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A CONJECTURE OF GRÜNBAUM ON COMMON TRANSVERSALS

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1. Introduction.

A finite family $\mathscr{A}$ of convex sets in the plane is said to have property T if the family admits a common transversal, that is, if there is a straight line which intersects every member of $\mathscr{A}$. The family $\mathscr{A}$ has property $\mathrm{T}(m)$ if every m-membered subfamily of $\mathscr{A}$ has property T. Results on common transversal and references may be found in [1] and [4].

Counterexamples, see [4] and [8], show that there is no m such that $\mathrm{T}(m)$ implies T for an arbitrary family of convex sets. However, with additional restrictions on members of $\mathscr{A}$ positive results have been obtained. Thus T(3) implies T for families of parallel segments, Santalo proved that T(6) implies T for families of parallelograms with parallel sides (cf. [4]) and Grünbaum has shown that T(5) implies T for disjoint translates of a parallelogram (see [2]).

Grünbaum conjectured [2] that T(5) implies T for disjoint families of translates of a fixed arbitrary convex set. (A disjoint family is a family whose members are mutually disjoint.) This conjecture is still open. A weaker conjecture, to be called the weak Grünbaum conjecture, is that there exists a universal k_0 such that $\mathrm{T}(k_0)$ implies T for any disjoint family of at least k_0 translates of a fixed arbitrary convex set. (This conjecture appears as an open problem in Lay's book [7].)

A positive solution to Grünbaum's weak conjecture is given in Theorem 1.

THEOREM 1. *There exists a positive integer k_0, $k_0 \leqq 128$, such that* $\mathrm{T}(k_0)$ *implies* T *for any disjoint family of at least k_0 translates of an arbitrary compact convex set.*

The proof of Theorem 1 relies on a theorem of Hadwiger [3] and cf. [4]:

HADWIGER'S THEOREM. *Let $\mathscr{A}$ be a disjoint family of at least three convex sets and suppose that*

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(1.2) *the members of* $\mathscr{A}$ *can be linearly ordered in such a way that any* 3 *members of* $\mathscr{A}$ *admit a common transversal meeting them in the specified order.*

Then $\mathscr{A}$ *satisfies property* T.

A simple consequence of Hadwiger's Theorem is that T(3) implies T for a family of at least 3 parallel segments, since the segments satisfy (1.2). In order to apply Hadwiger's Theorem to the proof of Theorem 1 the concept of a geometric permutation shall be used.

A common transversal for a disjoint family $\mathscr{A}$ of n convex sets intersects the members of A in a definite order, up to reversal, and therefore determines a permutation p of $\mathscr{A}$ and its reversal. Such a permutation pair, $\{p, -p\} = \tilde{p}$, shall be called a geometric permutation of $\mathscr{A}$ (or a G.P. of $\mathscr{A}$). The set of all G.P.'s of $\mathscr{A}$ shall be denoted by $P_{\mathscr{A}}$. G.P.'s have been studied in [5] and [6].

Properties of G.P.'s for translates and some definitons are presented in section 2. A proof of Theorem 1 is given in section 3, followed by related results and problems in section 4.

Unless stated otherwise $\mathscr{A}$ denotes a finite disjoint family of convex compact sets in the plane.

2. G.P.'s, definitions and consistent permutations.

In [5] it was shown that

THEOREM 2. *If* $\mathscr{A}$ *consists of at least* 11 *disjoint translates (of a fixed convex set), then*

$$|P_{\mathscr{A}}| \leqq 8.$$

Let l and m be two directed straight lines meeting at the point 0 and dividing the plane into four quadrants; Q_1, Q_2, Q_3, and Q_4 (Q_1 is bounded by the half-lines of l and m after 0, $Q_1 \cap Q_2$ is a half-line of m, and $Q_1 \cap Q_3 = Q_2 \cap Q_4 = 0$). The set S *crosses* a quadrant Q if $l \cap Q \cap S \neq \varnothing, m \cap S \neq \varnothing$, and 0 is not in S. The following lemma appears in [5, Lemma 3 of section 3]:

LEMMA 1. *Let* S_1, S_2, S_3 *and* S_4 *be translates of the convex set* S, *if* S_1 *and* S_2 *cross the first quadrant and* S_3 *and* S_4 *cross the second quadrant, then*

(2.1) $$\text{either } S_1 \cap S_2 \neq \varnothing \text{ or } S_3 \cap S_4 \neq \varnothing.$$

However, in going over the proof of the lemma in [5] it is clear that a stronger result has been proved, namely:

LEMMA 2. *Let S_1, S_3, S_4 be disjoint translates of the convex set S, with S_1 crossing the first quadrant Q_1 and S_2 and S_3 crossing the second quadrant Q_2. Then it is impossible that on the half line $Q_1 \cap Q_2$:*

$$\text{(2.2)} \qquad \begin{gathered} \textit{Both } S_3 \cap Q_1 \cap Q_2 \textit{ and } S_4 \cap Q_1 \cap Q_2 \textit{ are between} \\ S_1 \cap Q_1 \cap Q_2 \textit{ and } 0. \end{gathered}$$

Some definitions. The permutation

$$\begin{pmatrix} 1\ 2 \ldots . k \\ i_1, \ldots, i_k \end{pmatrix}$$

shall be denoted by $(i_1, \ldots, i_k)$. For a permutation $p = (i_1, \ldots, i_k)$ define set $p = \{i_1, \ldots, i_k\}$ (that is, $j \in \operatorname{set} p$ if $j = i_t$ for an integer t, $1 \leqq t \leqq k$). The permutation q is a *subpermutation* of the permutation p (or is *contained* in p), or $q \subset p$, if set $q \subset$ set p and if the members of q appear in the same order in p and in q. $((2, 7, 3)$ is contained in $(1, 2, 5, 7, 8, 3, 6))$. If R is a family of permutations of subsets of a set K and $L \subset K$, then

$$R(L) = \{p \in R : \operatorname{set} p = L\}$$

(e.g., if $R = \{(1,2,3,4), (3,4), (4,3), (4,5,6), (5,6,4)\}$, then $R(\{4,5,6)\} = \{(4,5,6), (5,6,4)\})$.

If p is a permutation and $S \subset \operatorname{set} p$, then $q = P|S$ (p restricted to S) is defined by $q \subset p$ and set $q = S$. (Thus $(1,2,7,6,3,4)|\{2,7,4,3\} = (2,7,3,4)$.) If P is a family of permutations then $P|S = \{p|S : p \in P\}$.

Consistent permutations. A family P of permutations, of a k-set shall be called *r-consistent* (r a non-negative integer) if

$$r = 0 \quad \text{and} \quad |P| = 1$$

or if $r > 0$ and there exists integers l and m, $l < m$ and $m = l + r + 1$ such that for any two permutations $p_1 = (i_1, \ldots, i_k) \neq p_2 = (j_1, \ldots, j_k)$ of P:

$$\text{(2.3)} \qquad i_t = j_t \quad \text{for any } t \leqq l \text{ or } t \geqq m\,.$$

(For example: $\{(1,2,6,7,8,9),\ (1,2,7,6,8,9),\ (1,2,6,7,9,8)\}$ is 4-consistent but not 3-consistent.) Note that, r-consistent implies $r+1$ consistent and 1-consistent implies 0-consistent. P is r-consistent then $P|\ L$ is also r-consistent. It is not difficult to see that

(2.4) P is r-consistent if any two permutations of P are r-consistent.

Lemmas 1 and 2 imply:

LEMMA 3. *If A is a family of translates and if two common transversals determine the* G.P.'s *$\tilde{p}$ and $\tilde{p}'$, then it is possible to choose $p \in \tilde{p}$ and $p' \in \tilde{p}'$ and integers v and w so that*

(2.5) *p and p' are 5-consistent.*

(2.6) *If q is obtained from p by deleting v and w and q' is obtained from p' by deleting v and w, then $q=q'$.*

Lemma 3 hold since it can be assumed (Lemma 1), with l and m as the transversals, that each of the even quadrants is crossed by at most one translate, say by A_v and A_w. Let p and p' be the permutations determined by l and m. Deletion of A_v and A_w leaves only sets A_i that cross the odd quadrants or contain 0. These sets appear in the same order in l and m so that $q=q'$ and (2.6) holds. Similar reasoning, using Lemma 2 yields (2.5).

3. Proof of Theorem 1.

Let $\mathscr{A}=\{A_1,\ldots,A_n\}$ be a family of at least 128 translates satisfying T(128) and let S be a fixed subset of $N=\{1,2,\ldots n\}$ of cardinality 11. Define

(3.1) $\tilde{Q}=\{\tilde{p}=\{p,-p\}:\tilde{p}$ is a G.P. of set p, a subset of A, set $p\subset S$ and $|\text{set } p| \leqq 128\}$.

Let $\tilde{q}_0=\{q_0,-q_0\}$ with set $q=S$ and let

(3.2) $Q = \{p:\tilde{p}\in Q$ and $p|S$ is 5-consistent with $q_0\}$.

It follows from (2.5), (2.6), and $|S|=11$ that if $p,p'\in Q$, then $p|\ L$ and $p'|\ L$ are 5-consistent and if

(3.3) $(i,j,k,l,m) \subset p$, then $(m,l,j,i) \not\subset p'$.

Since the permutations described arise from G.P.'s

(3.4) If $q\subset p\in Q$ and if set $q\supset S$, then $q\in Q$.

Partition the 2-sets of N into 2 colors:

(3.5) Color the 2-set $u=\{i,j\}$ *black* if there exists a set $T=T(u)\subset N$ that contains u, such that $|T(u)|=7$ and such that i and j appear in the same order in all the permutations of $Q(S\cup T(u))$.

(3.6) Color $u=\{i,j\}$ *white* if for any $N\supset T\supset u$, $|T|\leqq 7$: i appears both before j and after j in different permutations of $Q(S\cup T)$.

It shall be shown that

(3.7) the union of all the white 2-sets is contained in a set W of cardinality 5,

and that

(3.8) there exists a permutation p_0 of W so that for any subset U of N of cardinality 14 there exists a permutation p of Q such that set $p = S \cup W \cup U$ and $p \supset p_0$.

The next step in the proof is to order N, or equivalently A, so that Hadwiger's Theorem may be applied. The final step is the proof of (3.7) and (3.8).

The ordering of N. Define the linear order $\prec$ of N as follows:
If $\{i, j\}$ is black and i is before j in all the permutations of

(3.9) $Q(S \cup W \cup T(u))$, then $i \prec j$.

(3.10) If $\{i, j\}$ is white and i precedes j in p_0, then $i \prec j$.

To show that $\prec$ is transitive: suppose that $i \prec j$ and $j \prec k$ and that $\{i, j\}$ and $\{j, k\}$ are black. (The other possibilities are treated similarly.)

Assume that $k \prec i$ (and reach a contradiction). If $\{i, j\}$ is black, then in all the permutations of

$$Q(S \cup T(\{i, j\}) \cup T(\{j, k\}) \cup T(\{i, k\})) .$$

The integer i is before j, j is before k, and i is *after* k, a contradiction since

$$|S \cup \mathrm{T}\{(i, j\}) \cup \mathrm{T}(\{j, k\}) \cup \mathrm{T}(\{i, j\})| \leqq 11+7+7+7 = 32 < 128$$

and A satisfies $\mathrm{T}(128)$ (so that i is before k).

Therefore, $\{i, k\}$ has to be white, and

$$|S \cup W \cup \mathrm{T}(\{i, j\} \cup \mathrm{T}(\{j, k\})| \leqq 11+5+7+7 = 30$$

implies the existence of a permutation of $Q(S \cup W \cup \mathrm{T}(\{i, j\}) \cup \mathrm{T}(\{j, k\}))$ that contains p_0. In such a permutation i is before j, and j before k, but k before i, a contradiction. So $i \prec k$ and the relation is transitive.

It remains to show that if $i \prec j \prec k$, then there exists a common transversal for A_i, A_j, A_k in that order: suppose that $\{i, j\}$ is black and that $\{j, k\}$ is white (the other possibilities are treated similarly). Let q be a permutation of $Q(S \cup W \cup \mathrm{T}(\{i, j\}))$ that contains p_0. In q, i is before j (due to $\mathrm{T}(\{i, j\})$) and j is before k (due to p_0).

The common transversal corresponding to q has the desired property.

PROOF OF (3.7). Assume that the white 2-sets are not contained in a set of cardinality 5. Let l be the smallest integer ($6 \leqq l \leqq 7$) such that the white 2-sets $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ are contained in L, of cardinality l, but not in a set of smaller cardinality.

Observe the permutations p of L which are subpermutations of permutations of $Q(S \cup L)$ with set $p - L$. Suppose (after relabeling) that one of them is $p_1 = (1, 2, 3, \ldots, l)$. The minimality of L and the definition (3.6), of the white pairs imply the existence of $p_2 = (j_1, \ldots, j_l)$ and $p_3 = (i_1, \ldots, i_l)$, subpermutations of $Q(S \cup L)$ with set $p_2 = \text{set}\, p_3 = L$ and such that $j_1 \neq 1$ and $i_l \neq l$ (1 is contained in one of the k 2-sets and so is l, otherwise, L would not be minimal). Since pair p_2, p_1 and the pair p_3, p_1 are 5-consistent and $|L| < 5$, $j_l = l$ and $i_1 = 1$ so that p_2 and p_3 are not 5-consistent, a contradiction.

PROOF OF (3.8). Let $q_1, \ldots, q_m$ be the permutations of W contained in permutations of Q. By Theorem 2, $m \leqq 8$. If (3.8) does not hold, then there exists for $1 \leqq i \leqq m$ a set N_i of cardinality 14, so that q_i is not contained in any permutation of $Q(S \cup W \cup N_i)$. But then no permutation of $Q(S \cup W \cup N_1 \cup N_2 \cup \ldots \cup N_m)$ contains q_1 or $q_2 \ldots$ or q_m so that $Q(S \cup W \cup N_1 \cup N_2 \cup \ldots \cup N_m) = \varnothing$, contradicting

$$|S \cup W \cup N_1 \cup N_2 \cup \ldots \cup N_m| \leqq 11 + 5 + 8 \cdot 14 = 128 .$$

4. Related problems and results.

Additional results shall be given (without proof) and some open problems discussed.

RESULTS.

1) An α-family is a disjoint family of compact convex sets, each of diameter at most 1 and area at least α.

Using techniques similar to those applied in proving Theorem 1 it is possible to prove

THEOREM 3. *For each $\alpha > 0$ there exists a positive integer $k = k(\alpha)$ such that $T(k)$ implies T for any α-family of at least k sets.*

2) Lewis [9], conjectured that for any convex compact set C which is not a segment, there is an integer $k_0 = k_0(C)$ such that $T(k_0)$ implies T for any disjoint family of sets congruent to C. He also constructed (in [8]) for any $k \geqq 3$ a disjoint family of k congruent segments satisfying $T(k-1)$ but failing to have property T.

Note that Theorem 3 proves the Lewis conjecture. (Assume without loss of generality that the diameter of C is equal to 1 and that the area of C is α.)

3) The value of 128 given in Theorem 1 can certainly be improved. However, the aim of this paper is to give a qualitative result. In order to decrease the 128 to a number close to 5 additional results on G.P.'s are needed. This will be the subject of another paper.

4) $|P_{\mathscr{A}}| \leqq |\mathscr{A}| \cdot (|\mathscr{A}|-1)/2$ and for $n \geqq 4$ there exists disjoint families $\mathscr{A}$ with $|\mathscr{A}| = n$ and $|P_{\mathscr{A}}| \geqq 2n-2$ ([5] and [6]).

5) $|P_{\mathscr{A}}| \leqq |\mathscr{A}|$ if $\mathscr{A}$ is a family of disjoint segments and there exists a disjoint family of segments $\mathscr{A}$ with $|P_{\mathscr{A}}| = |\mathscr{A}|$ for $|\mathscr{A}| = n \geqq 3$ (see [6]).

PROBLEMS.

6. Reduce the gap between the upper and lower bounds in 4).

7. Obtain results similar to 3) and 4) for restricted families of convex sets (discs, congruent discs, etc.)

8. Extend, if possible Hadwiger's Theorem to higher dimensions (that is, to R^k with $k>2$).

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