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QUASICONFORMALITY OF PSEUDO-CONFORMAL TRANSFORMATIONS AND DEFORMATIONS OF HYPERSURFACES IN $\mathbf{C}^{n+1}$

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Summary.

Quasiconformality (with respect to a Riemannian metric generated by a contact form and the natural complex structure) of pseudo-conformal transformations and deformations of a real hypersurface of codimension 1 in $\mathbf{C}^{n+1}$ is investigated.

Introduction and outline results.

Conformality and holomorphy have the same source: complex analytic transformations in the space $\mathbf{C}^1$. Biholomorphic transformations of several complex variables are no more conformal with respect to the Euclidean metric. Yet, in multidimensional spaces there are relations between the two notions. In particular, there are metrics with respect to which every biholomorphic transformation is conformal. The Bergman metric is an example. Also in the case of real hypersurfaces of real codimension 1 in $\mathbf{C}^{n+1}$ there are relations between conformality and biholomorphy. Under suitable assumptions (formulated below) a hypersurface M in $\mathbf{C}^{n+1}$ can be endowed with a contact metric structure (contact structure is an odd dimensional analogue of the complex structure). The requirement for a transformation of a contact metric manifold to be conformal implies very strong restrictions (cf. [12]). The situation is different if we require conformality on some distribution $\mathscr{D}$ only. The distribution $\mathscr{D}$ of real dimension $2n$ is defined as the assignment to every point of M the maximal J-invariant subspace of the tangent space at this point. Some information on transformations which are conformal on a distribution one can find in [11].

In this paper we construct a Riemannian metric on a real hypersurface M in $\mathbf{C}^{n+1}$ with the property that every transformation of M locally extendable to a biholomorphic transformation of a neighbourhood of M

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in $\mathbf{C}^{n+1}$ is conformal on the distribution $\mathscr{D}$ (Lemma 1). The Riemannian metric is generated by a contact form η on M and the natural complex structure J in $\mathbf{C}^{n+1}$. We also study conformality, or more generally, quasiconformality of pseudo-conformal transformations, i.e. transformations of M which preserve the distribution $\mathscr{D}$ and whose differentials commute with J. It is well known that under some smoothness conditions pseudo-conformal mappings of a hypersurface M contained in the complex space $\mathbf{C}^{n+1}$ have biholomorphic extensions to a neighbourhood of M in $\mathbf{C}^{n+1}$. In this sense they are restrictions of biholomorphic mappings of domains in this space. Many important properties of pseudo-conformal mappings can be found in [6], [7], [10], [5], [8] and [13]. We study geometrical properties of pseudo-conformal transformations and estimate their rank of quasiconformality (Theorem 1).

For pseudo-conformal deformations Z (which are infinitesimal versions of pseudo-conformal transformations) we find the precise form of Ahlfors' operator SZ (Lemma 2) and observe that its restriction to the complex distribution $\mathscr{D}$ has a particularly simple form. Namely, it is a multiplicity of the Riemannian metric tensor (Corollary to Lemma 2). Finally we estimate the rank of quasiconformality of a deformation Z (Theorem 2). Pseudo-conformal deformations of domains in the Euclidean space R^k, $k>2$, were systematically investigated by Ahlfors in [1], [2], [3].

All manifolds and mappings in this paper are assumed to be smooth, i.e. of the class C^∞.

1. Hypersurfaces with contact structures in $\mathbf{C}^{n+1}$.

By a hypersurface in $\mathbf{C}^{n+1}$ we mean an oriented real submanifold of dimension $k=2n+1$ in $\mathbf{C}^{n+1}=\mathsf{R}^{2n+2}$. For every point p of such a hypersurface M we denote by T_pM the tangent space of M at p. Of course T_pM is a vector subspace of the vector space $T_p\mathbf{C}^{n+1}\cong\mathsf{R}^{2n+2}$. In $T_p\mathbf{C}^{n+1}$ we have the natural complex structure J. Consequently, in T_pM we can distinguish the unique J-invariant subspace $\mathscr{D}_p$ of dimension $2n$. Obviously J, restricted to the linear space $\mathscr{D}_p$, is a complex structure in this space, which we also denote by J. The assignment $p\mapsto\mathscr{D}_p$ defines a distribution $\mathscr{D}$ on M, which will be called the hyperdistribution with complex structure [13]. Because M is oriented there is a family H of 1-forms η on M which annihilate $\mathscr{D}$, that is which satisfy the condition

$$\eta(X) = 0 \quad \text{if and only if } X\in\mathscr{D} \tag{1}$$

for all vector fields X on M, where the inscription $X\in\mathscr{D}$ means $X_p\in\mathscr{D}_p$

for every point $p \in M$. Every two forms from H differ by a (multiplicative) non-vanishing function.

Assume that among the forms of the family H there exists a contact form, i.e. such a form η that the condition

$$\eta \wedge (d\eta)^n \neq 0 \tag{2}$$

holds. For example, if M is a smooth boundary of strictly pseudo-convex domain such a form always exists.

Now fix such a form η. The conditions

$$\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi, X) = 0, \quad X \in \mathscr{X}(M) \tag{3}$$

where $\mathscr{X}(M)$ denotes the family of all smooth vector fields on M, define a unique vector field ξ on M. Next define the tensor field f on M (being a counterpart of the complex structure) of the type $(1,1)$ as follows

$$fX = J(X - \eta(X)\xi), \quad X \in \mathscr{X}(M). \tag{4}$$

It is easy to see that the system (f, ξ, η) satisfies the following conditions:

$$f^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad f \circ \xi = 0, \quad \eta \circ f = 0, \tag{5}$$

where I is the identity operator.

Define the Levi form G on M being a tensor field of the type (0,2) as follows:

$$G(X, Y) = d\eta(fX, Y), \quad X, Y \in \mathscr{X}(M). \tag{6}$$

Then G is symmetric. Indeed, first assume that $X, Y \in \mathscr{D}$. By the integrability condition for J in $\mathbf{C}^{n+1}$ ($[\cdot,\cdot]$ is the Lie bracket):

$$[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y] = 0,$$

J-invariance of $\mathscr{D}$ and the fact that $fX = JX$ (compare the definition (4)) we get

$$[fX, fY] - [X, Y] - f[X, fY] - f[fX, Y] = 0.$$

Consequently,

$$\eta([fX, fY]) = \eta([X, Y])$$

Replacing Y by fY we obtain

$$\eta([fX, Y]) = \eta([fY, X])$$

which follows that

$$G(X,Y) = G(Y,X)\,. \tag{7}$$

Now assume that X and Y are arbitrary vector fields belonging to $\mathscr{X}(M)$. Then we can write

$$X = X_{\mathscr{D}} + \eta(X)\xi, \quad Y = Y_{\mathscr{D}} + \eta(Y)\xi\,,$$

where $X_{\mathscr{D}}, Y_{\mathscr{D}} \in \mathscr{D}$. By the second part of (3) we obtain

$$d\eta(fX, Y) = d\eta(fX_{\mathscr{D}}, Y_{\mathscr{D}})$$

and, consequently, (7) holds also in the case of arbitrary fields $X, Y \in \mathscr{X}(M)$.

Suppose that the form G is positively defined on $\mathscr{D}$, that is that for any point $p \in M$ and any vector $X_p \in \mathscr{D}_p$, $X_p \neq 0$, we have $G(X_p, X_p) > 0$. Then the tensor field g of the type (0,2) defined on M by the formula

$$g(X,Y) = G(X,Y) + \eta(X)\eta(Y), \quad X, Y \in \mathscr{X}(M) \tag{8}$$

is symmetric and positively defined, and thus can be regarded as a Riemannian scalar product on M.

Observe that

$$\eta(X) = g(X, \xi)$$

and

$$g(fX, Y) = -g(X, fY)\,,$$

which together with conditions (5) say that the system (f, ξ, η, g) is a contact metric structure on M (cf. [4]).

In this way, a contact form on a hypersurface M in $\mathbf{C}^{n+1}$ may generate a Riemannian scalar product g on M. We are going to investigate conformality or – more general – quasiconformality (with respect to g) of transformations and deformations which have holomorphic extensions from M to domains of the ambient space $\mathbf{C}^{n+1}$.

2. Quasiconformality of pseudo-conformal transformations.

Assume that M and g are as above. Let $F: M \to M$ be a transformation, i.e. a smooth diffeomorphism of the manifold M onto itself.

F is said to be a pseudo-conformal transformation if it satisfies the following conditions (cf. [13]):

$$DFX \in \mathscr{D} \quad \text{if and only if } X \in \mathscr{D}\,, \tag{9}$$

$$DF(JX) = J(DFX) \quad \text{for } X \in \mathscr{D}\,. \tag{10}$$

It is well known that under some smoothness conditions pseudo-conformal mappings have biholomorphic extensions to some open neighbourhoods of M in $\mathbf{C}^{n+1}$. In this sense they are restrictions to M of biholomorphic mappings of domains in $\mathbf{C}^{n+1}$.

Now we are going to study some geometrical properties of pseudo-conformal transformations of a hypersurface M with a metric g mentioned above.

We are interested especially in deformations of angles, and then we are going to estimate the rank of quasiconformality. To this purpose let us introduce the notion of quasiconformality.

Let M be a Riemannian manifold of dimension m with a metric tensor g. Let $F: M \to M$ be a transformation. We say that F is a *K-quasi-conformal transformation*, $1 \leqq K < \infty$, if the norm $\|B^F\|$ of the tensor field B^F is bounded on M by K (cf. [9]), i.e. if

$$(11) \qquad \|B_p^F\| \leqq K, \quad p \in M ,$$

where the tensor field B^F of the type (0,2) is defined as follows

$$(12) \qquad B^F(X, Y) = g(J_F^{-1/m} DFX, J_F^{-1/m} DFY) \quad X, Y \in \mathscr{X}(M)$$

(J_F is the Jacobian of F), and where the norm $\|B_p\|$ of the tensor field B at p is defined as follows

$$\|B_p\| = \sup |B_p(X_p, X_p)|^{1/2} .$$

The supremum is taken over all vectors $X_p \in T_pM$ with the length not greater than 1, i.e. all vectors X_p satisfying $\|X_p\| = g(X_p, X_p)^{1/2} \leqq 1$.

A transformation F: $M \to M$ is said to be quasiconformal if it is K-quasiconformal for some K.

For a geometrical interpretation of the above definition there could be shown that quasiconformal transformations map infinitesimal spheres onto infinitesimal ellipsoids with bounded ratio of the greatest semiaxis to the smallest one in the whole manifold. Consequently, "a deformation of angles" is also bounded. This explains the name of the transformations.

More detailed information on quasiconformal transformations on manifolds may be found in the paper [9].

Now we are going to investigate how pseudo-conformal transformations change tensors η, ξ, f, G and g.

Let M be a hypersurface in $\mathbf{C}^{n+1}$ and (f, ξ, η, g) be as before. Let F: $M \to M$ be a pseudo-conformal transformation.

Let $F^*\eta$ be the 1-form on M defined by

$$F^*\eta(X_p) = \eta(DFX_p), \quad X_p \in T_pM .$$

By the conditions (1) and (9) we get that there exists a non-vanishing function a on M such that

$$F^*\eta = a\eta . \tag{13}$$

If we now consider the vector field $DF\xi$ and decompose it onto the sum of two orthogonal terms such that one of them belongs to $\mathscr{D}$ and the other is parallel to ξ, then, by the first condition in (3), we see that there exists on M a unique vector field A belonging to $\mathscr{D}$ such that

$$DF\xi_p = a(p)\xi_{F(p)} + DF(fA_p), \quad p \in M .$$

Since for every vector field X the vector field $X - \eta(X)\xi$ belongs to $\mathscr{D}$, $f \mid \mathscr{D} = J$ and J commutes with DF on $\mathscr{D}$ (cf. the relation (10)), then

$$(DF_p \circ f_p)(X_p) = f_{F(p)}DF_p(X_p) + \eta_p(X_p)DF_pA_p .$$

Consequently, we obtain the following relation

$$DF \circ f = f \circ DF + \eta \otimes DF(A) . \tag{15}$$

Relations (13), (14), and (15) enable us to get a transformation formula of the scalar product g. Let us first compute, with their help, how the form $d\eta$ changes under the transformation F. For arbitrary vector field X and Y on M we have

$$d\eta(DFX, DFY) \circ F = ad\eta(X, Y) + da \wedge \eta(X, Y) ,$$

which gives the following transformation formula for $d\eta$:

$$d\eta(DFX, DFY) \circ F = ad\eta(X, Y) + da \wedge \eta(X, Y), \quad X, Y \in \mathscr{X}(M) \tag{16}$$

or, shortly,

$$F^*d\eta = ad\eta + da \wedge \eta .$$

By (15) and (16), an analogous computation shows that for arbitrary vector fields X and Y on M we have

$$\begin{aligned} G(DFX, DFY) \circ F &= aG(X, Y) + da(fX)\eta(Y) - \\ &\quad - a\eta(X)d\eta(A, Y) - da(A)\eta(X)\eta(Y), \end{aligned} \tag{17}$$

what, by the symmetry of G, implies

$$da(fX) = -ad\eta(A, X)$$

or, in particular

$$da(A) = -aG(A, A).$$

In this way the formula (17) takes the form

$$G(DFX, DFY)\circ F = a[G(X,Y)+G(fA,X)\eta(Y)+\eta(X)G(fA,Y)+ \\ +G(fA,fA)\eta(X)\eta(Y)] .$$

Consequently, by (8), (13) and (18), we get the following transformation formula for the metric g:

$$g(DFX, DFY)\circ F = ag(X,Y)+a[g(fA,X)\eta(Y)+\eta(X)g(fA,Y)+ \\ +(\|fA\|^2+a-1)\eta(X)\eta(Y)] \tag{19}$$

or, equivalently, with the help of G:

$$g(DFX, DFY)\circ F = a[G(X,Y)+G(fA,X)\eta(Y)+\eta(X)G(fA,Y)+ \\ +\big(G(fA,fA)+a\big)\eta(X)\eta(Y)] . \tag{19'}$$

It is easy to see, that if X and Y belong to $\mathscr{D}$, then

$$g(DFX, DFY)\circ F = ag(X,Y) . \tag{20}$$

Consequently, we obtain the following

LEMMA 1. *Every pseudo-conformal transformation $F\colon M \to M$ is conformal on the distribution $\mathscr{D}$, that is it preserves angles between vectors belonging to $\mathscr{D}$.*

Notice that the transformation formula (19′) will permit to estimate the rank of quasi-conformality of a pseudo-conformal transformation F if the Jacobian J_F of F is known.

To compute J_F take a point $p \in M$. Since the vector field ξ is orthogonal to $\mathscr{D}$ and $\|\xi\| = 1$, then

$$J_F(p) = J_{DF_p} = J_{DF_p|\mathscr{D}_p}\cdot a(p) .$$

But, by the formula (20), the Jacobian $J_{DF_p|\mathscr{D}_p}$ of the linear mapping $DF_p|\mathscr{D}_p$ (the restriction of DF_p to $\mathscr{D}_p$) equals $a(p)^n$. Consequently,

$$J_F(p) = a(p)^{n+1}, \quad p \in M . \tag{21}$$

In this way, for an arbitrary pseudo-conformal transformation F, we express all transformation formulas through two parameters: the function a and the vector field A ($A \in \mathscr{D}$), which are uniquely determined by F. They will be called the *characteristic parameters of* F.

Now we shall estimate the rank of quasiconformality of F through its characteristic parameters. We can prove the following

THEOREM 1. *If $F: M \to M$ is a pseudo-conformal transformation such that its characteristic parameters a and A are bounded from above and a is greater than a positive constant on M, then F is K-quasiconformal, where*

$$K \leqq \sup_{p \in M} \{[a(p)]^{-1/(4n-4)}[1+a(p)+\|A(p)\|^2]^{1/2}\} .$$

PROOF. Let $m = \dim M = 2n+1$. By definition (12), we obtain

$$\|B^F\| = \|DF\|/J_F^{1/m} ,$$

and it is enough to estimate the function $\|DF\|/J_F^{1/(2n+1)}$ (cf. Definition (11)). By the formula (19′) we obtain

$$g(DFX, DFX) \circ F = a[G(X,X) + 2G(fA, X)\eta(X) + \\ + G(fA, fA)\eta^2(X) + a\eta^2(X)], \quad X \in \mathscr{X}(M) .$$

If we decompose X into two components $X = X_{\mathscr{D}} + X_\xi$, where $X_{\mathscr{D}}$ belongs to $\mathscr{D}$ and X_ξ is parallel to ξ, then, by properties (3) and (6) of η and G, and, by definition (8) of g we derive

$$\|DFX\|^2 \circ F = a[\|X_{\mathscr{D}}\|^2 + 2G(fA, X_{\mathscr{D}})\|X_\xi\| + (\|A\|^2 + a)\|X_\zeta\|^2] .$$

Using the Schwarz inequality, we obtain

$$\|DFX\|^2 \circ F \leqq a[(\|X_{\mathscr{D}}\| + \|A\| \; \|X_\xi\|)^2 + a\|X_\xi\|^2] .$$

Now, assume that $\|X\| = 1$. Since $X_{\mathscr{D}}$ and X_ξ are orthogonal, $\|X_{\mathscr{D}}\|^2 + \|X_\xi\|^2 = 1$. The maximum of the expression $x + Ay$ with the condition $x^2 + y^2 = 1$ equals $1 + A^2$. Consequently,

$$\|DFX\|^2 \circ F \leqq a[1 + a + \|A\|^2], \quad X \in \mathscr{X}(M), \; \|X\| = 1 .$$

This implies

$$\|DF\| \leqq a^{1/2}(1 + a + \|A\|^2)^{1/2} ,$$

and then, by formula (21), we obtain

$$\|DF\|/J_F^{1/(2n+1)} \leqq a^{-1/(4n+2)}(1 + a + \|A\|^2)^{1/2},$$

which is equivalent to our assertion.

3. Pseudo-conformal deformations.

Assume that M is a hypersurface in $\mathbf{C}^{n+1}$ equipped with a system (f, ξ, η, g) described above. Let Z be a vector field on the hypersurface M. We say that Z is a pseudo-conformal deformation if the local group of local transformations generated by Z is a group of pseudo-conformal transformations.

Similarly to pseudo-conformal transformations, the pseudo-conformal deformations are, under some smoothness conditions, restrictions of holomorphic deformations of a neighbourhood of M in $\mathbf{C}^{n+1}$ (cf. [13]).

It may be also proved (cf. [13]) that Z is a pseudo-conformal deformation of M if and only if there exist a function α on M and a vector field V belonging to $\mathscr{D}$ such that

$$(22) \qquad \mathscr{L}_Z\eta = \alpha\eta, \quad \mathscr{L}_Z\xi = -\alpha\xi - fV. \quad \mathscr{L}_Z f = -\eta\otimes V.$$

The function α and the field V will be called the characteristic parameters of the field Z.

We are going to estimate the rank of quasiconformality of Z. The measure of quasiconformality of a deformation Z on a Riemannian manifold M is the norm of Ahlfors' operator SZ. SZ is a symmetric tensor field of the type (0,2) on M with zero trace, being an infinitesimal version (in the direction Z) of the tensor field B in the formula (12) defined as follows

$$(23) \quad SZ(X, Y) = (\mathscr{L}_Z g)(X, Y) - \frac{2}{n}\operatorname{div} Z g(X, Y), \qquad X, Y \in \mathscr{X}(M).$$

We accept the following definition (cf. [9]).

A deformation Z on M will be called a *k-quasiconformal deformation*, $0 \leqq k < \infty$, if

$$\|SZ\| \leqq k$$

on M.

It can be proved (see [9]) that every complete k-quasiconformal deformation on a Riemannian manifold M generates such a one-parameter family of transformations F_t, $t \in \mathrm{R}$, of M that F_t is an $\exp(\frac{1}{2}k^2|t|)$-quasiconformal transformation. Of course, 0-quasiconformal deformations are conformal, i.e. they are conformal Killing vector fields.

The formula (22) enables us to express SZ in a form which will be convenient to estimations. Let us first compute $\mathscr{L}_Z G$:

$$\mathscr{L}_Z G(X, Y) = \alpha G(X, Y) + d\alpha(fX)\eta(Y) - \eta(X)d\eta(V, Y).$$

By the symmetry of G we obtain

$$d\alpha(fX) = -d\eta(V, X).$$

Consequently,

$$\begin{aligned}\mathscr{L}_Z G(X,Y) &= \alpha G(X,Y)-[\eta(X)d\eta(V,Y)+\eta(Y)d\eta(V,X)]\\ &= \alpha G(X,Y)+[\eta(X)G(fV,Y)+G(fV,X)\eta(Y)]\,.\end{aligned}$$

Next, by (21), we obtain

$$\operatorname{div} Z = (n+1)\alpha\,.$$

Finally, by the two last formulas and the definition (23) we derive

$$\begin{aligned}SZ(X,Y) &= \alpha G(X,Y)+2\alpha\eta(X)\eta(Y)-\\ &\quad -\frac{2n+2}{2n+1}\alpha g(X,Y)+\eta(X)g(fV,Y)+g(fV,X)\eta(Y)\\ &= -\frac{\alpha}{2n+1}g(X,Y)+\alpha\eta(X)\eta(Y)+\eta(X)g(fV,Y)+\\ &\quad +g(fV,X)\eta(Y)\,.\end{aligned}$$

In this way we proved the following

LEMMA 2. *If Z is a pseudo-conformal transformation of a hypersurface M in $\mathbf{C}^{n+1}$ and α, V are its characteristic parameters, then*

$$SZ = -\frac{\alpha}{2n+1}g+\alpha\eta\otimes\eta+\eta\otimes g(fV,\cdot)+g(fV,\cdot)\otimes\eta\,. \tag{24}$$

COROLLARY. *If M and Z are as in Lemma* 1, *then for all vector fields X, Y belonging to $\mathscr{D}$*

$$SZ(X,Y) = -\frac{\alpha}{2n+1}g(X,Y)\,.$$

We can now estimate the norm of SZ. Namely, we shall prove the following:

THEOREM 2. *Let M be a hypersurface in $\mathbf{C}^{n+1}$. Every pseudo-conformal deformation Z of M with bounded characteristic parameters and V is a k-quasiconformal deformation, where*

$$k \leqq \sup_M\left(\frac{2n}{2n+1}|\alpha|+\|V\|\right)^{1/2}.$$

PROOF. Let X be a vector field on M with the property $\|X\|=1$. Then $X=X_{\mathscr{D}}+X_{\xi}$, where $X_{\mathscr{D}}$ belongs to $\mathscr{D}$ and X_{ξ} is parallel to ξ. Of course,

$$\|X_{\mathscr{D}}\|^2+\|X_{\xi}\|^2 = 1 . \tag{25}$$

By the formula (24) in Lemma 1, we obtain

$$SZ(X,X) = -\frac{\alpha}{2n+1}+\alpha\|X_{\xi}\|^2+2\|X_{\xi}\|g(fV,X_{\mathscr{D}}) .$$

Using the Schwarz inequality and taking the supremum over all X satisfying (25), we derive

$$\|SZ\| \leqq \left(\frac{2}{2n+1}|\alpha|+\|V\|\right)^{1/2}$$

which, by the definition of quasiconformality of the deformation Z, completes the proof.

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