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INTERPOLATION WITH A PARAMETER FUNCTION

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Abstract.

The (Lions–Peetre) real interpolation spaces $\bar{A}_{\theta,q}$ are defined by using the function norm

$$\phi(\varphi) = \left(\int_0^\infty (\varphi(t)/t^\theta)^q \frac{dt}{t}\right)^{1/q}.$$

By replacing t^θ by a more general (parameter) function $\varrho = \varrho(t)$ we obtain the spaces $\bar{A}_{\varrho,q}$. In this paper we shall point out the fact that most of the classical (and some new) theorems for the spaces $\bar{A}_{\theta,q}$ can be formulated also for the more general spaces $\bar{A}_{\varrho,q}$. Sometimes we only need to adjust some recent results to the present situation but sometimes we must give separate proofs of our statements. Every result is given in a form which is very adjusted to immediate applications. This paper can be seen as a follow-up and unification of several results of this kind in the literature.

0. Introduction.

The (Lions–Peetre) real interpolation spaces $\bar{A}_{\theta,q}$ (the spaces of means) were introduced in [16]. We refer to the books [3], [15] or [32] for the theory and bibliography concerning these spaces. The spaces $\bar{A}_{\theta,q}$ are defined by using the "function norm"

$$\phi_{\theta,q}(\varphi) = \left(\int_0^\infty (\varphi(t)/t^\theta)^q \frac{dt}{t}\right)^{1/q}$$

(see section 2). If we replace $\phi_{\theta,q}$ by a more general function norm, then we obtain more general interpolation spaces. The study of such spaces was initiated in the fundamental paper [25]. Later on and in particular in the very last years the theory has been developed in an astounding way. We refer to [4], [6], [11], [22] and [23] and the references given there.

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The theory for the spaces $\bar{A}_{\theta,q}$ has been used as a powerful tool for applications in many branches of mathematics. However, many new beautiful results have not yet found so many applications as expected. In this paper we shall only consider the interpolation spaces $\bar{A}_{\varrho,q}$ which are obtained by replacing t^{θ} in the definition of $\phi_{\theta,q}$ by a parameter function $\varrho=\varrho(t)\in Q(0,1)$, which means that, for some $\varepsilon>0$, $\varrho(t)t^{-\varepsilon}$ is increasing and $\varrho(t)t^{-1+\varepsilon}$ is decreasing. On the one hand, this generalization seems to be quite sufficient for many applications. On the other hand, the present investigation reveals that most of the classical (and some new) theorems for the spaces $\bar{A}_{\theta,q}$ can be formulated also for the spaces $\bar{A}_{\varrho,q}$. Every result is given in a form which is very adjusted to immediate applications.

The results obtained in this paper can be seen as a follow-up and unification of some investigations by Kaliguna [13], Gustavsson [7], Heinig [9], Maligranda [18], Merucci [20], etc. However, in many cases our proofs are much simpler. Sometimes it is even sufficient to adjust some recent results to the present situation.

The paper is organized in the following way. In section 1 we discuss a useful class of (parameter-) functions. In particular the (close) relations to the function classes $\mathscr{P}^{+-}$ (see [8]) and B_{ψ} (see [7]) are pointed out. Section 2 is used to give some basic interpolation terminology. For the reader's convenience we also formulate some well-known results in a form which is suited for our purposes. In order not to interrupt our discussions later on we state some technical lemmas in section 3. In section 4 we discuss reiteration results. Our starting point is to use some important estimates from [4] and [22]. Our results are more general than the corresponding results in [7], [9] and [20]. In particular, we need *not* in general assume that we have some a priori separation condition between the actual interpolation spaces. In section 5 we generalize Wolff's theorem (see [33]) to the considered situation. In section 6 we apply some our results and obtain well-known and also new results concerning interpolation spaces between Lorentz spaces. Some results concerning interpolation between the sum $(\Sigma(\bar{A}))$ and the intersection $(\Delta(\bar{A}))$ can be found in section 7 (compare with [18]). In particular, we point out an elementary description of the spaces $(L^p+L^\infty, L^p\cap L^\infty)_{\varrho,q}$. Finally, we give some concluding remarks in section 8.

CONVENTIONS. The equivalence $a\approx b$ means that $c_1 a\leqq b\leqq c_2 a$ for some positive constants c_1 and c_2. Two quasi-normed spaces, A and B, are considered as equal and we write $A=B$ whenever their quasi-norms are equivalent. The relation $A\subset B$ means that we have a continuous embedding. If nothing else is postulated all considered spaces are quasi-

Banach spaces. C denotes any positive constant (not the same in different appearances).

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1. On an elementary class of functions on $]0,\infty[$.

Let a_0 and a_1 be real numbers such that $a_0 < a_1$. The class $Q[a_0,a_1]$ consists of all function $\varphi(t)$ on $]0,\infty[$ such that $\varphi(t)t^{-a_0}$ is nondecreasing and $\varphi(t)t^{-a_1}$ is nonincreasing. Moreover, we say that $\varphi(t)$ belongs to the class $Q(a_0,a_1)$, whenever $\varphi(t) \in Q[a_0+\varepsilon, a_1-\varepsilon]$ for some $\varepsilon > 0$. The notation $\varphi(t) \in Q(a_0,-)$ means that $\varphi(t) \in Q(a_0,b)$ for *some* real number b. We shall also permit hybrid cases, for example $Q[a_0,b_0)$ or $Q[a_0,-)$. (See e.g. [29] or [31].)

EXAMPLE 1.1. Let $a_0 < a < a_1$ and let b and c be arbitrary real numbers. Then

$$\varphi(t) = t^a(\log(B+t))^b(\log(C+1/t))^c$$

belongs to the class $Q(a_0,a_1)$ whenever B and C are sufficiently large constants (any $B > e^{2|b|/\delta}$ and $C > e^{2|c|/\delta}$, $\delta = \min(a-a_0, a_1-a)$, will do).

First we state the following elementary (but useful) lemma.

LEMMA 1.1. *Let* $\varphi(t) \in Q[a_0,a_1]$. *Then*

(a) $\varphi(t^\alpha) \in Q[a_0\alpha, a_1\alpha]$, *if* $\alpha > 0$,
$\varphi(t^\alpha) \in Q[a_1\alpha, a_0\alpha]$, *if* $\alpha < 0$.

(b) *the inverse* $\varphi^{-1}(t)$ *exists and* $\varphi^{-1}(t) \in Q[a_1^{-1}, a_0^{-1}]$, *whenever* $a_0 > 0$.

(c) $t^\alpha(\varphi(t))^\beta \in Q[\alpha+a_0\beta, \alpha+a_1\beta]$, *if* $\alpha \in \mathrm{R}$, $\beta > 0$,
$t^\alpha(\varphi(t))^\beta \in Q[\alpha+a_1\beta, \alpha+a_0\beta]$, *if* $\alpha \in \mathrm{R}$, $\beta < 0$.

PROOF. The proof of Lemma 1.1 only consists of some straightforward applications of the definition of the class $Q[a_0,a_1]$ so we leave the details to the reader. (Part (b) ought to be compared with Lemma 1.2 in [8]).

EXAMPLE 1.2. Let $\varrho^*(t) = t\varrho(1/t)$. Then, by Lemma 1.1(a) and (c), $\varrho(t) \in Q(0,1)$ if and only if $\varrho^*(t) \in Q(0,1)$.

EXAMPLE 1.3. Let $\varphi(t) \in Q[a_0,a_1]$. Then there exists a function $\varrho(t) \in Q[0,1]$ and a concave function $k(t)$ so that

$$\varphi(t) = t^{a_0}\varrho(t^{a_1-a_0}) \quad \text{and} \quad \varphi(t) \approx t^{a_0}k(t^{a_1-a_0}) .$$

Example 1.3 is an easy consequence of Lemma 1.1 (a) and (c) and the well-known fact that every function $\varrho(t) \in Q[0,1]$ is quasi-concave (see Peetre [26] and [3, p. 117]).

PROPOSITION 1.2. *Let $\psi(t)$ be a function on $]0,\infty[$. The following conditions are equivalent*:

(a) $\psi(st) \leqq C \max(s^{a_0}, s^{a_1})\psi(t)$, *for* $s>0$, $t>0$.

(b) $\psi(t)$ *is equivalent to some function* $\varphi(t) \in Q[a_0, a_1]$.

(c) $\psi(t) \approx \alpha t^{a_0} + \beta t^{a_1} + \int_0^\infty \min(st^{a_0}, t^{a_1})\, d\mu(s)$, *where* $\alpha \geqq 0$, $\beta \geqq 0$ *and* $\mu(s)$ *is a nondecreasing function on* $]0,\infty[$ *satisfying* $\lim_{s\to\infty} \mu(s) < \infty$ *and* $\lim_{s\to 0+} s\mu(s) = 0$.

PROOF. If (b) holds, then $c_0\varphi(t) \leqq \psi(t) \leqq c_1\varphi(t)$ for some positive constants c_0, c_1 and

$$\varphi(st) \leqq \max(s^{a_0}, s^{a_1})\varphi(t) .$$

Thus (a) is satisfied with $C = c_1/c_0$. The implication (a) $\Rightarrow$ (b) follows by choosing

$$\varphi(t) = \sup_{s>0} \left(\psi(st)/\max(s^{a_0}, s^{a_1})\right) .$$

Let (b) be satisfied. Then, by Example 1.3 and the usual representation formula by Peetre (see [3, p. 117]),

$$\varphi(t) \approx t^{a_0}\left(\alpha + \beta t^{a_1-a_0} + \int_0^\infty \min(s, t^{a_1-a_0})\, d\mu(s)\right) .$$

Hence (b) $\Rightarrow$ (c). Finally, if (c) holds, then

$$\psi(t) \approx \varphi(t) = t^{a_0}k(t^{a_1-a_0}) ,$$

where k is concave and thus, in particular, $\varphi(t) \in Q[a_0, a_1]$. Therefore (c) $\Rightarrow$ (b) and the proof is complete.

In order to be able to compare our results later on with some similar results in the literature we shall now compare the class $Q(0,1)$ with the similar function classes $\mathscr{P}^{+-}$ (see [8]) and B_ψ (see [13] or [7]).

The class $\mathscr{P}^{+-}$ consists of all functions $\varphi(t)$ in $Q[0,1]$ such that

$$\bar{\varphi}(t) \overset{\text{(def)}}{=} \sup_{s>0} \left(\varphi(st)/\varphi(s)\right) = o(\max(1,t)) \quad \text{as } t \to 0 \text{ and } t \to \infty .$$

The class B_ψ consists of all continuously differentiable functions ψ on $]0,\infty[$ satisfying

(1.1)
$$0 < \inf_{t>0} \frac{t\psi'(t)}{\psi(t)} \leqq \sup_{t>0} \frac{t\psi'(t)}{\psi(t)} < 1 .$$

PROPOSITION 1.3.

(a) $B_\psi \subset Q(0,1) \subset \mathscr{P}^{+-}$.

(b) *If* $\varphi(t) \in \mathscr{P}^{+-}$, *then there exists a function* $\psi(t) \in B_\psi$ *such that* $\varphi(t) \approx \psi(t)$.

PROOF. Let $\psi(t) \in B_\psi$. The condition (1.1) implies that $\psi(t)\, t^{-\alpha}$ is nondecreasing and $\psi(t)\, t^{-\beta}$ is nonincreasing, where

$$\alpha = \inf_{t>0} \frac{t\psi'(t)}{\psi(t)} \quad \text{and} \quad \beta = \sup_{t>0} \frac{t\psi'(t)}{\psi(t)} .$$

Thus $\psi(t) \in Q(0,1)$. Moreover the condition $\psi(t) \in Q(0,1)$ implies that, for some $\varepsilon > 0$ and every $s > 0$, $t > 0$,

$$\frac{\varphi(st)}{\varphi(t)} \leqq \max(t^\varepsilon, t^{1-\varepsilon}) ,$$

which, in its turn, implies that $\varphi(t) \in \mathscr{P}^{+-}$. This completes the proof of (a).

If $\varphi(t) \in \mathscr{P}^{+-}$ then it is well-known that $\varphi(t)$ satisfies (a) in Proposition 1.2 with $a_0 = \varepsilon$, $a_1 = 1 - \varepsilon$ for some $\varepsilon > 0$. Therefore, $\bar{\varphi}(t) = O(t^\varepsilon, t^{1-\varepsilon})$. We put

$$\psi(t) = \int_0^\infty \min(1, s/t)\varphi(t)\frac{dt}{t} .$$

By making some calculations (compare with [7, p. 293]) we find that $\psi(t) \approx \varphi(t)$ and $\psi(t) \in B_\psi$.

REMARK 1.1. We owe the arguments in the proof of part (b) to Gustavsson [7].

2. Some basic terminology and results.

If nothing else is postulated we shall always use the following additional conventions in the sequel: p_0, p_1, q_0, q_1, p, q, θ, p_θ and q_θ are parameters satisfying $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, p, q \leqq \infty$, $0 < \theta < 1$,

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} .$$

$\varrho = \varrho(t)$, $\varrho_0 = \varrho_0(t)$ and $\varrho_1 = \varrho_1(t)$ denote functions in $Q(0,1)$.

We let $\bar{A}=(A_0,A_1)$ denote a compatible quasi-Banach pair (i.e. A_0 and A_1 are quasi-Banach spaces, which both are continuously embedded in some Hausdorff topological vector space). We put $\Sigma(\bar{A})=A_0+A_1$ and $\Delta(\bar{A})=A_0 \cap A_1$. For $a \in \Sigma(\bar{A})$ and $t>0$ we define the K-functional

$$K(t,a) = K(t,a,A_0,A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0}+t\|a_1\|_{A_1}) .$$

The J-functional

$$J(t,a) = J(t,a,A_0,A_1) = \max (\|a\|_{A_0}, t\|a\|_{A_1})$$

can be defined for every $a \in \Delta(\bar{A})$.

We put

$$\phi_{\varrho,q}(\varphi) = \left(\int_0^\infty (\varphi(t)/\varrho(t))^q \frac{dt}{t}\right)^{1/q}$$

and say that $a \in \bar{A}_{\varrho,q;K}$ whenever $\|a\|_{\varrho,q;K}=\phi_{\varrho,q}(K(t,a))<\infty$.

Furthermore, we say that $a \in \bar{A}_{\varrho,q;J}$ whenever a can be represented as $a=\int_0^\infty u(t)t^{-1}dt$ (convergence in $\Sigma(\bar{A})$), where $u(t)$ is measurable with values in $\Delta(\bar{A})$ and $\phi_{\varrho,q}(J(t,u(t)))<\infty$. We equip $\bar{A}_{\varrho,q;J}$ with the quasi-norm

$$\|a\|_{\varrho,q;J} = \inf_u \phi_{\varrho,q}(J(t,u(t))) ,$$

where infimum is taken over all permissable representations of a.

If $\varrho(t)=t^\theta$, then, as usual, we write $\bar{A}_{\theta,q;K}$ instead of $\bar{A}_{t^\theta,q;K}$ etc.

Let X be a quasi-Banach space such that $\Delta(\bar{A})\subset X\subset\Sigma(\bar{A})$. We say that X is of the class $C_K(\varphi,\bar{A})$ if

$$K(t,a,\bar{A}) \leqq C\varrho(t)\|a\|_X, \quad a \in X ,$$

and of the class $C_J(\varrho,\bar{A})$ if

$$\|a\|_X \leqq \frac{C}{\varrho(t)} J(t,a,\bar{A}), \quad a \in \Delta(\bar{A}) .$$

We put $C(\varrho,\bar{A})=C_K(\varrho,\bar{A}) \cap C_J(\varrho,\bar{A})$.

The proofs of the following examples are standard.

EXAMPLE 2.1. $\bar{A}_{\varrho,q;K}$ is of the class $C_K(\varrho,\bar{A})$ and $\bar{A}_{\varrho,q;J}$ is of the class $C_J(\varrho,\bar{A})$. (Cf. [3, p. 64] and [7, p. 295].)

EXAMPLE 2.2. X is of the class $C_K(\varrho,\bar{A})$ if and only if $\Delta(\bar{A})\subset X\subset\bar{A}_{\varrho,\infty}$ and X is of the class $C_J(\varrho,\bar{A})$ if and only if $\bar{A}_{\varrho,q}\subset X\subset\Sigma(\bar{A})$, for some $q\leqq 1$. (Cf. [3, p. 66].)

We shall also note some important (well-known) theorems.

THEOREM 2.1 (*The equivalence theorem*).

$$\bar{A}_{\varrho,q;K} = \bar{A}_{\varrho,q;J}.$$

In the sequel we write $\bar{A}_{\varrho,q}$ instead of $\bar{A}_{\varrho,q;K}$ or $\bar{A}_{\varrho,q;J}$.

THEOREM 2.2. (*The interpolation theorem*). *Let* (A_0, A_1) *and* (B_0, B_1) *be compatible quasi-Banach pairs. Assume that* T *is a bounded sublinear operator form* A_i *to* B_i, $i=0,1$, *with quasi-norms* M_0 *and* M_1, *respectively. Then* T *is a bounded operator from* $\bar{A}_{\varrho,q}$ *to* $\bar{B}_{\varrho,q}$ *with a bound* $M \leqq M_0 \bar{\varrho}(M_1/M_0)$, *where* $\bar{\varrho}(s) = \sup_{t>0} \varrho(st)/\varrho(t)$.

PROOF. Cf. [7, p. 295] and [9, p. 248].

THEOREM 2.3 ("*The power theorem*"). *If* $0<p<\infty$, *then*

$$(A_0^p, A_1^p)_{\varrho,q}^{1/p} = (A_0, A_1)_{\varrho_1, qp},$$

where $\varrho_1(t) = (\varrho(t^p))^{1/p}$.

PROOF. This is an easy consequence of the usual relation between K- and L-functionals first discovered by Peetre [24, p. 28]. See also [3, pp. 68–69].

REMARK 2.1. For the general case we only obtain the estimate

$$\|a\|_{(A_0^{p_0}, A_1^{p_1})_{\varrho,q}} \approx \left(\int_0^\infty \left(\frac{(K(t,a))^{p_0}}{\varrho\left(t^{p_1}(K(t,a))^{p_0-p_1}\right)} \right)^q \frac{dt}{t} \right)^{1/q}$$

For the case $\varrho(t) = t^n, \eta = \theta p_\theta / p_1$, this estimate is equivalent to the usual form of the power theorem (see [3, p. 68]).

Let A' denote the dual space of the Banach space A.

THEOREM 2.4 (*The duality theorem*). *Let* (A_0, A_1) *be a Banach pair such that* $\Delta(\bar{A})$ *is dense in both* A_0 *and* A_1. *Then, for* $1 \leqq q < \infty$,

$$(A_0, A_1)'_{\varrho,q} = (A'_0, A'_1)_{\varrho_1, q'},$$

where $\varrho_1(t) = 1/\varrho(1/t)$ *and* $(1/q) + (1/q') = 1$.

PROOF. This can be carried out by generalizing the proof in [3, p. 54]. See also [4, p. 188]. Much more general versions of Theorem 2.4 can be found in [6, p. 19] and [11].

3. Lemmas.

Let $L^p(\omega)$, $0<p\leqq\infty$, $\omega=\omega(x)\geqq 0$ denote the space of all functions $f(x)$ on a measure space (Ω,μ) satisfying $(\int_\Omega(|f(x)|\omega(x))^p\,d\mu(x))^{1/p}<\infty$.

LEMMA 3.1. *If* $0<p<\infty$ *and* $\varrho\in Q(0,1)$, *then*

$$(L^p(\omega_0),L^p(\omega_1))_{\varrho,p} = L^p(\omega_0/\varrho(\omega_0/\omega_1))\,. \tag{3.1}$$

REMARK 3.1. The following complement of (3.1) holds: If $0<p,q<\infty$ and $\gamma=1/q-1/p$ then, for sufficiently small $\varepsilon>0$,

$$(L^p(\omega_0),L^p(\omega_1))_{\varrho,q} = \begin{cases} \bigcap_{\psi\in Q_\varepsilon} L^p(\omega_0/(\varrho(\omega_0/\omega_1)\psi^\gamma(\omega_0/\omega_1))), & \text{if } q>p\,,\\ \bigcup_{\psi\in Q_\varepsilon} L^p(\omega_0/(\varrho(\omega_0/\omega_1)\psi^\gamma(\omega_0/\omega_1))), & \text{if } q<p\,, \end{cases} \tag{3.2}$$

where Q_ε is the class of functions ψ satisfying $\psi\in Q[-\varepsilon,\varepsilon]$ and $\int_0^\infty \psi(t)t^{-1}dt=1$.

(3.1) and (3.2) are special cases of Example 7.2 in [28].

LEMMA 3.2. *Let* $0<q\leqq\infty$, $0<r<\infty$ *and* $\psi(t)\in Q(-,-)$. *Let* $h(t)$ *be a positive and nonincreasing function on* $]0,\infty[$.

(a) *If* $\varphi(t)\in Q(-,0)$, *then*

$$\left(\int_0^\infty (\varphi(t))^q\left(\int_0^t (h(u)\psi(u))^r\frac{du}{u}\right)^{q/r}\frac{dt}{t}\right)^{1/q} \leqq C\left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q\frac{dt}{t}\right)^{1/q}.$$

(b) *If* $\varphi(t)\in Q(0,-)$, *then*

$$\left(\int_0^\infty (\varphi(t))^q\left(\int_t^\infty (h(u)\psi(u))^r\frac{du}{u}\right)^{q/r}\frac{dt}{t}\right)^{1/q} \leqq C\left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q\frac{dt}{t}\right)^{1/q}.$$

(C depends only on q and the constants involved in the definition of φ and ψ.)

For the case $q\geqq r$ the lemma is a special case of the usual optimal estimates by Muckenhoupt [21]. A proof of the general case can be found in [29].

LEMMA 3.3. *Let* $\varrho_0(t)$, $\varrho_1(t)$, *and* $\varrho(t)$ *be in the class* $Q(0,1)$ *and put* $\tau(t)=\varrho_1(t)/\varrho_0(t)$.

a) *If* $\tau(t)\in Q(0,-)$ *or* $\tau(t)\in Q(-,0)$, *then* $\varrho_2(t)=\varrho_0(t)\varrho(\tau(t))\in Q(0,1)$.

b) *Assume that* $\tau(t)$ *is differentiable. Then* $\tau(t) \in Q(0,-)$ *if and only if, for some* $\varepsilon > 0$,

$$\frac{t\cdot\tau'(t)}{\tau(t)} \geqq \varepsilon \,. \tag{3.3}$$

PROOF. According to our assumptions there exists an $\varepsilon>0$ so that $\varrho_i(t)t^{-\varepsilon}$ is increasing and $\varrho_i(t)t^{-1+\varepsilon}$ is decreasing $(i=0,1)$. Assume that $\tau(t)\in Q(0,-)$. Then, in particular, $\tau(t)$ is increasing which, in its turn, implies that $\varrho(\tau(t))/\tau(t)$ is decreasing and $\varrho(\tau(t))$ is increasing. We conclude that

$$\varrho_2(t)t^{-1+\varepsilon} = \varrho_1(t)t^{-1+\varepsilon}\varrho(\tau(t))/\tau(t)$$

is decreasing, and

$$\varrho_2(t)t^{-\varepsilon} = \varrho_0(t)t^{-\varepsilon}\varrho(\tau(t))$$

is increasing. Thus $\varrho_2(t)\in Q(0,1)$. The proof of the case when $\tau(t)\in Q(-,0)$ is quite similar.

Assume that $\tau(t)$ is differentiable. Then $\tau(t)t^{-\varepsilon}$ is increasing exactly when $t^{-\varepsilon}(\tau'(t)-\varepsilon\tau(t)/t)\geqq 0$ which, in its turn, is equivalent to (3.3). The proof is complete.

LEMMA 3.4. *Let* $\phi(t) \in Q(0,1)$ *and* $\theta(t) \in Q(0,1)$. *Then, for every* $t>0$, *we have a unique solution* $\xi=\xi(t)$, $\psi=\psi(t)$ *of the system*

$$\begin{cases} \xi = \phi(\psi) \\ \psi = \theta(t/\xi)\cdot\xi \end{cases} \tag{3.4}$$

and $\xi(t) \in Q(0,1)$ *and* $\psi(t) \in Q(0,1)$. *Moreover, if* $\phi(t)$ *in* (3.4) *is replaced by* $\phi_\alpha(t)=\phi(\alpha t)$, $\alpha>0$, *and if the corresponding (unique) solutions are denoted* $\xi=\xi_\alpha(t)$ *and* $\psi=\psi_\alpha(t)$, *then* $\xi_\alpha(t)\approx\xi(t)$ *and* $\psi_\alpha(t)\approx\psi(t)$.

PROOF. We note that $V_\phi=D_\phi=]0,\infty[$ and consider a fixed $\xi \in]0,\infty[$. We use (3.4) and Lemma 1.1 (b) to see that $\psi=\phi^{-1}(\xi)$ and

$$t = t(\xi) = \theta^{-1}(\phi^{-1}(\xi)/\xi)\cdot\xi \,. \tag{3.5}$$

Moreover, $\theta^{-1}(t) \in Q(1,-)$ and $\phi^{-1}(\xi)/\xi \in Q(0,-)$. By making some straightforward calculations we can therefore conclude that $\theta^{-1}(\phi^{-1}(\xi)/\xi) \in Q(0,-)$. Thus $t(\xi) \in Q(1,-)$. We define $\xi(t)$ as the (unique) inverse of $t(\xi)$ and use Lemma 1.1 (b) once more conclude that $\xi(t)\in Q(0,1)$. We put $\psi=\psi(t)=\theta(t/\xi(t))\cdot \xi(t)$. It is easy to see that $\psi(t)\in Q(0,1)$.

Now we assume that $\xi = \phi_\alpha(\psi)$ in (3.4) so that $\psi = (1/\alpha)\phi^{-1}(\xi)$. The corresponding function in (3.5) is

$$t_\alpha(\xi) = \theta^{-1}\big((1/\alpha)\phi^{-1}(\xi)/\xi\big)\cdot\xi \,. \tag{3.6}$$

Since $\theta \in Q(0,1)$ we have the estimate $\theta^{-1}(t) \approx \theta^{-1}\big((1/\alpha)t\big)$. Therefore, by (3.5)–(3.6), $t_\alpha(\xi) \approx t(\xi)$, which, in its turn, implies that $\xi_\alpha(t) \approx \xi(t)$ ($\xi_\alpha(t)$ is the inverse of $t_\alpha(\xi)$). Finally, we have

$$\psi_\alpha(t) = \frac{1}{\alpha}\phi^{-1}\big(\xi_\alpha(t)\big) \approx \phi^{-1}\big(\xi_\alpha(t)\big) \approx \phi^{-1}\big(\xi(t)\big) = \psi(t) \,.$$

This completes the proof.

4. Reiteration and Holmstedt's formula.

We use the notation $\alpha = (\alpha_\nu)_\nu$ for any sequence with Z as index set. By $l^q(\omega)$, where $\omega = (\omega_\nu)_\nu$, we denote the space of all sequences $\alpha = (\alpha_\nu)_\nu$ such that

$$\left(\sum_{-\infty}^{\infty} |\alpha_\nu \omega_\nu|^q\right)^{1/q} < \infty$$

($l^q(\omega)$ is a special case of $L^q(\omega)$.)

Let E be an interpolation space with respect to the pair

$$\bar{l}^\infty = \big(l^\infty, l^\infty((2^{-\nu})_\nu)\big) \,.$$

Then we say that $a \in \bar{A}_{E;K}$ whenever $\|(K(2^\nu, a, \bar{A}))_\nu\|_E < \infty$. The following important theorem has independently been found by Brudnyĭ-Krugljak [4] and Nilsson [22, p. 301].

THEOREM 4.1. *Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach pair and $\bar{E} = (E_0, E_1)$ any pair of interpolation spaces between $\bar{l}_\infty$. Then, for all $t > 0$ and $a \in \Sigma(\bar{A})$,*

$$K(t, a, \bar{A}_{E_0;K}, \bar{A}_{E_1;K}) \approx K\big(t, (K(2^\nu, a, \bar{A}))_\nu, \bar{E}\big) \,. \tag{4.1}$$

REMARK 4.1. The assumption $E_i \subset \Sigma \bar{c}_0$, $i = 0,1$, used in the original proof in [22] is superfluous.

In particular, the assumptions in Theorem 4.1 are obviously satisfied when $E_i = l^{q_i}(\omega_i)$, where $\omega_i = \big(1/\varrho_i(2^\nu)\big)_\nu$, $i = 0,1$. Moreover, for this case we have $\bar{A}_{E_i;K} = \bar{A}_{\varrho_i, q_i}$, $i = 0,1$. We conclude that the formula (4.1) carries the following reiteration information:

PROPOSITION 4.1.

$$(\bar{A}_{\varrho_0,q_0}, \bar{A}_{\varrho_1,q_1})_{\varrho,q} = \bar{A}_{E;K},$$

where

$$E = \left(l^{q_0}(\omega_0), l^{q_1}(\omega_1)\right)_{\varrho,q} \text{ and } \omega_i = \left(1/\varrho_i(2^\nu)\right)_\nu, \quad i = 0,1.$$

Proposition 4.1 implies that our general interpolation problem is reduced to interpolation between weighted l^p-sequences. Unfortunately, for the general case such descriptions can be rather complicated (see [27], [28] and the references given there). However for some (diagonal) cases we have very simple descriptions. For example, if $\varrho(t)=t^\theta$ and $q=q_\theta$ we have

$$E = l^{q_\theta}\left((\alpha_\nu)_\nu\right), \ \alpha_\nu = \left(\frac{1}{\varrho_0(2^\nu)}\right)^{1-\theta}\left(\frac{1}{\varrho_1(2^\nu)}\right)^\theta$$

(see [3, p. 119]). Thus we have

EXAMPLE 4.1.

$$(\bar{A}_{\varrho_0,q_0}, \bar{A}_{\varrho_1,q_1})_{\theta,q_\theta} = \bar{A}_{\varrho_2,q_\theta}, \quad \text{where } \varrho_2 = \varrho_0^{1-\theta}\varrho_1^\theta\,. \tag{4.2}$$

REMARK 4.2. For the special case $\varrho_0(t)=\varrho_1(t)=t^{\theta_0}$ the formula (4.2) reduces to the reiteration formula (see [3, p. 51])

$$(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_0,q_1})_{\theta,q_\theta} = \bar{A}_{\theta_0,q_\theta}\,.$$

By using Lemma 3.1 we also obtain

EXAMPLE 4.2. $(\bar{A}_{\varrho_0,q}, \bar{A}_{\varrho_1,q})_{\varrho,q} = \bar{A}_{\varrho_2,q}$, where $\varrho_2 = \varrho_0/\varrho(\varrho_1/\varrho_0)$.

By using Remark 3.1 and other descriptions obtained in [27] and [28] we can in the same way obtain concrete descriptions of the spaces $(\bar{A}_{\varrho_0,q_0}, \bar{A}_{\varrho_1,q_1})_{\varrho,q}$ for *all* cases even when we do *not* impose some a priori "separation condition" between the parameter functions ϱ_0 and ϱ_1. However, these descriptions will be too complicated for our purposes so we shall in the sequel assume that $\tau(t) \in Q(0,-)$ (or $\tau(t) \in Q(-,0)$), where $\tau(t)=\varrho_1(t)/\varrho_0(t)$. (This restriction corresponds to the usual condition $\theta_0 \neq \theta_1$ in the parameter case, see [3, p. 50].)

We also remark that the celebrated Holmstedt's formula (see [10] or [3, p. 52]) is another consequence of Theorem 4.1 (see [22]). More generally, we can state

PROPOSITION 4.2. *Let* $\tau(t)=\varrho_1(t)/\varrho_0(t)$. *If* $\tau(t)\in Q(0,-)$, *then*

$$K(t,a,\bar{A}_{\varrho_0,q_0},\bar{A}_{\varrho_1,q_1}) \approx \left(\int_0^{\eta(t)}\left(\frac{K(s,a,\bar{A})}{\varrho_0(s)}\right)^{q_0}\frac{ds}{s}\right)^{1/q_0}+t\left(\int_{\eta(t)}^{\infty}\left(\frac{K(s,a,\bar{A})}{\varrho_1(s)}\right)^{q_1}\frac{ds}{s}\right)^{1/q_1}, \tag{4.3}$$

where $\eta(t)$ *is the inverse of* $\tau(t)$.

PROOF. The proof only consists of modifications of the arguments used by Nilsson [22, pp. 310–311] for the (Holmstedt's) case $\varrho_i(t)=t^{\theta_i}$, $i=0,1$, so we omit the details.

REMARK 4.3. According to Proposition 1.3 (a) we see that Proposition 4.2 implies the corresponding result by Heinig [9, Theorem 2.1]. (The proof in [9] is carried out on the model of Holmstedt's original proof.)

REMARK 4.4. For the extreme cases the formulas corresponding to (4.3) read:

$$K(t,a,\bar{A}_{\varrho_0,q_0},A_1) \approx \left(\int_0^{\eta(t)}\left(\frac{K(s,a,\bar{A})}{\varrho_0(s)}\right)^{q_0}\frac{ds}{s}\right)^{1/q_0}, \tag{4.4}$$

where $\eta(t)$ is the inverse of $t/\varrho_0(t)$, and

$$K(t,a,A_0,\bar{A}_{\varrho_1,q_1}) \approx t\left(\int_{\eta(t)}^{\infty}\left(\frac{K(s,a,\bar{A})}{\varrho_1(s)}\right)^{q_1}\frac{ds}{s}\right)^{1/q_1}, \tag{4.5}$$

where $\eta(t)$ is the inverse of $\varrho_1(t)$.

We remark that in [22] we can also find a (reiteration) formula for the J-functional (corresponding to (4.1) for the K-functional). Moreover, the reiteration results formulated in [22] are much more general (but also more intricate) than our next proposition too.

PROPOSITION 4.3. *Let* A_0, A_1, X_0, X_1 *be quasi-Banach spaces such that* $\Delta(\bar{A})\subset X_i\subset\Sigma(\bar{A})$, $i=0,1$. *We put*

$$\varrho_2(t) = \varrho_0(t)\varrho(\varrho_1(t)/\varrho_0(t)),$$
$$\varrho_3(t) = \varrho_0(t)\varrho(t/\varrho_0(t)),$$
$$\varrho_4(t) = \varrho(\varrho_1(t))$$

and assume that X_i *is of the class* $C(\varrho_i,\bar{A})$, $i=0,1$, *respectively. Then*

(4.6) $$(X_0, A_1)_{\varrho,q} = (A_0, A_1)_{\varrho_3,q},$$

(4.7) $$(A_0, X_1)_{\varrho,q} = (A_0, A_1)_{\varrho_4,q},$$

and, if in addition $\tau(t)=\varrho_1(t)/\varrho_0(t) \in Q(0,-)$, *then*

(4.8) $$(X_0, X_1)_{\varrho,q} = (A_0, A_1)_{\varrho_2,q}.$$

REMARK 4.5. For the Banach case a direct proof of the formula (4.8) has been carried out by Gustavsson [7]. This proof can be carried over to our quasi-Banach case without difficulty (we can use Lemma 3.2 instead of Minkowski's inequality). Also the formulas (4.6) and (4.7) can be proved by using the method by Gustavsson ($\varrho_1(t)$) corresponds to t in the proof of (4.6) and $\varrho_0(t)$ corresponds to 1 in the proof of (4.7)). Finally we notice that, by Lemma 1.3 and Lemma 3.3, it is no real restriction to prove Proposition 4.3 under the apparent more restrictive assumptions used in [7].

REMARK 4.6. If X_i is (only) of the class $C_K(\varrho_i, \bar{A})$, $i=0,1$, respectively, then the formulas (4.6)–(4.8) hold with "$=$" replaced by "$\subset$". Inclusions in the opposite direction hold when X_i is of the class $C_J(\varrho_i, \bar{A})$, $i=0,1$, respectively. These statements follow at once by analysing the method in the proof in [7].

REMARK 4.7. The formula (4.8) holds as well when the condition $\tau(t) \in Q(0,-)$ is replaced by the "symmetric" condition $1/\tau(t) \in Q(0,-)$. This fact follows from the following elementary calculation:

$$(X_0, X_1)_{\varrho,q} = (X_1, X_0)_{\varrho^*,q} = \bar{A}_{\varrho_2^*,q},$$

where

$$\varrho^*(t) = t\varrho(1/t)$$

and

$$\varrho_2^*(t) = \varrho_1(t)\varrho^*(\varrho_0(t)/\varrho_1(t)) = \varrho_0(t)\varrho(\varrho_1(t)/\varrho_0(t)).$$

In particular for our test case $\varrho_i = t^{\theta_i}$, $i=0,1$, we only need to impose the usual restriction $\theta_0 \neq \theta_1$.

According to Example 2.1 and Remark 4.7 we also have

COROLLARY 4.4. *Let* $\varrho_i(t)$, $i=0,\ldots,4$, *be defined as in Proposition* 4.3. *Then*

$$(\bar{A}_{\varrho_0,q_0}, A_1)_{\varrho,q} = \bar{A}_{\varrho_3,q}, \tag{4.9}$$

$$(A_0, \bar{A}_{\varrho_1,q_1})_{\varrho,q} = \bar{A}_{\varrho_4,q}, \tag{4.10}$$

and, if in addition $\varrho_1(t)/\varrho_0(t) \in Q(0,-)$ *or* $\varrho_0(t)/\varrho_1(t) \in Q(0,-)$, *then*

$$(\bar{A}_{\varrho_0,q_0}, \bar{A}_{\varrho_1,q_1})_{\varrho,q} = \bar{A}_{\varrho_2,q}. \tag{4.11}$$

REMARK 4.8. By comparing with Example 2.2 we see that Corollary 4.4 and Theorem 4.3 are in fact essentially equivalent.

REMARK 4.9. For the case $q<\infty$ Heinig [9, Theorem 2.2] has carried out another proof of the formula (4.11). A much more elementary proof of this kind (of *all* formulas in Corollary 4.4) can be obtained by using the (Holmstedt's) formulas (4.3)–(4.5), making a change of variables $(u=\eta(t))$ and applying Lemma 3.2 in a suitable way. Moreover, according to the description

$$\left(l^{q_0}\left(\left(\frac{1}{\varrho_0(2^\nu)}\right)_\nu\right), l^{q_1}\left(\left(\frac{1}{\varrho_1(2^\nu)}\right)_\nu\right)\right)_{\varrho,q} = l^q((\omega_\nu)_\nu),$$

where $\omega_\nu = 1/\varrho_0(2^\nu)\varrho(\varrho_1(2^\nu)/\varrho_0(2^\nu))$, which can be deduced from the general descriptions in [28], we can also consider the formula (4.11) as a special case of Proposition 4.1.

5. Wolff's theorem.

In this section we assume that A_i, $i=1,2,3,4$ are quasi-Banach spaces satisfying $A_1 \cap A_4 \subset A_2 \cap A_3$. Moreover, let $\theta(t)$, $\phi(t) \in Q(0,1)$ and let $\xi(t)$, $\psi(t)$ denote the unique functions in the class $Q(0,1)$ satisfying

$$\psi(t) = \theta\left(\frac{t}{\xi(t)}\right)\xi(t) \quad \text{and} \quad \xi(t) = \phi(\psi(t)).$$

(See Lemma 3.4.) We can state the following dual propositions.

PROPOSITION 5.1. *If* A_2 *is of the class* $C_K(\phi(t), A_1, A_3)$ *and* A_3 *is of the class* $C_K(\theta(t), A_2, A_4)$, *then* A_2 *is of the class* $C_K(\xi(t), A_1, A_4)$ *and* A_3 *is of the class* $C_K(\psi(t), A_1, A_4)$.

PROPOSITION 5.2. *If* A_2 *is of the class* $C_J(\phi(t), A_1, A_3)$ *and* A_3 *is of the class* $C_J(\theta(t), A_2, A_4)$, *then* A_2 *is of the class* $C_J(\xi(t), A_1, A_4)$ *and* A_3 *is of the class* $C_J(\psi(t), A_1, A_4)$.

REMARK 5.1. According to Example 2.2 we see that Propositions 5.1 and 5.2 coincide with Corollaries 2 and 1, respectively, in [12, pp. 288–289] for the case when A_i, $i=1,2,3,4$, are Banach spaces and $\theta(t)=t^\theta$, $\phi(t)=t^\phi$, $\psi(t)=t^\psi$, and $\xi(t)=t^\xi$.

We can now state the following more general version of Wolff's theorem (see [33]).

THEOREM 5.3. *Assume that* $A_2=(A_1,A_3)_{\phi(t),p}$ *and* $A_3=(A_2,A_4)_{\theta(t),q}$. *Then*

$$A_2 = (A_1,A_4)_{\xi(t),p} \quad \text{and} \quad A_3 = (A_1,A_4)_{\psi(t),q} .$$

PROOF. Our assumptions imply that A_2 is of the class $C(\phi(t),A_1,A_3)$ and A_3 is of the class $C(\theta(t),A_2,A_4)$. Therefore we can use Propositions 5.1 and 5.2 to conclude that A_2 is of the class $C(\xi(t),A_1,A_4)$ and A_3 is of the class $C(\psi(t),A_1,A_4)$. Thus we can use (4.7) and (4.6) in Proposition 4.3 to conclude that

$$A_2 = (A_1,A_3)_{\phi(t),p} = (A_1,A_4)_{\phi(\psi(t)),p} = (A_1,A_4)_{\xi(t),p}$$

and

$$A_3 = (A_2,A_4)_{\theta(t),q} = (A_1,A_4)_{\xi(t)\theta(t/\xi(t)),q} = (A_1,A_4)_{\psi(t),q} ,$$

respectively. The proof is complete.

For the case when A_i, $i=1,2,3,4$, are Banach spaces the proofs of Propositions 1 and 2 can be carried out by generalizing the arguments in [12] as indicated in [12, pp. 289–290]. The proof of our more general case can be carried out by generalizing the original arguments by Wolff in a suitable way.

PROOF OF PROPOSITION 5.1. According to our assumptions we can choose a_i^1, $i=1,2,3,4$, so that $a=a_2^1+a_4^1$, $a_2^1=a_1^1+a_3^1$,

$$\|a_2^1\|_{A_2}+u\|a_4^1\|_{A_4} \leqq C\theta(u)\|a\|_{A_3} ,$$

and

$$\|a_1^1\|_{A_1}+v\|a_3^1\|_{A_3} \leqq C\phi(v)\|a_2^1\|_{A_2} \leqq C\phi(v)\theta(u)\|a\|_{A_3} .$$

In particular, we find

$$\|a_3^1\|_{A_3} \leqq C\theta(u)\frac{\phi(v)}{v}\|a\|_{A_3} \tag{5.1}$$

and

$$\|a_1^1\|_{A_1}+t\|a_4^1\|_{A_4} \leqq C\left(\theta(u)\phi(v)+t\frac{\theta(u)}{u}\right)\|a\|_{A_3}. \tag{5.2}$$

Let α be a small positive number which we shall define exactly later on. For every $t>0$ we can easily see that the system

$$\begin{cases} \phi(v)u = t \\ \theta(u)\dfrac{\phi(v)}{v} = \alpha \end{cases} \tag{5.3}$$

has a unique solution $u=u_\alpha(t)$, $v=v_\alpha(t)$. We put $\psi_\alpha(t)=\alpha v_\alpha(t)$ and $\xi_\alpha(t) =\phi\big((1/\alpha)\psi_\alpha(t)\big)$. Then, by (5.3), we find that $\xi_\alpha(t)$, $\psi_\alpha(t)$ is the unique solution of the system

$$\begin{cases} \xi_\alpha(t) = \phi_{1/\alpha}(\psi_\alpha(t)) \\ \psi_\alpha(t) = \theta\left(\dfrac{t}{\xi_\alpha(t)}\right)\xi_\alpha(t), \end{cases}$$

where $\phi_{1/\alpha}(t)=\phi\big((1/\alpha)t\big)$. Moreover, by Lemma 3.4, $\xi_\alpha(t)\approx\xi(t)$ and $\psi_\alpha(t)\approx\psi(t)$.

We combine (5.1)–(5.3) and obtain

$$\|a_3^1\|_{A_3} \leqq C\alpha\|a\|_{A_3}$$

and

$$\|a_1^1\|_{A_1}+t\|a_4^1\|_{A_4} \leqq 2C\psi_\alpha(t)\|a\|_{A_3}.$$

Following Wolff we now use the same arguments with a replaced by a_3^1. Then, recursively, we obtain, for $n=1,2,3,\ldots$, $a_i^n \in A_i$, $i=1,2,3,4$, so that

$$a = a_3^n+\sum_{k=1}^{n} a_1^k+\sum_{k=1}^{n} a_4^k, \tag{5.4}$$

where

$$\|a_3^n\|_{A_3} \leqq (\alpha C)^n\|a\|_{A_3} \tag{5.5}$$

and

$$\|a_1^n\|_{A_1}+t\|a_4^n\|_{A_4} \leqq 2C(C\alpha)^{n-1}\psi_\alpha(t)\|a\|_{A_3}. \tag{5.6}$$

Since A_1 and A_4 are quasi-Banach spaces there exists a $\lambda\geqq 1$ so that

(5.7)

$$\left\|\sum_{n=1}^{\infty} a_1^n\right\|_{A_1} \leqq \sum_{n=1}^{\infty} \lambda^n\|a_1^n\|_{A_1} \quad\text{and}\quad \left\|\sum_{n=1}^{\infty} a_4^n\right\|_{A_4} \leqq \sum_{n=1}^{\infty} \lambda^n\|a_4^n\|_{A_4}.$$

Now we put $a_1=\sum_{n=1}^{\infty} a_1^n$ and $a_4=\sum_{n=1}^{\infty} a_4^n$ and fix α so that $C\alpha\lambda<1$. Then, by (5.4)–(5.7), we conclude that $a=a_1+a_4$, $a_1 \in A_1$, $a_4 \in A_4$, and

$$\|a_1\|_{A_1}+t\|a_4\|_{A_4} \leqq 2\lambda C \sum_{n=0}^{\infty} (C\alpha\lambda)^n \psi_\alpha(t)\|a\|_{A_3}.$$

Moreover, $\psi_\alpha(t)\approx\psi(t)$, so we have proved that A_3 is of the class $C_K(\psi(t), A_1, A_4)$. In a similar (symmetric) way we can prove that A_2 is of the class $C_K(\xi(t), A_1, A_4)$. This fact can also be seen by observing that, by hypothesis, $A_2 \subset (A_1, A_3)_{\phi(t),\infty}$ and thus, by reiteration (see Remark 4.6 and (4.7)) we find $A_2 \subset (A_1, A_4)_{\phi(\psi(t)),\infty}$. Therefore A_2 is of the class $C_K(\xi(t), A_1, A_4)$.

Proof of Proposition 5.2. Let $a \in A_1 \cap A_4$. Then, by our assumptions,

$$\|a\|_{A_2} \leqq C\|a\|_{A_1}/\phi(\|a\|_{A_1}/\|a\|_{A_3}),$$

and

$$\|a\|_{A_3} \leqq C\|a\|_{A_2}/\theta(\|a\|_{A_2}/\|a\|_{A_4}).$$

Since $t/\theta(t)$ is nondecreasing we therefore obtain

$$\|a\|_{A_3} \leqq C_0\|a\|_{A_1}/N(a), \tag{5.8}$$

where C_0 is a fixed constant and

$$N(a) = \phi(\|a\|_{A_1}/\|a\|_{A_3})\cdot\theta\big(\|a\|_{A_1}/(\|a\|_{A_4}\cdot\phi(\|a\|_{A_1}/\|a\|_{A_3}))\big).$$

It is sufficient to prove that (5.8) implies

$$\|a\|_{A_3} \leqq C\|a\|_{A_1}/\psi(\|a\|_{A_1}/\|a\|_{A_4}). \tag{5.9}$$

We assume the contrary and choose a large constant C_1 and an element a_0 so that

$$\psi(\|a_0\|_{A_1}/\|a_0\|_{A_4}) \geqq C_1(\|a_0\|_{A_1}/\|a_0\|_{A_3}).$$

We put $v_0=\|a_0\|_{A_1}/\|a_0\|_{A_4}$, $u_0=\|a_0\|_{A_1}/\|a_0\|_{A_3}$ and

$$\psi(v_0) = C_2 u_0. \tag{5.10}$$

According to our assumptions we also have, for some ε, $0<\varepsilon<1/2$, and every c, $0<c<1$,

$$c^{1-\varepsilon} \leqq \frac{\phi(ct)}{\phi(t)} \leqq c^{\varepsilon} \quad \text{and} \quad c^{1-\varepsilon} \leqq \frac{\theta(ct)}{\theta(t)} \leqq c^{\varepsilon}.$$

Thus

$$\phi(u_0) = \phi\big((1/C_2)\psi(v_0)\big) \geqq (1/C_2)^{1-\varepsilon}\phi\big(\psi(v_0)\big) = (1/C_2)^{1-\varepsilon}\xi(v_0)\,,$$

and, since $\theta(t)t^{-1}$ is decreasing,

$$\begin{aligned}\phi(u_0)\theta\big(v_0/\phi(u_0)\big) &\geqq (1/C_2)^{1-\varepsilon}\xi(v_0)\theta\big(C_2^{1-\varepsilon}v_0/\xi(v_0)\big)\\ &\geqq C_2^{-(1-\varepsilon)^2}\xi(v_0)\theta\big(v_0/\xi(v_0)\big) = C_2^{-\varepsilon(1-\varepsilon)}\psi(v_0)\,.\end{aligned}$$

Therefore, by (5.8),

$$\psi(v_0) \leqq C_0\cdot C_2^{(1-\varepsilon)^2}\cdot u_0\,.$$

This inequality contradicts (5.10) as soon as $C_2 > C_0^{1/(2\varepsilon-\varepsilon^2)}$. Thus (5.9) holds and the proof is complete.

6. Interpolation between Lorentz spaces.

For the measurable function a on a measure space (Ω,μ) we define the nonincreasing rearrangement a^* in the usual way (see e.g. [3, p. 7]). The Lorentz space $\Lambda^q(\varphi)$, $0<q\leqq\infty$, $\varphi(t)\geqq 0$, is defined to be the collection of all functions a satisfying

$$\|a\|_{\Lambda^q(\varphi)} = \left(\int_0^l (a^*(t)\varphi(t))^q\frac{dt}{t}\right)^{1/q} < \infty,$$

where $l=m(\Omega)$ and with the usual interpretation of the integral when $q=\infty$.

It is well-known that, for $0<p<\infty$,

$$K(t,a,L^p,L^\infty) \approx \left(\int_0^{t^p}(a^*(u))^p\,du\right)^{1/p} \tag{6.1}$$

(see [14] or [3, p. 109]). Thus, we find

$$\|a\|_{(L^p,L^\infty)_{\varrho,q}} \approx \left(\int_0^\infty\left(\frac{1}{\varrho(t^{1/p})}\right)^q\left(\int_0^t (a^*(u))^p\,du\right)^{q/p}\frac{dt}{t}\right)^{1/q}.$$

By Lemma 1.1 we see that $1/\varrho(t^{1/p}) \in Q(-1/p,0)$. Therefore, by Lemma 3.2 (a) and the trivial estimate

$$\int_0^t (a^*(u))^p\,du \geqq t(a^*(t))^p\,,$$

we obtain

LEMMA 6.1. *Let* $0<p<\infty$, $0<q\leqq\infty$ *and* $\varrho\in Q(0,1)$. *Then*

$$(L^p,L^\infty)_{\varrho,q} = \Lambda^q\big(t^{1/p}/\varrho(t^{1/p})\big)\,.$$

REMARK 6.1. According to Proposition 1.3 (a) we see that Lemma 6.1 implies Theorem 1.3 in [9] and Lemma 3.1 in [7].

PROPOSITION 6.2. *Let* $\varphi_i(t) \in Q(0, -)$, $i=0,1$, *and* $0<p<\infty$. *Then*

(a) $(\Lambda^{q_0}(\varphi_0), L^\infty)_{\varrho,q} = \Lambda^q(\varphi)$,
where $\varphi(t)=\varphi_0(t)/\varrho(\varphi_0(t))$.

(b) *If, in addition* $\varphi_1(t) \in Q(0,1/p)$, *then*
$(L^p, \Lambda^{q_1}(\varphi_1))_{\varrho,q} = \Lambda^q(\varphi)$,
where $\varphi(t)=t^{1/p}/\varrho(t^{1/p}/\varphi_1(t))$.

(c) *If, in addition* $\varphi_0(t)/\varphi_1(t) \in Q(0, -)$ *or* $\varphi_0(t)/\varphi_1(t) \in Q(-,0)$, *then*
$(\Lambda^{q_0}(\varphi_0), \Lambda^{q_1}(\varphi_1))_{\varrho,q} = \Lambda^q(\varphi)$,
where $\varphi(t)=\varphi_0(t)/\varrho(\varphi_0(t)/\varphi_1(t))$.

PROOF. First we prove (c). Put $\varrho_i(t)=t/\varphi_i(t^p)$ and choose p so small that $\varrho_i(t)\in Q(0,1)$, $i=0,1$. According to (4.11) in Corollary 4.4 and Lemma 6.1 we obtain

$$(6.2)\qquad \begin{aligned}(\Lambda^{q_0}(\varphi_0), \Lambda^{q_1}(\varphi_1))_{\varrho,q} &= ((L^p, L^\infty)_{\varrho_0,q_0}, (L^p, L^\infty)_{\varrho_1,q_1})_{\varrho,q} \\ &= (L^p, L^\infty)_{\varrho_0\varrho(\varrho_1/\varrho_0),q} = \Lambda^q(\varphi)\,,\end{aligned}$$

where $\varphi(t)=\varphi_0(t)/\varrho(\varphi_0(t)/\varphi_1(t))$. In order to prove (b) we first note that, by Lemma 1.1, the condition $\varphi_1(t) \in Q(0,1/p)$ implies that $\varrho_1(t) =t/\varphi_1(t^p) \in Q(0,1)$. Therefore the proof follows as above by using Lemma 6.1 and (4.10) in Corollary 4.4. In a similar way we see that (a) is an easy consequence of Lemma 6.1 and (4.9).

REMARK 6.2. The formula in (c) can also be found in Theorem 4.4.1 in [20]. However, our separation conditions are somewhat less restrictive.

Finally we shall give two examples which in particular proves that our method can be used even when we have *no* a priori separation condition between our weight functions φ_0 and φ_1.

EXAMPLE 6.1. Let $\varphi_i \in Q(0, -)$, $i=0,1$. Then

$$(\Lambda^q(\varphi_0), \Lambda^q(\varphi_1))_{\varrho,q} = \Lambda^q(\varphi)\,,$$

where $\varphi(t)=\varphi_0(t)/\varrho(\varphi_0(t)/\varphi_1(t))$.

According to the calculations (6.2), Example 6.1 is a consequence of Example 4.2 and Lemma 6.1. Similarly, in view of Example 4.1, we obtain

EXAMPLE 6.2. Let $\varphi_i \in Q(0, -)$, $i=0,1$. Then

$$(\Lambda^{q_0}(\varphi_0),\Lambda^{q_1}(\varphi_1))_{\theta,q_\theta} = \Lambda^{q_\theta}(\varphi_0^{1-\theta}\varphi_1^\theta)\,.$$

REMARK 6.3. A much more general version of Example 6.2 can be found in [30]. In particular, we have descriptions of the spaces $(\Lambda^{q_0}(\varphi_0), \Lambda^{q_1}(\varphi_1))_{\theta,q}$ also for the most troublesome case when $q \neq q_\theta$ *and* $\varphi_0=\varphi_1$ (or φ_0 is only "close to" φ_1).

Finally we notice that

$$\Lambda^q(t^{1/p}(1+|\log t|)^\alpha) = L^{p,q}(\log L)^\alpha\,,$$

where $L^{p,q}(\log L)^\alpha$ are the (Lorentz–Zygmund) spaces introduced and carefully studied by Bennett–Rudnick [2]. These spaces represent a natural scale of spaces, which generalizes the usual L^p-, $L^{p,q}$-, and $L^p(\log^+ L)^\alpha$-scales of spaces (if $\mu(\Omega)<\infty$, then $L^{p,p}(\log L)^\alpha = L^p(\log^+ L)^{\alpha p}$). In particular, these observations prove that Example 6.2 implies descriptions in [1] and [7, p. 304]. Compare also with [8, p. 49]. In the same way we see that the description in [19, p. 278] is a special case of part (c) of Proposition 6.1.

7. Interpolation between the sum and the intersection.

Per Nilsson has (in a private communication) pointed out to me that estimate (4.1) also implies the following nice information: If $0<t\leq 1$, then

$$K(t,a,\Sigma(\bar{A}),\Delta(\bar{A})) \approx K(t,a,\bar{A})+tK(t^{-1},a,\bar{A}) \tag{7.1}$$

and

$$K(t,a,\Sigma(\bar{A}),A_1) \approx K(t,a,\bar{A})\,. \tag{7.2}$$

Elementary proofs of these formulas have also been carried out by Maligranda [17] (in fact, it is proved in [17] that for the Banach case (7.2) holds even with $\approx$ replaced by $=$).

By using (7.1) and an elementary argument we find that

$$\|a\|^q_{(\Sigma(\bar{A}),\Delta(\bar{A}))_{\varrho,q}} \approx \int_0^1 \left(\frac{K(t,a,\bar{A})+tK(t^{-1},a,\bar{A})}{\varrho(t)}\right)^q \frac{dt}{t} + K(1,a,\bar{A}) \int_1^\infty \left(\frac{1}{\varrho(t)}\right)^q \frac{dt}{t}\,.$$

Thus, we easily obtain the estimate

$$\|a\|_{(\Sigma(\bar{A}),\Delta(\bar{A}))_{\varrho,q}} \approx \left(\int_0^1 \left(\frac{K(t,a,\bar{A})}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left(\frac{K(t,a,\bar{A})}{\varrho^*(t)}\right)^q \frac{dt}{t}\right)^{1/q}, \tag{7.3}$$

where $\varrho^*(t)=t\varrho(1/t)$. We can use (7.2) in a similar way to see that

$$\|a\|_{(\Sigma(\bar{A}),A_1)_{\varrho,q}} \approx \left(\int_0^1 \left(\frac{K(t,a,\bar{A})}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q}$$

and

$$\|a\|_{(\Sigma(\bar{A}),A_0)_{\varrho,q}} \approx \left(\int_0^1 \left(\frac{tK(t^{-1},a,\bar{A})}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q} \approx \left(\int_1^\infty \left(\frac{K(t,a,\bar{A})}{\varrho^*(t)}\right)^q \frac{dt}{t}\right)^{1/q}.$$

In particular, we have

Proposition 7.1.

$$(\Sigma(\bar{A}),\Delta(\bar{A}))_{\varrho,q} = (\Sigma(\bar{A}),A_0)_{\varrho,q} \cap (\Sigma(\bar{A}),A_1)_{\varrho,q}.$$

At least for the Banachcase we also have the relations

$$\bar{A}_{\min(\varrho,\varrho_0),q} = \bar{A}_{\varrho,q} \cap \bar{A}_{\varrho_0,q} \quad \text{and} \quad \bar{A}_{\max(\varrho,\varrho_0),q} = \bar{A}_{\varrho,q} + \bar{A}_{\varrho_0,q}.$$

See [5, p. 169]. Therefore, by (7.3), we also obtain

Proposition 7.2. *Let $\bar{A}=(A_0,A_1)$ be a Banach pair and $1\leqq q\leqq\infty$. If $\varrho\in Q(0,1/2]$, then*

$$(\Sigma(\bar{A}),\Delta(\bar{A}))_{\varrho,q} = \bar{A}_{\varrho,q}+\bar{A}_{\varrho^*,q},$$

and

$$(\Sigma(\bar{A}),\Delta(\bar{A}))_{\varrho^*,q} = \bar{A}_{\varrho,q} \cap \bar{A}_{\varrho^*,q}.$$

Remark 7.1. According to our Proposition 1.3 we see that this is just another way of formulating Proposition 3 in [18].

In view of (6.1) another consequence of (7.3) is that

$$\|a\|_{(L^p+L^\infty,L^p\cap L^\infty)_{\varrho,q}}$$

(7.4)
$$\approx \int_0^1 \left(\frac{1}{\varrho(t^{1/p})}\right)^q \left(\int_0^t (a^*(u))^p\,du\right)^{q/p}\frac{dt}{t} + \int_1^\infty \left(\frac{1}{\varrho^*(t^{1/p})}\right)^q \left(\int_0^t (a^*(u))^p\,du\right)^{q/p}\frac{dt}{t}\,.$$

If $\varphi(t)=(\varrho(t^{1/p}))^{-1}$, $0\leqq t\leqq 1$, and $\varphi(t)=(\varrho^*(t^{1/p}))^{-1}$, $t\geqq 1$, then it is easy to see that $\varphi(t)\in Q(-1/p,0)$. Therefore we can apply Lemma 3.2 to the estimate (7.4) and get the following

EXAMPLE 7.1. Let $0<p<\infty$, $0<q\leqq\infty$ and $\varrho(t)\in Q(0,1)$. Then

$$(L^p+L^\infty, L^p\cap L^\infty)_{\varrho,q} = \Lambda^q(t^{1/p}/\varrho_1(t^{1/p}))\,,$$

where $\varrho_1(t)=\varrho(t)$ if $0<t\leqq 1$, and $\varrho_1(t)=\varrho^*(t)$ if $t\geqq 1$.

REMARK 7.2. Example 7.1 ought to be compared with Lemma 6.1. Moreover, if $\varrho\in Q(0,1/2]$, then $\varrho_1(t)=\max(\varrho(t),\varrho^*(t))$ and if $\varrho\in Q[1/2,1)$, then $\varrho_1(t)=\min(\varrho(t),\varrho^*(t))$. Thus Example 7.1 is a special case of Proposition 7.2 in these cases.

8. Concluding remarks.

Let $L^p(A,\omega)$ denote the space of A-valued, strongly measurable functions $a(x)$ on a measure space (Ω,μ) satisfying

$$\left(\int_\Omega (\|a(x)\|_A\omega(x))^p\,d\mu(x)\right)^{1/p} < \infty\,.$$

In particular, if $\omega(x)\equiv 1$, $\Omega=]0,\infty[$ and $d\mu(x)=dt/t$ (where dt denotes the Lebesque-measure) then we have the spaces denoted $L^p_*(A)$ in the literature.

We remark that our descriptions (3.1) and (3.2) are special cases of the general descriptions of the spaces $(L^{p_0}(A_0,\omega_0),L^{p_1}(A_1,\omega_1))_{\varrho,q}$ (and more general spaces of this type) which have been obtained in [27] (the case $\varrho(t)=t^\theta$) and [28]. We emphasize the fact that we need not here exclude the troublesome (off-diagonal) case when $q\neq p_\theta$. We say that $a(x)$ *belongs to the (Lions–Peetre) generalized "space of means" and write* $a(x)\in \underline{S}(A_0,A_1,p_0,p_1,\varrho(t))$, whenever

$$\inf_{a=a_0(t)+a_1(t)}\left(\left\|\frac{a_0(t)}{\varrho(t)}\right\|^{p_0}_{L^{p_0}_*(A_0)}+\left\|\frac{a_1(t)}{\varrho(t)/t}\right\|^{p_1}_{L^{p_1}_*(A_1)}\right)^{1/p_\theta} < \infty\,.$$

We can generalize the proof in [3, pp. 71–72] and obtain at least the following connection:

EXAMPLE 8.1.

$$\underline{S}(A_0, A_1, p_0, p_1, \varrho(t)) = \left(A_0^{p_0}, A_1^{p_1}\right)_{\varrho_1, 1}^{1/p_0},$$

where $\varrho_1(t) = (\varrho(h^{-1}(t)))^{p_0}$ and $h^{-1}(t)$ is the inverse of $h(t) = (\varrho(t))^{p_0 - p_1} t^{p_1}$.

REMARK 8.1. In the case when $p_0 = p_1 = p$ we have $\varrho_1(t) = (\varrho(t^{1/p}))^p$. Therefore, by Theorem 2.3,

$$\underline{S}(A_0, A_1, p, p, \varrho(t)) = (A_0, A_1)_{\varrho(t), p}.$$

In the case when $p_0 \neq p_1$ we can only get a more complicated description by using the K-functional (see Remark 2.1).

REMARK 8.2. The spaces $\underline{S}$ correspond to the K-method. In a similar way we can generalize the (Lions–Peetre) spaces $\bar{S}$ which correspond to the J-method. Moreover, we can generalize the proof in [3, p. 70] to see that $\underline{S} = \bar{S}$ in this case, too.

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