2(x+\alpha), \quad x \in \mathsf{R}^d. \end{aligned} \tag{5.54}$$

Using that $\Phi_1/m$ is bounded, $m \in R[cq]$, and $qK \in L^1(\mathsf{R}^d)$ we find, for some $C$,

$$\Psi_1(x) \leqq Cm(x), \quad x \in \mathsf{R}^d. \tag{5.55}$$

Combining (5.54), (5,52), and (5.55) and using $m \in R[cq]$ we have proved that $\Phi_2(x)/m(x)$, $x \geqq X^{(3)}$ is bounded. From (5.48) with $n = 2$ and (5.53) it then follows that $\Phi_2/m$ is bounded.

We will apply Theorem 3 with $m$ defined by (5.53), $F = K$, $\Phi = \Phi_2 - \Phi_1$ and $w \equiv c$ and we will check that the conditions of Theorem 3 are satisfied.

The conditions $(\Phi * K)/m$ bounded and (5.13) follow from $\Phi/m$ bounded, $m \in R[cq]$ and $qK \in L^1(\mathsf{R}^d)$. If we further use the definition of $m$ it is immediate that (4.1) holds true with $s = a$ and $X^{(0)} = X^{(3)}$. Since $P(I) < \infty$, $h_P(x)^{-1}\prod_{j=1}^d (1+|x_j|)$ is bounded and hence (5.12) is satisfied. The assumption (5.46) yields that $\Psi_2(x) - \Psi_1(x) = o\big(\Psi_1(x)\big)$, $x \to \infty$. Combining with (5.55) we have proved that $\Phi * K(x) = o\big(m(x)\big)$, $x \to \infty$.

It remains to check the Tauberian condition (5.35). Using $\Phi_2/m$ positive and bounded for $x \geqq X^{(1)}$ and the assumption $\tau(\Phi_2, \Phi_2)=0$ we find $\tau(\Phi_2, m)=0$. Since $\tau(-\Phi_1, \Phi_1)=0$ by assumption we have $\tau(-\Phi_1, m)=0$. Thus

$$\tau(\Phi, m) = \tau(\Phi_2 - \Phi_1, m) \geqq \tau(\Phi_2, m) + \tau(-\Phi_1, m) = 0$$

and hence $\tau(\Phi, m)=0=\chi(\Phi, m)$ and (5.35) is satisfied. We may apply Theorem 3 and obtain

$$\Phi(x) = o(m(x)), \quad x \to \infty,$$

or

$$\Phi_2(x) - \Phi_1(x) = o(\Phi_1(x)), \quad x \to \infty,$$

which is equivalent to (5.47). Thus Theorem 4 is proved.

To give an example of a kernel $K$ satisfying the conditions of Theorem 4 for every $a$, let $\|x\|$ denote the Euclidean norm and choose $K(x)=e^{-\|x\|^2}$.

To derive Korenblum's theorem from Theorem 4, consider the kernel $K_0$ defined by

$$K_0(x) = \prod_{j=1}^{d} e^{-x_j} \exp(-e^{-x_j}), \quad x \in \mathsf{R}^d. \tag{5.56}$$

Then $q_{a,b} K_0 \in L^1(\mathsf{R}^d)$ for $a < 1$ and for every $b \in \mathsf{R}^d_+$ and

$$\hat{K}_0(\xi) = \prod_{j=1}^{d} \Gamma(1 + i\xi_j).$$

Choosing, for instance, $W(r) = e^{\pi d r}$, $r \geqq r_0$, we find that $K_0 \in \mathscr{A}(W, P, \alpha, b)$ for every $P$, $\alpha$, and $b$. Using the kernel $K_0$ in Theorem 4 we obtain the following special case of Theorem 4.

THEOREM $4_0$. *Let $A_n \colon \mathsf{R}^d_+ \to \mathsf{R}^1_+$, $n = 1,2$ and suppose for some $a \in \mathsf{R}^d_+$, $a < 1$,*

$$t^a A_n(t) \text{ almost increasing}, \quad n = 1,2. \tag{5.57}$$

*Let*

$$\psi_n(s) = \int_{\mathsf{R}^d_+} e^{-st} A_n(t)\,dt, \quad s \in \mathsf{R}^d_{++}, \quad n = 1,2.$$

*Suppose*

$$\lim_{h\to 1+} \overline{\lim_{t\to\infty}} \sup_{t\leqq\tau\leqq ht} \frac{A_1(\tau)}{A_1(t)} = 1 \tag{5.58}$$

*and*

$$\lim_{h\to 1+} \underline{\lim_{t\to\infty}} \inf_{t\leqq\tau\leqq ht} \frac{A_2(\tau)}{A_2(t)} = 1. \tag{5.59}$$

*If*

$$\frac{\psi_2(s)}{\psi_1(s)} \to 1, \quad s \to \boldsymbol{0}, \quad s \in \mathsf{R}^d_{++},$$

*then*

$$\frac{A_2(t)}{A_1(t)} \to 1, \quad t \to \infty. \tag{5.60}$$

PROOF. Substituting $s_j = e^{-x_j}$, $t_j = e^{y_j}$, $j = 1,2,\ldots,d$, and letting

$$\Phi_n(x) = A_n(e^{x_1}, e^{x_2}, \ldots, e^{x_d}), \quad n = 1,2,$$

we find

$$\psi_n(s) = e^{\sum_{j=1}^d x_j} K_0 * \Phi_n(x), \quad n = 1,2.$$

Thus, Theorem $4_0$ follows from Theorem 4.

If we suppose $A_2 \nearrow$, then Condition (5.59) in Theorem $4_0$ is trivially satisfied and may be omitted. If, further, $A_1 \nearrow$ and $d = 1$ in Theorem $4_0$, then (5.58) may be written $A_1(\tau)/A_1(t) \to 1$ as $t \to \infty$ and $1 < \tau/t \to 1$, and we obtain, but for a partial integration, Korenblum's theorem for Laplace transforms ([3, Theorem 2, p. 174]). In particular, it follows that the conditions $A_n \nearrow$, $n = 1,2$, in Korenblum's theorem may be replaced by the weaker conditions (5.57) and (5.59).

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