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GROWTH PROPERTIES OF ANALYTIC AND PLURISUBHARMONIC FUNCTIONS OF FINITE ORDER

RAGNAR SIGURÐSSON

CONTENTS

Preface.

In this paper we are going to study the growth of analytic functions in $\mathbf{C}^n$ of finite order and of finite type. We use two growth characteristics in our study, the classical indicator function introduced by Phragmén and Lindelöf and the limit set of $\log|f|$ recently introduced by Azarin [3]. If f is of order $\varrho>0$ and of finite type, then the indicator function i_f of f is defined as the least upper semi-continuous majorant of $\overline{\lim}_{t\to\infty} t^{-\varrho}\log|f(t.)|$. It is plurisubharmonic in $\mathbf{C}^n$ and positively homogeneous of order ϱ. The limit set of $\log|f|$ is defined as the set of all plurisubharmonic functions in $\mathbf{C}^n$ that are limits in the sense of distributions of sequences of the form $\{t_j^{-\varrho}\log|f(t_j.)|\}$, where $t_j \to \infty$.

The classical indicator theorem gives a characterization of plurisubharmonic functions in $\mathbf{C}^n$, which are indicator functions of some analytic function. It states that if p is plurisubharmonic in $\mathbf{C}^n$ and positively homogeneous of order $\varrho>0$, then there exists an analytic function f in $\mathbf{C}^n$ such that $i_f=p$.

In the theory of entire functions of one variable, analytic functions f of general order $\varrho>0$ are commonly constructed with the aid of Hadamard's product theorem. The asymptotic behavior of f is then described in terms of pointwise convergence of $t^{-\varrho}\log|f(t.)|$ as $t\to\infty$ outside a certain exceptional set. In the case of several variables these methods must be abandoned. In this paper we construct the functions f with the aid of Hörmander's existence theory for the Cauchy–Riemann system and we describe the asymptotic behavior of f by considering the family $\{t^{-\varrho}\log|f(t.)|;\ t>0\}$ of plurisubharmonic functions in the distribution topology.

The paper is divided into three chapters. In Chapter 1 we deal with functions of general order $\varrho>0$ and of finite type. We begin by studying the properties of the limit sets, then we deal with the problem of constructing an analytic function f with the limit set of $\log|f|$ equal to a prescribed set of plurisubharmonic functions, and finally we give a refinement of the indicator theorem consisting of an L^2-estimate of the analytic function f.

One of the main motivations for the theory of entire analytic functions is the fact that Fourier–Laplace transforms of analytic functionals are analytic functions of exponential type. In Chapter 2 we study indicator functions of Fourier–Laplace transforms of hyperfunctions with compact support. If u is a hyperfunction with compact support, then it can be represented by an analytic functional with support contained in $\mathbf{R}^n$. If K denotes the convex hull of the support of u and H is the supporting function of K, then for every $\varepsilon > 0$ there exists a positive constant C_ε such that the Fourier–Laplace transform $\hat{u}$ of u satisfies

$$|\hat{u}(\zeta)| \leqq C_\varepsilon \exp\big(H(\operatorname{Im}\zeta)+\varepsilon|\zeta|\big) \quad \text{for } \zeta \in \mathbf{C}^n .$$

This implies $i_{\hat{u}}(\zeta) \leqq H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n$. We prove that $i_{\hat{u}}(\zeta) = H(\operatorname{Im}\zeta)$ for all $\zeta \in \mathbf{C}\mathbf{R}^n$. The main result of the chapter is a description of the set of all $\zeta \in \mathbf{C}^n$ with $i_{\hat{u}}(\zeta) = H(\operatorname{Im}\zeta)$ in terms of the analytic singularities of u at the supporting planes of K. The proof is based on a Paley–Wiener type theorem for analytic singularities, which is due to Hörmander.

In Chapter 3 we study Fourier–Laplace transforms of distributions u with compact support. Then $\hat{u}$ satisfies a growth estimate of the form

$$|\hat{u}(\zeta)| \leqq C(1+|\zeta|)^N \exp\big(H(\operatorname{Im}\zeta)\big) \quad \text{for } \zeta \in \mathbf{C}^n ,$$

where C and N are positive constants and H is the supporting function of the convex hull K of the support of u. This implies that every function p in the limit set of $\log|\hat{u}|$ satisfies $p(\zeta) \leqq H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n$. In the case $n=1$, the theorem of Ahlfors and Heins [1] gives that $t^{-1}\log|\hat{u}(t.)| \to H(\operatorname{Im}.)$ in the sense of distributions as $t \to \infty$. An analogous result does not hold if $n>1$ as Vauthier [22] has shown. However, there is a certain regularity in the growth of $\hat{u}$ near $\mathbf{C}\mathbf{R}^n$, for we show that all the functions p in the limit set of $\log|\hat{u}|$ satisfy $p(\zeta) = H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}\mathbf{R}^n$. The proof is based on a theorem of Hörmander [8] on the asymptotic behavior of $\hat{u}$ near $\mathbf{C}\mathbf{R}^n$. We prove a variant of the indicator theorem for Fourier–Laplace transforms of distributions with compact support. A natural problem which arises in this context is to construct distributions with the limit set of $\log|\hat{u}|$ equal to a prescribed set of plurisubharmonic functions. We do not solve this problem in general, but we generalize the construction given in Vauthier [22] by proving the existence of a distribution u with the limit set of $\log|\hat{u}|$ equal to the convex hull of finitely many plurisubharmonic functions. Finally, we show that there exist distributions u with $i_{\hat{u}}$ discontinuous. The existence of discontinuous indicator functions was first proved by Lelong [14].

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Chapter 1. General functions of order ϱ.

1.0. Introduction.

Let f be an entire analytic function in $\mathbf{C}^n$ of general order $\varrho > 0$ and of finite type. Then f satisfies a growth estimate of the form

$$|f(z)| \leqq \exp(\tau + \sigma|z|^\varrho), \quad z \in \mathbf{C}^n,$$

where τ and σ are positive constants. In this chapter we are going to study the asymptotic behavior of f by considering the limit points at infinity of the family $\{t^{-\varrho}\log|f(t.)|;\ t \geqq 1\}$ in the distribution topology in $\mathbf{C}^n$, and by considering the indicator function i_f of f defined by

$$i_f(z) = \overline{\lim_{w\to z}}\ \overline{\lim_{t\to\infty}}\ t^{-\varrho}\log|f(tw)|, \quad z \in \mathbf{C}^n.$$

Since the function $\log|f|$ is plurisubharmonic, it is natural to consider plurisubharmonic functions p of order $\varrho > 0$ and of finite type. These functions satisfy a growth condition of the form

$$p(z) \leqq \tau + \sigma|z|^\varrho, \quad z \in \mathbf{C}^n.$$

For every $t > 0$, we define the operator $T_t\colon L^1_{\text{loc}}(\mathbf{C}^n) \to L^1_{\text{loc}}(\mathbf{C}^n)$ *by* $T_t q(z) = t^{-\varrho} q(tz)$ for $z \in \mathbf{C}^n$. If p is plurisubharmonic in $\mathbf{C}^n$ and of order $\varrho > 0$ and of finite type, then we let $L(p)$ denote the set of all plurisubharmonic functions in $\mathbf{C}^n$ that are limits in $L^1_{\text{loc}}(\mathbf{C}^n)$ to sequences of the form $\{T_{t_j} p\}$, where $t_j \to \infty$. The set $L(p)$ is called the limit set of p.

In section 1.1 we define the order, type, limit set, and indicator function of a plurisubharmonic function p in $\mathbf{C}^n$. If p is of order $\varrho > 0$ and of finite type, then the set $L(p)$ turns out to be compact, connected, and invariant under T_t for all $t > 0$. Its elements all vanish at the origin and are bounded by $\sigma|z|^\varrho$ for some $\sigma > 0$. The indicator function of p is the least upper semi-continuous majorant of all the elements in $L(p)$. In the special case when p is positively homogeneous of order one and satisfies $p(z) \leqq \sigma|y|$ for $z = x + iy \in \mathbf{C}^n$, where σ is a positive constant, then $p(x+iy) \leqq p(iy)$ with equality when x and y are proportional. Furthermore $\mathbf{R}^n \ni y \to p(iy)$ is a supporting function.

In section 1.2 we deal with the problem of constructing a plurisubharmonic function with prescribed limit set. We prove that every compact, connected set M of plurisubharmonic functions, that are positively homogeneous of order ϱ, is the limit set of a plurisubharmonic function of order ϱ. In the general case we are not able to replace the homogeneity condition by invariance of the set M under T_t. However, we can prove, that for every compact invariant set M of plurisubharmonic functions in $\mathbf{C}^n$ that vanish at the origin and are bounded by $\sigma|z|^\varrho$ for some $\sigma>0$, there exists a plurisubharmonic function p with $M \subset L(p)$ and $L(p)$ contained in the union of all line segments with endpoints in M. In the special case when M is convex, M is equal to the limit set.

The result generalizes those of Azarin [3] for the case of subharmonic functions of non-integral order in the plane. Only a slight modification of our method gives a generalization of his results for subharmonic functions in $\mathbf{R}^n$. The idea in Azarin's construction is to choose a sequence $\{q_k\}$ in M forming a dense subset in the weak topology in $\mathscr{D}'$, such that every element appears infinitely many times in it. Then p is defined so that its Riesz mass is equal to $\sum \varphi_k \mu_k$, where $\{\varphi_k\}$ is a partition of unity in $\mathbf{R}^n$ and μ_k is the Riesz mass of $x \mapsto s_k^{-\varrho} q_k(s_k x)$ for certain numbers s_k. Instead of dealing with the Riesz masses we use the sequence $\{q_k\}$ directly in our construction.

In section 1.3 we prove that for every plurisubharmonic function p in $\mathbf{C}^n$ of order ϱ and of finite type there exists an analytic function f in $\mathbf{C}^n$ such that $T_t p - T_t \log|f| \to 0$ in $L^1_{\text{loc}}(\mathbf{C}^n)$ as $t \to \infty$. This is also a generalization of a theorem of Azarin [2] for $n=1$. The theorem improves the results of section 1.2 so that they hold with $p = \log|f|$, where f is an analytic function in $\mathbf{C}^n$. As a consequence we give two refinements of the well-known indicator theorem, which states that every plurisubharmonic function in $\mathbf{C}^n$, which is positively homogeneous of order ϱ, is the indicator function of some analytic function. It was proved by Pólya [19] for $n=1$ and $\varrho=1$, Bernstein [5], [6] for $n=1$ and $\varrho>0$, Kiselman [12] for $n \geqq 1$ and $\varrho=1$, and finally Martineau [16], [17] for $n \geqq 1$ and $\varrho>0$.

In section 1.4 we give the third refinement of the indicator theorem. This refinement consists of an L^2-estimate of the analytic function f with a weight depending on the indicator function. In section 3.1 we use this estimate to characterize the plurisubharmonic functions in $\mathbf{C}^n$, which are indicator functions of Fourier–Laplace transforms of distributions with compact support. In the proof we follow the lines of Kiselman [13] and Martineau [16], [17].

1.1. Basic definitions.

Let p be a plurisubharmonic function in $\mathbf{C}^n$. Then the order of p is defined by

$$\overline{\lim_{r\to\infty}} \log^+\left(\max_{|z|=r} p(z)\right)/\log r .$$

If the order of p is finite and equal to $\varrho>0$, we define the type of p by

$$\overline{\lim_{r\to\infty}}\, r^{-\varrho}\left(\max_{|z|=r} p(z)\right).$$

If $\varrho>0$, then for every $t>0$ we define the operator

$$T_t\colon L^1_{\mathrm{loc}}(\mathbf{C}^n) \to L^1_{\mathrm{loc}}(\mathbf{C}^n)$$

by

$$T_t q(z) = t^{-\varrho} q(tz) \quad \text{for all } z \in \mathbf{C}^n .$$

If p is plurisubharmonic in $\mathbf{C}^n$ of order ϱ and of finite type, we let $L(p)$ denote the set of all plurisubharmonic functions in $\mathbf{C}^n$ that are limits in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$, or equivalently in $\mathscr{D}'(\mathbf{C}^n)$, of sequences of the form $\{T_{t_j}p\}$, where $t_j\to\infty$.

PROPOSITION 1.1.1. *Let ϱ be a positive real number and let p be a plurisubharmonic function of order ϱ and of finite type. Then the set $\{T_t p;\, t\geqq 1\}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$. Every element in $L(p)$ vanishes at the origin and is bounded by $\sigma|z|^\varrho$ for some $\sigma>0$. The set $L(p)$ is compact, connected, and invariant under T_t for all $t>0$.*

PROOF. Since p is of order ϱ and of finite type, there exist positive constants τ and σ such that

$$p(z) \leqq \tau+\sigma|z|^\varrho, \quad z\in\mathbf{C}^n . \tag{1.1.1}$$

Let $\{t_j\}$ be a sequence of positive real numbers $\geqq 1$. If $\{t_j\}$ has a subsequence with finite limit, then $\{T_{t_j}p\}$ has a convergent subsequence because the mapping $\mathbf{R}_+\ni t\mapsto T_t p\in L^1_{\mathrm{loc}}$ is continuous. Hence we can suppose that $t_j\to\infty$. By (1.1.1) the functions in $\{T_t p; t\geqq 1\}$ have a uniform upper bound in every compact subset of $\mathbf{C}^n$. If $\varepsilon>0$ and $r=1/t$, then

$$\begin{aligned} t^{-\varrho}\int_{|z|\leqq\varepsilon} p(tz)\,d\lambda &\geqq t^{-\varrho}\int_{|z|\leqq\varepsilon} p(trz)\,d\lambda \\ &= t^{-\varrho}\int_{|z|\leqq\varepsilon} p(z)\,d\lambda \to 0, \quad \text{as } t\to\infty . \end{aligned} \tag{1.1.2}$$

Hence $\{T_{t_j}p\}$ does not converge to $-\infty$ uniformly in all compact subsets of $\mathbf{C}^n$. Theorem 4.1.9 in Hörmander [11] gives that $\{T_{t_j}p\}$ has a convergent subsequence and that $\{T_t p;\, t\geqq 1\}$ is relatively compact.

By (1.1.1) all the elements in $L(p)$ are bounded by $\sigma|z|^\varrho$ and (1.1.2) gives that they vanish at the origin.

The set $L(p)$ is compact, because it is the intersection of the closure in L^1_{loc} of the sets $\{T_t p;\ t \geqq N\}$ for all $N \geqq 1$.

Suppose now that $L(p)$ is not connected. Then $L(p)$ can be written as the union of two disjoint non-empty closed sets A and B. Let U and V be disjoint open neighborhoods of A and B respectively in L^1_{loc}. Since A and B are non-empty there exist sequences $\{s_j\}$ and $\{t_j\}$ such that $s_j < t_j$, $s_j \to \infty$, $T_{s_j} p \in U$, and $T_{t_j} p \in V$. The mapping $\mathsf{R}_+ \ni t \mapsto T_t p \in L^1_{\mathrm{loc}}$ is continuous. Hence its image is connected. This implies that there exists a sequence $\{u_j\}$ with $s_j < u_j < t_j$ such that $T_{u_j} p \notin U \cup V$. Since $\{T_t p;\ t \geqq 1\}$ is relatively compact, the sequnce $\{T_{u_j} p\}$ has a convergent subsequence and its limit is neither in A nor in B, a contradiction. Hence $L(p)$ is connected.

If $\lim_{j\to\infty} T_{t_j} p = q$, then

$$T_t q = \lim_{j\to\infty} T_t T_{t_j} p = \lim_{j\to\infty} T_{tt_j} p ,$$

because the mapping $T_t\colon L^1_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$ is continuous for all $t > 0$. This proves the proposition.

For a plurisubharmonic function p in C^n of order $\varrho > 0$ and of finite type we define the indicator function j_p of p as the least upper semi-continuous majorant of $\overline{\lim}_{t\to\infty} T_t p$, that is

$$j_p(z) = \overline{\lim_{w\to z}}\ \overline{\lim_{t\to\infty}}\ t^{-\varrho} p(tw), \quad z \in \mathsf{C}^n .$$

Then j_p is plurisubharmonic in C^n and positively homogeneous of order ϱ.

PROPOSITION 1.1.2. *Let ϱ be a positive real number and let p be a plurisubharmonic function of order ϱ and of finite type. Then the indicator function j_p of p is the least upper semi-continuous majorant of all the functions in $L(p)$, that is*

$$j_p(z) = \overline{\lim_{w\to z}} \sup \{q(w)\ ;\ q \in L(p)\}, \quad z \in \mathsf{C}^n . \tag{1.1.3}$$

PROOF. It is obvious that $q \leqq j_p$ for every $q \in L(p)$. On the other hand, for every z we can choose a sequence $t_j \to \infty$ such that

$$t_j^{-\varrho} p(t_j z) \to \overline{\lim_{t\to\infty}}\, t^{-\varrho} p(tz)$$

and $T_{t_j} p \to q \in L(p)$ in L^1_{loc}. Then $q(z) \geqq \overline{\lim}_{t\to\infty} t^{-\varrho} p(tz)$. Hence j_p is bounded by the right-hand side of (1.1.3) and the proposition is proved.

The following proposition shows that positively homogeneous subharmonic functions in the complex plane are continuous. Lelong [14] has shown, that for every $n > 1$ and every $\varrho > 0$, there exists a discontinuous plurisubharmonic function in $\mathbf{C}^n$, which is positively homogeneous of order ϱ. Thus the proposition has no counterpart for plurisubharmonic functions.

PROPOSITION 1.1.3. *Let ϱ be a positive real number. Let p be a subharmonic function in* $\mathbf{C}$ *of order ϱ and of finite type.*

i) *Let $z \mapsto z^{1/\varrho}$ be an analytic ϱ-root in some sector S of* $\mathbf{C}$. *Then the function*

$$z \mapsto \left(\varlimsup_{t\to\infty} T_t p\right)(z^{1/\varrho})$$

is convex in S.

ii) *If $\varrho = 1$, then p is the supporting function of some compact subset K in in* $\mathbf{C}$, *that is*

$$p(z) = \sup_{w\in K} \operatorname{Re}(z\bar{w}), \quad z \in \mathbf{C}\,.$$

Part i) follows with an application of the Phragmén–Lindelöf principle. For a proof see Hardy and Rogosinski [7]. For a proof of ii), see Hörmander [11, Theorem 4.3.2].

We define the order of an analytic function f in $\mathbf{C}^n$ as the order of the plurisubharmonic function $\log|f|$. If f is of order $\varrho > 0$, we define the type of f as the type of $\log|f|$. Finally we define the indicator function i_f of f by $i_f = j_{\log|f|}$.

If u is an analytic functional, then its Fourier–Laplace transform $\hat{u}$, defined by $\hat{u}(\zeta) = u(\exp(-i\langle\cdot,\zeta\rangle))$ for $\zeta \in \mathbf{C}^n$, is an analytic function of exponential type. That is, $\hat{u}$ is of order one and finite type. If u is carried by a compact subset of $\mathbf{R}^n$, then there exists a positive constant σ and for every $\varepsilon > 0$ a positive constant C_ε such that

$$|\hat{u}(\zeta)| \leqq C_\varepsilon \exp(\sigma|\operatorname{Im}\zeta| + \varepsilon|\zeta|), \quad \zeta \in \mathbf{C}^n\,.$$

This implies $i_{\hat{u}}(\zeta) \leqq \sigma|\operatorname{Im}\zeta|$ for $\zeta \in \mathbf{C}^n$.

PROPOSITION 1.1.4. *Let p be a plurisubharmonic function in* $\mathbf{C}^n$. *Suppose that p is positively homogeneous of order one and $p(\zeta) \leqq \sigma|\operatorname{Im}\zeta|$ for $\zeta \in \mathbf{C}^n$, where σ is a positive constant. Then $p(\xi + i\eta) \leqq p(i\eta)$ for all $\xi, \eta \in \mathbf{R}^n$ and equality holds if ξ and η are proportional. Furthermore, $\mathbf{R}^n \ni \eta \mapsto p(i\eta)$ is a supporting function.*

PROOF. Let $\xi, \eta \in \mathbf{R}^n$. The function $z \mapsto p(\xi + z\eta)$ is subharmonic in $\mathbf{C}$, bounded by $\sigma|\operatorname{Im} z\eta|$, and

$$\varlimsup_{t\to\infty} t^{-1} p(\xi + it\eta) = \varlimsup_{t\to\infty} p(\xi/t + i\eta) = p(i\eta)\,.$$

The Phragmén–Lindelöf principle gives $p(\xi+z\eta) \leqq \operatorname{Im} z p(i\eta)$, so the inequality holds. The subharmonic function $z \mapsto p(z\eta) - \operatorname{Im} z p(i\eta)$ is non-positive in the upper half plane and equal to zero for $z=i$. The maximum principle gives that it is identically equal to zero. If $\xi = x\eta$ for some $x \in \mathbf{R}$, we get $p(\xi+i\eta) = p((x+i)\eta) = p(i\eta)$.

By Theorem 4.3.2 in Hörmander [11] the second statement follows if we prove that $\mathbf{R}^n \ni \eta \mapsto p(i\eta)$ is convex. Let $\eta_1, \eta_2 \in \mathbf{R}^n$. The first part of the proof gives

$$p(i\eta_1 + z(\eta_2 - \eta_1)) \leqq (1 - \operatorname{Im} z) p(i\eta_1) + \operatorname{Im} z p(i\eta_2)$$

for all z with $\operatorname{Im} z = 0$ or $\operatorname{Im} z = 1$. The left hand side is bounded from above as a function of z in the strip $\{z \in \mathbf{C};\ 0 \leqq \operatorname{Im} z \leqq 1\}$. Hence the three lines theorem gives that the inequality holds for all z in the strip. We take $z = it$ with $t \in [0,1]$ and get

$$p(i((1-t)\eta_1 + t\eta_2)) \leqq (1-t)p(i\eta_1) + tp(i\eta_2) .$$

This completes the proof.

1.2. Plurisubharmonic functions with prescribed limit sets.

If p is a plurisubharmonic function of order ϱ we know from section 1.1 that all functions in the limit set $L(p)$ of p vanish at the origin and are bounded by $\sigma|z|^\varrho$ for some $\sigma > 0$. Furthermore the set $L(p)$ is compact, connected and invariant under T_t for all $t > 0$. Azarin and Giner [4] have constructed a set of subharmonic functions satisfying these properties which is not a limit set. We are not able to characterize the limit sets, but we have:

THEOREM 1.2.1. *Let ϱ and σ be positive real numbers. Let M be a set of plurisubharmonic functions q in $\mathbf{C}^n$ with $q(0)=0$ and $q(z) \leqq \sigma|z|^\varrho$, and suppose that M is compact and invariant under T_t for all $t > 0$.*

i) *There exists a plurisubharmonic function p of order $\leqq \varrho$ and of finite type, such that $M \subset L(p) \subset N$, where*

$$N = \{\vartheta q_1 + (1-\vartheta) q_2 \ ;\ \vartheta \in [0,1],\ q_1, q_2 \in M\}$$

is the union of all line segments with endpoints in M.

ii) *If M is connected and all its elements are positively homogeneous of order ϱ, then p can be chosen so that $M = L(p)$.*

In the proof we choose a sequence $\{q_k\}$ in M, such that its elements form a dense subset of M and every element appears infinitely many times in the sequence. Then we define a partition of unity $\{\varphi_k\}$ in $\mathbf{C}^n$. We choose positive real numbers τ_k depending on the partition of unity and set $p_k = T_{\tau_k} q_k$. Then we let r_k be a certain regularization of p_k and show that p can be chosen of the form

$$p = \sum \varphi_k r_k + \Phi ,$$

where Φ is a C^∞ plurisubharmonic function in $\mathbf{C}^n \setminus \{0\}$ with Levi form dominating the Levi form of the sum and $T_t \Phi \to 0$ in L^1_{loc} as $t \to \infty$.

We begin with some preliminary constructions. The first step is to regularize plurisubharmonic functions in $\mathbf{C}^n$. Let $0 \leqq \alpha \in C_0^\infty(\mathbf{C}^{n^2})$ and $\int \alpha \, d\lambda = 1$, where $d\lambda$ denotes the Lebesgue measure in $\mathbf{C}^{n^2}$. We identify $\mathbf{C}^{n^2}$ with the space of all $n \times n$ matrices with complex elements, let I denote the identity matrix and set

$$\alpha_\delta(A) = \delta^{-2n^2} \alpha\big((A - I)/\delta\big) \quad \text{for } \delta \in (0,1) .$$

If q is plurisubharmonic in $\mathbf{C}^n$, we define $R_\delta q$ by

$$\text{(1.2.1)} \quad R_\delta q(z) = \int q(Az) \alpha_\delta(A) \, d\lambda(A) = \int q(z + \delta A z) \alpha(A) \, d\lambda(A) .$$

Here Az denotes matrix multiplication.

LEMMA 1.2.2. *Let ϱ be a positive real number. Let M be a set of plurisubharmonic functions in $\mathbf{C}^n$, that are of order $\leqq \varrho$ and of finite type. Suppose that M is compact in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ and invariant under T_t for all $t > 0$. Let $q \in M$ and define $R_\delta q$ by* (1.2.1). *Then*:

i) *$R_\delta q$ is a plurisubharmonic function in $\mathbf{C}^n$ of order $\leqq \varrho$ and of finite type. We have*

$$R_\delta q - q \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{C}^n)$$

uniformly for $q \in M$ as $\delta \to 0$.

ii) *$R_\delta q \in C^\infty(\mathbf{C}^n \setminus \{0\})$. For every multi-index β there exists a positive constant C_β such that*

$$\text{(1.2.2)} \qquad |D^\beta R_\delta q(z)| \leqq C_\beta \delta^{-2n^2 - |\beta|} |z|^{\varrho - |\beta|}$$

for all $z \in \mathbf{C}^n \setminus \{0\}$, $q \in M$, and $\delta \in (0,1)$, where $D = (\partial/\partial z, \partial/\partial \bar{z})$.

iii) *$R_\delta q(z) \to q(z)$ for all $z \in \mathbf{C}^n$ as $\delta \to 0$.*

PROOF. i) Since $\alpha \geqq 0$, the function $R_\delta q$ is plurisubharmonic in $\mathbf{C}^n$. Since q is of order $\leqq \varrho$ and of finite type, there exist constants τ and σ such that $q(z) \leqq \tau + \sigma |z|^\varrho$. Thus

$$R_\delta q(z) \leqq \tau + \sigma \Big(\int \| I + \delta A \|^\varrho \alpha(A) \, d\lambda(A) \Big) |z|^\varrho ,$$

where $\|B\|$ denotes the operator norm of the matrix B, and it follows that $R_\delta q$ is of order $\leqq \varrho$ and of finite type.

If K is a compact subset of $\mathbf{C}^n$, then

$$\text{(1.2.3)} \quad \int_K |R_\delta q - q|\, d\lambda \leqq \iint_K |q(z+\delta Az) - q(z)|\, d\lambda(z)\alpha(A) d\lambda(A)\,.$$

The mapping $S\colon L^1_{\text{loc}} \times \mathrm{GL}_n \to L^1_{\text{loc}}$, $S(q, A) = q(Az)$, is continuous, where

$$\mathrm{GL}_n = \{A \in \mathbf{C}^{n^2}\ ;\ \det A \neq 0\}\,.$$

There exists δ_0 such that $\operatorname{supp} \alpha_\delta$ is contained in a compact subset N of GL_n for all $\delta < \delta_0$. The uniform continuity of S in $M \times N$ now gives that for every $\varepsilon > 0$ there exists δ_1 such that

$$\int_K |q(z+\delta Az) - q(z)|\, d\lambda(z) < \varepsilon$$

for all $q \in M$, $A \in \operatorname{supp} \alpha$, and $\delta < \delta_1$. Now (1.2.3) gives that $R_\delta q \to q$ in L^1_{loc} uniformly for $q \in M$ as $\delta \to 0$.

ii) Let $z_0 \in \mathbf{C}^n \setminus \{0\}$. The linear mapping $\mathbf{C}^{n^2} \ni A \mapsto Az_0 \in \mathbf{C}^n$ is surjective and there exists a linear mapping $L\colon \mathbf{C}^{n^2} \to \mathbf{C}^{n^2-n}$ which is bijective from $\{A;\, Az_0 = 0\}$. Then

$$\text{(1.2.4)} \qquad \mathbf{C}^{n^2} \ni A \mapsto \big(Az_0, L(A)\big) \in \mathbf{C}^n \oplus \mathbf{C}^{n^2-n} \cong \mathbf{C}^{n^2}$$

is a bijection and it remains one if z_0 is replaced by z in a sufficiently small neighborhood U of z_0. Let $\Psi_z\colon \mathbf{C}^{n^2} \to \mathbf{C}^{n^2}$ denote the inverse for $z \in U$. We let A' denote the projection of A on the first n variables, change variables in the first integral in (1.2.1) and get

$$\text{(1.2.5)} \qquad R_\delta q(z) = \int q(A')\alpha_\delta\big(\Psi_z(A)\big)|\det \Psi_z|^2\, d\lambda(A)\,,$$

for all $z \in U$. Since $q \in L^1_{\text{loc}}$ and $\alpha \in C_0^\infty$ it follows that $R_\delta q \in C^\infty(\mathbf{C}^n \setminus \{0\})$.

We begin by proving (1.2.2) when $|z| = 1$. We let D^β operate under the integral sign in (1.2.5). Since $0 < \delta < 1$, the supports of α_δ are contained in a compact subset of $\mathbf{C}^{n^2}$. If V is a compact neighborhood of z with $V \subset U$, then there exists a constant C'_β and a compact subset K of $\mathbf{C}^n$ such that

$$|D^\beta R_\delta q(z)| \leqq C'_\beta \delta^{-2n^2-|\beta|} \int_K |q(w)|\, d\lambda(w), \qquad z \in V\,.$$

Since M is compact the integral is bounded. By the Borel–Lebesgue lemma, (1.2.2) holds for $|z| = 1$, $q \in M$, and $\delta \in (0, 1)$.

For $z \in \mathbf{C}^n\setminus\{0\}$ we choose $s=1/|z|$. Then

$$D^\beta R_\delta q(z) = D^\beta\big(s^{-\varrho}(T_{1/s}R_\delta q(sz))\big) = s^{-\varrho+|\beta|}\big(D^\beta(R_\delta T_{1/s}q)\big)(sz)\,.$$

Now M is invariant under T_t for all $t>0$, so (1.2.2) follows.

iii) Since q is subharmonic, every z with $q(z)>-\infty$ is a Lebesgue point of q. With the same notation as in ii) we have

$$\begin{aligned}|R_\delta q(z)-q(z)| &\leqq \int |q(z+\delta A')-q(z)|\alpha(\Psi_z(A))|\det \Psi_z|^2\,d\lambda(A)\\ &\leqq C_z\int_K |q(z+\delta w)-q(z)|d\lambda(w)\to 0 \quad \text{as } \delta\to 0\,,\end{aligned}$$

where C_z is a positive constant and K is a ball in $\mathbf{C}^n$ with center at the origin. If $q(z)=-\infty$, then the semi-continuity gives that for every $N>0$ there exists $\varepsilon>0$ and $\delta_0>0$ such that $q\big((I+\delta A)w\big)<-N$ for all $A\in \operatorname{supp}\alpha$, $\delta<\delta_0$, and $w \in \mathbf{C}^n$ with $|z-w|\leqq\varepsilon$. The mean value theorem gives

$$R_\delta q(z) \leqq \int\int_{|z-w|\leqq\varepsilon} q\big((I+\delta A)w\big)\,d\lambda(w)\alpha(A)\,d\lambda(A)\Big/\int_{|w-z|\leqq\varepsilon} d\lambda \leqq -N\,.$$

Hence $R_\delta q(z)\to -\infty$ as $\delta\to 0$. This completes the proof.

Now we construct a partition of unity in $\mathbf{C}^n$. Let $\{\beta_k\}$ be an increasing sequence of real numbers with $\beta_0=1$ and $\sigma_k=\beta_k/\beta_{k-1}$ increasing to ∞. Let $\chi \in C^\infty(\mathbf{R})$ with $0\leqq\chi\leqq 1$, $\chi(x)=1$ if $x<1/3$ and $\chi(x)=0$ if $x>2/3$. Set

$$\begin{aligned}\varphi_0(z) &= \chi(\log|z|/\log\sigma_1)\\ \varphi_k(z) &= \chi\big(\log(|z|/\beta_k)/\log\sigma_{k+1}\big)-\chi\big(\log(|z|/\beta_{k-1})/\log\sigma_k\big)\end{aligned}$$

for $k\geqq 1$. Since $\chi\big(\log(|z|/\beta_k)/\log\sigma_{k+1}\big)$ is equal to 1 for $|z|\leqq\beta_k\sigma_{k+1}^{1/3}$ and vanishes for $|z|\geqq\beta_k\sigma_{k+1}^{2/3}$, and since $\beta_{k-1}\sigma_k=\beta_k$, we have

i) $0\leqq\varphi_k \in C_0^\infty(\mathbf{C}^n)$ and $\sum\varphi_k=1$.

ii) $\operatorname{supp}\varphi_k\subset\{z\in\mathbf{C}^n\ ;\ \beta_k\sigma_k^{-2/3}\leqq|z|\leqq\beta_k\sigma_{k+1}^{2/3}\}$, $\operatorname{supp}\varphi_k\cap\operatorname{supp}\varphi_j=\emptyset$ if $k\neq 0$ and $|j-k|>1$.

iii) $\varphi_k=1$ in $\{z\in\mathbf{C}^n\ ;\ \beta_k\sigma_k^{-1/3}\leqq|z|\leqq\beta_k\sigma_{k+1}^{1/3}\}$.

iv) For every multi-index $\beta\neq 0$ there exists a positive constant C_β such that

$$|D^\beta\varphi_k(z)| \leqq C_\beta(\log\sigma_k)^{-1}|z|^{-|\beta|}\,, \tag{1.2.6}$$

where $D=(\partial/\partial z,\partial/\partial\bar z)$.

v) If $K\subset\mathbf{C}^n\setminus\{0\}$ is a non-empty compact set, then there exists k_0 such that

$$\varphi_k(\beta_k z) = 1, \quad z \in K, \ k \geqq k_0 .$$

If $t_j \to \infty$, $\beta_{k_j} \leqq t_j \leqq \beta_{k_j+1}$, then there exists j_0 such that

$$\varphi_{k_j}(t_j z) + \varphi_{k_j+1}(t_j z) = 1, \quad z \in K, \ j > j_0 .$$

The final step in this preliminary discussion is the following lemma which is only a variant of Lemme 3 in Martineau [16].

LEMMA 1.2.3. *Let ϱ be a positive real number. For every continuous function $\gamma > 0$ on $\bar{\mathsf{R}}_+$, with $\gamma(r) \to 0$ as $r \to \infty$, there exists a plurisubharmonic function Φ in C^n of class C^∞ in $\mathsf{C}^n \setminus \{0\}$ such that*

$$\sum_{j,k} \partial^2\Phi(z)/\partial z_j \partial \bar{z}_k w_j \bar{w}_k \geqq \gamma(|z|)|z|^{\varrho-2}|w|^2 \tag{1.2.7}$$

for $w \in \mathsf{C}^n$, $|z| \geqq 1$, and $T_t\Phi(z)$ decreases to zero as $t \to \infty$ for all $z \in \mathsf{C}^n$. The function Φ can be chosen so that for every multi-index β,

$$|D^\beta \Phi(z)| \leqq C_\beta |z|^{\varrho - |\beta|}, \qquad z \in \mathsf{C}^n \setminus \{0\} , \tag{1.2.8}$$

where C_β is a positive constant.

PROOF. We are going to show that Φ can be chosen of the form

$$\Phi(z) = c \exp\big(\kappa(\log|z|^2)\big)|z|^\varrho, \quad z \in \mathsf{C}^n , \tag{1.2.9}$$

where κ is a real valued C^∞-function on the real axis and c is a positive constant. We begin by observing that if $\Phi(z) = \Psi(|z|^2)$, where Ψ is a function on the positive real axis, then

$$\sum_{j,k} \partial^2\Phi/\partial z_j \partial \bar{z}_k w_j \bar{w}_k = \Psi'(|z|^2)|w|^2 + \Psi''(|z|^2)|\langle z, \bar{w}\rangle|^2 .$$

Hence the expression

$$\min\{\Psi'(|z|^2), \Psi'(|z|^2) + |z|^2\Psi''(|z|^2)\}|w|^2 \tag{1.2.10}$$

is a lower bound for the Levi form of Φ. If we set $s = \log|z|^2$ and define the function ψ by $\psi(s) = \Psi(e^s)$, then (1.2.10) is equal to

$$e^{-s}\min\{\psi'(s), \psi''(s)\}|w|^2 . \tag{1.2.11}$$

The function $t \mapsto \log\big(\gamma(\exp(t/2))\big)$ has a finite limit as $t \to -\infty$ and converges to $-\infty$ as $t \to \infty$. Hence there exists an infinitely differentiable function κ on the real axis, which is decreasing, convex, satisfies $\kappa(t) \geqq \log\big(\gamma(\exp(t/2))\big)$, and converges to $-\infty$ as $t \to \infty$. Furthermore, κ can be chosen so that $\kappa' > -\varrho/4$. We set $c = \max\{4/\varrho, 16/\varrho^2\}$ and define Φ by (1.2.9). Then

$$\Psi(t) = c\exp(\kappa(\log t))t^{\varrho/2} \quad \text{and} \quad \psi(s) = c\exp(\kappa(s)+\varrho s/2)\,.$$

The lower bound for the Levi form of Φ given by (1.2.11) is

$$c\exp(\kappa(s)+(\varrho/2-1)s)\min\{\kappa'(s)+\varrho/2, \kappa''(s)+(\kappa'(s)+\varrho/2)^2\}|w|^2\,.$$

Since $\kappa' > -\varrho/4$, κ is convex and $c\min\{\varrho/4, \varrho^2/16\} = 1$, we get

$$\sum_{j,k} \partial^2\Phi/\partial z_j\partial\bar{z}_k w_j\bar{w}_k \geqq \exp(\kappa(s)+(\varrho/2-1)s)|w|^2$$
$$\geqq \exp(\kappa(\log|z|^2))|z|^{\varrho-2}|w|^2 \geqq \gamma(|z|)|z|^{\varrho-2}|w|^2\,.$$

Hence (1.2.7) holds. We have $T_t\Phi(z) = c\exp(\kappa(\log|tz|^2))|z|^{\varrho} \searrow 0$ as $t \to \infty$ for every $z \in \mathbf{C}^n$. If κ is chosen so that its derivatives of all orders are bounded, then the last statement follows. The lemma is proved.

PROOF OF THEOREM 1.2.1. i) Let $\{\varphi_k\}$ be the partition of unity defined before Lemma 1.2.3 and set $\tau_k = 1/\beta_k$. Let $\{q_k\}$ be a sequence in M with every element appearing infinitely many times in the sequence and forming a dense subset of M. Set $p_k = T_{\tau_k}q_k$. Let $\{\delta_k\}$ be a sequence of positive real numbers decreasing to zero such that

$$(\log\sigma_k)^{-1}\delta_k^{-2n^2-1} \to 0 \quad \text{as } k \to \infty\,,$$

and set $r_k = R_{\delta_k}p_k$, where R_δ is defined by (1.2.1).

We are going to choose p of the form

$$p = \sum \varphi_k r_k + \Phi\,, \tag{1.2.12}$$

where Φ is a plurisubharmonic function in $\mathbf{C}^n$ of class C^∞ in $\mathbf{C}^n\setminus\{0\}$ such that $T_t\Phi \to 0$ in L^1_{loc}. In order to show that it is possible to choose Φ such that p becomes plurisubharmonic we have to calculate the Levi form of the sum, which we denote by s. In a neighborhood of the set $\{z \in \mathbf{C}^n;\ \beta_k \leqq |z| \leqq \beta_{k+1}\}$ we have $s = \varphi_k r_k + \varphi_{k+1}r_{k+1}$ and the Levi form is equal to

$$\varphi_k \sum \partial^2 r_k/\partial z_l\partial\bar{z}_m w_l\bar{w}_m + \varphi_{k+1}\sum \partial^2 r_{k+1}/\partial z_l\partial\bar{z}_m w_l\bar{w}_m +$$
$$+2\,\mathrm{Re}(\langle\partial\varphi_k/\partial z, w\rangle\langle\partial(r_k-r_{k+1})/\partial\bar{z}, \bar{w}\rangle)+$$
$$+\sum \partial^2\varphi_k/\partial z_l\partial\bar{z}_m w_l\bar{w}_m(r_k-r_{k+1})\,.$$

Here we have used that $\varphi_k+\varphi_{k+1} = 1$ in a neighborhood of the set $\{z \in \mathbf{C}^n;\ \beta_k \leqq |z| \leqq \beta_{k+1}\}$. The first two terms are non-negative. By (1.2.2) and (1.2.6) the absolute value of the other terms can be estimated by

$$C(\log\sigma_k)^{-1}\delta_k^{-2n^2-1}|z|^{\varrho-2}|w|^2\,.$$

Since $(\log \sigma_k)^{-1}\delta_k^{-2n^2-1} \to 0$ as $k \to \infty$, there exists a positive continuous function γ on the positive real axis with $\gamma(x) \to 0$ as $x \to \infty$ and $\gamma(x) \geqq C(\log \sigma_k)^{-1}\delta_k^{-2n^2-1}$ if $\beta_k < x < \beta_{k+1}$. Then there exists a plurisubharmonic function Φ satisfying the conditions in Lemma 1.2.3, and it follows that (1.2.12) defines a plurisubharmonic function p. By Lemma 1.2.2 i), p is of order $\leqq \varrho$ and of finite type.

In order to prove that $M \subset L(p)$ it is sufficient to show that every element q in the sequence $\{q_k\}$ lies in $L(p)$, because $L(p)$ is closed and the elements in $\{q_k\}$ form a dense subset of M. Since q appears infinitely many times in $\{q_k\}$, there exists a subsequence $\{q_{k_j}\}$ with $q_{k_j} = q$. Set $t_j = \beta_{k_j}$. By Proposition 1.1.1 the set $\{T_t p;\, t \geqq 1\}$ is relatively compact in $L^1_{\text{loc}}(\mathbf{C}^n)$, so it suffices to prove that $T_{t_j} p \to q$ in $L^1_{\text{loc}}(\mathbf{C}^n \setminus \{0\})$. If $K \subset \mathbf{C}^n \setminus \{0\}$ is compact, then $\varphi_{k_j}(t_j z) = 1$ for all $z \in K$ and all sufficiently large j. Hence

$$T_{t_j} p(z) - q(z) = R_{\delta_{k_j}} q(z) - q(z) + T_{t_j} \Phi(z) .$$

Hence Lemma 1.2.2 i) and Lemma 1.2.3 give that $M \subset L(p)$.

In order to prove the second inclusion we need the following lemma which is a variant of Lemma 3.2.1 in Azarin [3].

LEMMA 1.2.4. *Set*

$$q^t = \sum_{k=0}^{\infty} \psi_k(t) T_{t\tau_k} q_k = \sum_{k=0}^{\infty} \psi_k(t) T_t p_k$$

for all $t > 0$, where $\{\psi_k\}$ is the partition of unity on the positive real axis satisfying $\varphi_k(z) = \psi_k(|z|)$. Then $T_t p - q^t \to 0$ in L^1_{loc} as $t \to \infty$.

PROOF. By Proposition 1.1.1 the set $\{T_t p;\, t \geqq 1\}$ is relatively compact in L^1_{loc}. For $t > 0$ at most two of the numbers $\psi_k(t)$ *are different from zero. Since* $T_{t\tau_k} q_k \in M$ and M is compact, the set $\{q^t;\, t \geqq 1\}$ is relatively compact in L^1_{loc}. By the compactness it follows that it is sufficient to prove convergence in $L^1_{\text{loc}}(\mathbf{C}^n \setminus \{0\})$.

Let K be a compact set of the form $K = \{z \in \mathbf{C}^n;\, \delta \leqq |z| \leqq \gamma\}$, where $0 < \delta < 1 < \gamma$. We have

$$(1.2.13) \quad \int_K |T_t p - q^t|\, d\lambda \leqq \sum_{k=0}^{\infty} \int_K |\psi_k(t|z|) - \psi_k(t)|\, |T_t r_k|\, d\lambda + \\ + \sum_{k=0}^{\infty} \psi_k(t) \int_K |T_t(r_k - p_k)|\, d\lambda + \int_K |T_t \Phi|\, d\lambda .$$

The last term converges to zero by Lemma 1.2.3. For $t \geqq 1$ we choose $k(t)$ such that $\beta_{k(t)} \leqq t \leqq \beta_{k(t)+1}$. For sufficiently large t all the terms in these sums are zero except at most the terms with $k=k(t)$ or $k=k(t)+1$, for $\beta_{k+1}/\beta_k \to \infty$. Hence it is sufficient to show that the terms in the sums converge to zero uniformly for $t>0$ as $k \to \infty$. Taylor's formula gives

$$|\psi_k(t|z|)-\psi_k(t)| \leqq \gamma t \sup_{\delta t<\xi<\gamma t} |\psi_k'(\xi)|$$

$$\leqq \frac{\gamma}{\delta} \sup_{\mathbf{R}_+} |x\psi_k'(x)| \leqq C_K(\log \sigma_k)^{-1} \to 0, \quad \text{as } k \to \infty .$$

By Lemma 1.2.2 i) the functions $T_t r_k$ are contained in a compact subset of L^1_{loc}. Hence the first sum in (1.2.13) converges to zero as $t \to \infty$. Lemma 1.2.2 i) gives that the second sum converges to zero. The proof is completed.

End of proof of theorem 1.2.1. Let $q \in L(p)$ and suppose that $T_{t_j}p \to q$, where $t_j \to \infty$. By Lemma 1.2.4, $q^{t_j} \to q$. There exists a sequence $\{k_j\}$ such that

$$q^{t_j} = \psi_{k_j}(t_j)(T_{t_j}p_{k_j}) + \psi_{k_j+1}(t_j)(T_{t_j}p_{k_j+1}) \tag{1.2.14}$$

and $\psi_{k_j}(t_j)+\psi_{k_j+1}(t_j)=1$. By replacing $\{t_j\}$ by a subsequence we can suppose that $\psi_{k_j}(t_j) \to \vartheta$ with $0 \leqq \vartheta \leqq 1$. This implies that $\psi_{k_j+1}(t_j) \to 1-\vartheta$. By replacing $\{t_j\}$ by a subsequence again we can suppose that $(T_{t_j}p_{k_j}) \to s_1 \in M$ and $(T_{t_j}p_{k_j+1}) \to s_2 \in M$, because M is compact and invariant under T_t for all $t>0$. This implies that

$$q = \lim_{j\to\infty} q^{t_j} = \vartheta s_1 + (1-\vartheta)s_2 \in N , \tag{1.2.15}$$

and shows that $L(p) \subset N$.

ii) If all the functions in M are positively homogeneous of order ϱ, then (1.2.14) becomes

$$q^{t_j} = \psi_{k_j}(t_j)q_{k_j} + \psi_{k_j+1}(t_j)q_{k_j+1} . \tag{1.2.14}$$

The space $L^1_{\text{loc}}(\mathbf{C}^n)$ is a Fréchet space. The following lemma shows that if M is connected, it is always possible to choose $\{q_k\}$ so that $q_k - q_{k+1} \to 0$ in L^1_{loc}. If we choose subsequences of $\{t_j\}$ as above, then $s_1 = s_2$. Hence

$$q = \lim_{j\to\infty} q^{t_j} = s_1 \in M . \tag{1.2.15$'$}$$

Thus the proof is completed by:

Lemma 1.2.5. *Let X be a compact connected metric space with metric d. Then there exists a sequence $\{x_k\}$ in X such that its elements form a dense subset of X, every element in the sequence appears infinitely many times, and $d(x_k, x_{k+1})$ decreases to zero as $k \to \infty$.*

PROOF. For $x \in X$ and $r>0$ we set

$$B(x,r) = \{y \in X \ ; \ d(x,y) \leqq r\} .$$

Since X is compact there exists a sequence $X_1 \subset X_2 \subset \dots$ of finite subsets of X such that $X = \cup B(x, 1/j)$ for every j, where the union is taken over all $x \in X_j$. First we show that it is possible to order the elements in X_j in a sequence $y_1, \dots, y_m$, with possible repetitions, such that $d(y_k, y_{k+1}) \leqq 2/j$. The first element is chosen arbitrarily. Suppose that $y_1, \dots, y_l$ have been chosen and $\{y_1, \dots, y_l) \neq X_j$. Then there exists $z \in X_j \setminus \{y_1, \dots, y_l\}$ with $B(y_k, 1/j) \cap B(z, 1/j)$ non-empty for some k with $1 \leqq k \leqq l$, because X is connected. If $k = l$ we set $y_{l+1} = z$. Otherwise we set

$$y_{l+1} = y_{l-1}, \quad y_{l+2} = y_{l-2}, \dots, \quad y_{2l-k} = y_k, \quad y_{2l-k+1} = z .$$

Since X_j is finite this process is completed in finitely many steps. It is clear that $d(y_k, y_{k+1}) \leqq 2/j$ for $1 \leqq k \leqq m-1$. Now we construct the sequence $\{x_k\}$. First we let $x_1, \dots x_{n_1}$ be an ordering of the elements in X_1 with $d(x_k, x_{k+1}) \leqq 2$, then we let $x_{n_1}, \dots x_{n_2}$ be an ordering of the elements of X_2 with $d(x_k, x_{k+1}) \leqq 1$. Continuing in this way we get a sequence $\{x_k\}$ with $d(x_k, x_{k+1})$ decreasing to zero as $k \to \infty$. The other conditions are clearly satisfied. The proof is completed.

If $p_1, \dots, p_k$ are plurisubharmonic functions in $\mathbf{C}^n$ of order ϱ and of finite type, we let $L(p_1, \dots, p_k)$ denote the set of all k-tuples of plurisubharmonic functions $(q_1, \dots, q_k)$ such that there exists a sequence $t_j \to \infty$ with $T_{t_j} p_l$ converging to q_l for $l = 1, \dots, k$. The set $L(p_1, \dots, p_k)$ is called the limit set of $(p_1, \dots, p_k)$. With an analogous proof as that of Proposition 1.1.1 it follows that $L(p_1, \dots, p_k)$ is a compact connected subset of $(L^1_{\text{loc}})^k$. Furthermore, $L(p_1, \dots, p_k)$ is invariant under T_t for all $t>0$, that is

$$(T_t q_1, \dots, T_t q_k) \in L(p_1, \dots, p_k) \quad \text{if } (q_1, \dots, q_k) \in L(p_1, \dots, p_k) ,$$

and there exists $\sigma > 0$ such that $q_l(z) \leqq \sigma |z|^\varrho$ for $z \in \mathbf{C}^n$ and $l = 1, \dots, k$ with equality at the origin.

PROPOSITION 1.2.6. *Let $j<k$ be positive integers and let $p_1, \dots, p_k$ be plurisubharmonic functions of order ϱ and of finite type. Then the projection $(q_1, \dots, q_k) \mapsto (q_1, \dots, q_j)$ of $L(p_1, \dots, p_k)$ on $L(p_1, \dots, p_j)$ is surjective. We have*

$$L(p_1 + p_2) = \{q_1 + q_2 \ ; \ (q_1, q_2) \in L(p_1, p_2)\} .$$

PROOF. Let $(q_1,\ldots,q_j) \in L(p_1,\ldots,p_j)$. Then there exists a sequence $t_l \to \infty$ such that $T_{t_l} p_m \to q_m$ in L^1_{loc} as $l \to \infty$ for all $m=1,\ldots,j$. By Proposition 1.1.1, there exists a subsequence $\{t_{l_\nu}\}$ of $\{t_l\}$ and plurisubharmonic functions $q_{j+1},\ldots,q_k$ such that

$$T_{t_{l_\nu}} p_m \to q_m \quad \text{in } L^1_{\text{loc}} \text{ for } m=j+1,\ldots,k\,.$$

Hence $(q_1,\ldots,q_k) \in L(p_1,\ldots,p_k)$. It is clear that

$$\{q_1+q_2\ ;\ (q_1,q_2) \in L(p_1,p_2)\} \subset L(p_1+p_2)\,.$$

The other inclusion follows with an application of Proposition 1.1.1 as above. The proof is complete.

With obvious modifications of the proof of Theorem 1.2.1 we get the following improvement of it:

THEOREM 1.2.7. *Let ϱ and σ be positive real numbers. Let M be a set of k-tuples of plurisubharmonic functions $(q_1,\ldots,q_k)$ in $\mathbf{C}^n$ with $q_j(0)=0$ and $q_j(z) \leqq \sigma|z|^\varrho$, and suppose that M is compact in $(L^1_{\text{loc}})^k$ and invariant under T_t for all $t>0$.*

i) *There exist plurisubharmonic functions $p_1,\ldots,p_k$ of order $\leqq \varrho$ and of finite type, such that $M \subset L(p_1,\ldots,p_k) \subset N$, where*

$$N = \{\vartheta Q_1 + (1-\vartheta)Q_2;\ \vartheta \in [0,1],\ Q_1,Q_2 \in M\}$$

is the union of all line segments with endpoints in M.

ii) *If M is connected and all its elements are positively homogeneous of order ϱ, then $p_1,\ldots,p_k$ can be chosen so that $M=L(p_1,\ldots,p_k)$.*

1.3. Asymptotic approximation of plurisubharmonic functions.

The main result of this section is:

THEOREM 1.3.1. *Let ϱ be a positive real number. Let p be a plurisubharmonic function in $\mathbf{C}^n$ of order ϱ and of finite type. Then there exists an analytic function f in $\mathbf{C}^n$ such that*

$$T_t p - T_t \log|f| \to 0$$

in $L^1_{\text{loc}}(\mathbf{C}^n)$ as $t \to \infty$.

The theorem gives that the plurisubharmonic functions p_j in Theorem 1.2.7 can be chosen of the form $p_j = \log|f_j|$, where f_j is an analytic function in $\mathbf{C}^n$. As a corollary we get the following refinement of the indicator theorem.

COROLLARY 1.3.2. *Let ϱ be a positive real number and let r be a plurisubharmonic function in $\mathbf{C}^n$ which is positively homogeneous of order ϱ. Then:*

i) *there exists an analytic function f in $\mathbf{C}^n$ such that*

$$T_t \log|f| \to r$$

in $L^1_{loc}(\mathbf{C}^n)$ as $t \to \infty$.

ii) *there exists an analytic function g in $\mathbf{C}^n$, such that the indicator function of g is equal to r, and the limit set of $\log|g|$ contains all plurisubharmonic functions in $\mathbf{C}^n$ that vanish at the origin and are majorized by r.*

PROOF. i) We have $T_t r = r$ for all $t>0$, so this follows from Theorem 1.3.1.

ii) Let M denote the set of all plurisubharmonic functions in $\mathbf{C}^n$ that vanish at the origin and are majorized by r. Then Theorem 1.2.1 and Theorem 1.3.1 give that there exists an analytic function g in $\mathbf{C}^n$ such that $L(\log|g|)=M$. This completes the proof.

In view of Proposition 1.1.2, the function $\log|g|$ has the largest limit set among all plurisubharmonic functions having r as an indicator function. For the proof of Theorem 1.3.1 we need some preliminary lemmas. The first one is a reduction to the case that p is a C^∞ function in $\mathbf{C}^n\setminus\{0\}$.

LEMMA 1.3.3. *Let ϱ and κ be positive real numbers. Let p be a plurisubharmonic function in $\mathbf{C}^n$ of order ϱ and of finite type. Then there exists a plurisubharmonic function q in $\mathbf{C}^n$ of class C^∞ in $\mathbf{C}^n\setminus\{0\}$ such that $T_t p - T_t q \to 0$ in $L^1_{loc}(\mathbf{C}^n)$ as $t \to \infty$. Furthermore, for every multi-index $\beta \neq 0$ there exists a positive constant C_β such that*

$$|D^\beta q(z)| \leqq C_\beta (\log(1+|z|))^{2n^2+|\beta|}|z|^{\varrho-|\beta|}\ , \tag{1.3.1}$$

for all $z \in \mathbf{C}^n$ with $|z|\geqq 1$, and the Levi form of q satisfies

$$\sum \partial^2 q(z)/\partial z_j \partial\bar{z}_k w_j \bar{w}_k \geqq |z|^{\varrho-\kappa-2}|w|^2\ , \tag{1.3.2}$$

for all $z \in \mathbf{C}^n$ with $|z|\geqq 1$ and $w \in \mathbf{C}^n$. Here $D=(\partial/\partial z, \partial/\partial\bar{z})$.

PROOF. Let $\{\delta_k\}$ be a sequence of positive real numbers decreasing to zero as $k \to \infty$. Set $r_k = R_{\delta_k} p$, where R_δ is defined by (1.2.1). Then Lemma 1.2.2 ii) gives that $r_k \in C^\infty(\mathbf{C}^n\setminus\{0\})$ and

$$|D^\beta r_k(z)| \leqq C'_\beta \delta_k^{-2n^2-|\beta|}|z|^{\varrho-|\beta|}, \quad z \in \mathbf{C}^n\setminus\{0\}\ , \tag{1.3.3}$$

where C'_β is a positive constant. We choose a partition of unity $\{\varphi_k\}$ as before Lemma 1.2.3 with $(\log \sigma_k)^{-1}\delta_k^{-2n^2-1} \to 0$ as $k \to \infty$. Then it follows from the proof of Theorem 1.2.1 that there exists a plurisubharmonic function Φ satisfying the conditions in Lemma 1.2.3 such that

$$q = \sum \varphi_k r_k + \Phi$$

is plurisubharmonic. In addition, q is of order ϱ and of finite type, and $q \in C^\infty(\mathbf{C}^n \setminus \{0\})$. If we replace $\gamma(x)$ in Lemma 1.2.3 by $\gamma(x)+x^{-\kappa}$, then (1.3.2) holds. In the set $\{z \in \mathbf{C}^n;\ \beta_{k-1} \leqq |z| \leqq \beta_k\}$ we have $q = \varphi_{k-1}r_{k-1} + \varphi_k r_k + \Phi$. By (1.2.6), (1.2.8), and (1.3.3) we have

$$|D^\beta q(z)| \leqq C_\beta \delta_k^{-2n^2-|\beta|}|z|^{\varrho-|\beta|}, \quad \beta_{k-1} \leqq |z| \leqq \beta_k .$$

If the sequence $\{\delta_k\}$ is chosen such that

$$\delta_k \geqq \big(\log(1+\beta_{k-1})\big)^{-1} ,$$

then (1.3.1) holds.

By Proposition 1.1.1 it is sufficient to prove that $T_t p - T_t q \to 0$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n \setminus \{0\})$. Let K be a compact subset of $\mathbf{C}^n \setminus \{0\}$. Let $t_j \to \infty$ and choose k_j such that $\beta_{k_j} \leqq t_j \leqq \beta_{k_j+1}$. For all sufficiently large j we have

$$\varphi_{k_j}(t_j z) + \varphi_{k_j+1}(t_j z) = 1, \quad z \in K .$$

Hence

$$T_{t_j}p - T_{t_j}q = \varphi_{k_j}(t_j z)T_{t_j}(p - r_{k_j}) + \varphi_{k_j+1}(t_j z)T_{t_j}(p - r_{k_j+1}) - T_{t_j}\Phi .$$

Lemma 1.2.2 i) and Lemma 1.2.3 give that $T_t p - T_t q \to 0$ in L^1_{loc}. The proof is completed.

LEMMA 1.3.4. *Let X be an open subset of R^N and let Y be an open relatively compact subset of X. Let u be a subharmonic function in X with $\int_X |u|\, d\lambda < \infty$. Let $Z(u,\delta)$ denote the set of all $x \in Y$ such that*

$$\text{(1.3.4)} \qquad \int_{|y|\leqq 1} u(x+\varepsilon y)\,d\lambda(y) \Big/ \int_{|y|\leqq 1} d\lambda \leqq u(x) + \varepsilon \int_X |u|\,d\lambda$$

for all $\varepsilon \in (0,\delta]$. Then there exists a positive constant C only depending on X and Y such that

$$\text{(1.3.5)} \qquad \lambda(Y \setminus Z(u,\delta)) \leqq C\delta$$

for all $\delta \leqq$ the distance from Y to the boundary of X.

PROOF. Let $0 \leqq \psi \in C_0^\infty(X)$ with $\psi = 1$ in a neighborhood of Y. Denote the left-hand side of (1.3.4) by $u_\varepsilon(x)$. If ε is smaller than the distance from $\operatorname{supp}\psi$ to the boundary of X, then

$$\int \psi u_\varepsilon \, d\lambda = \int \psi_\varepsilon u \, d\lambda = \int \psi u \, d\lambda + \int (\psi_\varepsilon - \psi) u \, d\lambda \, .$$

Taylor's formula gives

$$\psi_\varepsilon(x) - \psi(x) = \varepsilon\Big(\sum_j \int_{|y| \leqq 1} \partial_j \psi(x) y_j \, dy \Big/ \int_{|y| \leqq 1} dy\Big) + O(\varepsilon^2) = O(\varepsilon^2) \, .$$

Hence

$$\int_Y (u_\varepsilon - u) \, d\lambda \leqq \int \psi(u_\varepsilon - u) \, d\lambda \leqq C'\varepsilon^2 \int_X |u| \, d\lambda \, .$$

This inequality gives

$$\lambda\Big(\Big\{x \in Y \; ; \; u_\varepsilon(x) - u(x) > \varepsilon \int_X |u| \, d\lambda/2\Big\}\Big) \leqq 2C'\varepsilon \, .$$

This estimate and the fact that u_ε is an increasing function of ε now give

$$\begin{aligned}
\lambda(Y \setminus Z(u,\delta)) &= \lambda\Big(\Big\{x \in Y \; ; \; u_\varepsilon(x) - u(x) > \varepsilon \int_X |u| \, d\lambda \\
&\qquad\qquad \text{for some } \varepsilon \in (0,\delta]\Big\}\Big) \\
&\leqq \sum_{k=0}^\infty \lambda\Big(\Big\{x \in Y \; ; \; u_\varepsilon(x) - u(x) > \varepsilon \int_X |u| \, d\lambda \\
&\qquad\qquad \text{for some } \varepsilon \in (\delta/2^{k+1}, \delta/2^k]\Big\}\Big) \\
&\leqq \sum_{k=0}^\infty \lambda\Big(\Big\{x \in Y \; ; \; u_{\delta/2^k}(x) - u(x) > \delta \int_X |u| \, d\lambda/2^{k+1}\Big\}\Big) \\
&\leqq 2C' \sum_{k=0}^\infty \delta/2^k = 4C'\delta \, .
\end{aligned}$$

This completes the proof.

The last lemma we need relates L^2-norms of solutions of the $\bar\partial$-equation and estimates in the maximum norm. It is analogous to Lemma 15.1.8 in Hörmander [11].

LEMMA 1.3.5. *Let u be a C^1-function in $\mathbf{C}^n$ with*

$$\int |u|^2(1+|z|^2)^{-\mu}e^{-2\psi}\,d\lambda < \infty \tag{1.3.6}$$

and

$$|\bar\partial u(z)| \leqq Ce^{\psi(z)}, \quad z \in \mathbf{C}^n, \tag{1.3.7}$$

where μ and C are positive constants and ψ is a measurable function. Then there exists a positive constant C', such that

$$|u(z)| \leqq C'(1+|z|)^{\mu}\exp\left(\sup_{w\in B}\psi(z+w)\right), \tag{1.3.8}$$

where B is the closed unit ball in $\mathbf{C}^n$.

PROOF. Let $\chi \in C_0^\infty(B)$ with $\chi(z)=1$ if $|z|\leqq 1/2$, and set $\chi_z(w)=\chi(w-z)$ for $w, z \in \mathbf{C}^n$. Let E be the fundamental solution of the Laplace operator in $\mathbf{R}^{2n}$ of the form $E(z)=c|z|^{-2n+2}$ for $n>1$ and $E(z)=c\log|z|$ for $n=1$ with $c \in \mathbf{R}$. Then

$$\chi_z u = E * \Delta(\chi_z u) = 4\sum \partial_j E * \bar\partial_j(\chi_z u),$$

where $\partial_j=\partial/\partial z_j$ and $\bar\partial_j=\partial/\partial\bar z_j$. We have

$$\bar\partial_j(\chi_z u) = \chi_z\bar\partial_j u + u\bar\partial_j\chi_z \quad \text{and} \quad \chi_z(z)u(z) = u(z).$$

We have

$$\partial_j E * (\chi_z\bar\partial_j u)(z) = \int \partial_j E(-w)\chi(w)\bar\partial_j u(z+w)\,d\lambda(w).$$

Since $\partial_j E \in L^1_{\text{loc}}(\mathbf{C}^n)$ and $\operatorname{supp}\chi\subset B$, (1.3.7) gives an estimate of this term of the form (1.3.8). We have

$$\partial_j E * (u\bar\partial_j\chi_z) = \int \partial_j E(-w)\bar\partial_j\chi(w)u(z+w)\,d\lambda(w).$$

The function $\partial_j E$ is bounded in the support of $\bar\partial_j\chi$. If we multiply the first two factors in the integral by $(1+|z+w|^2)^{\mu/2}e^{\psi(z+w)}$ and the third factor by $(1+|z+w|^2)^{-\mu/2}e^{-\psi(z+w)}$ and use the Cauchy–Schwarz inequality, then (1.3.6) gives an estimate of the form (1.3.8) for this term also. This completes the proof.

PROOF OF THEOREM 1.3.1. Let κ and τ be positive real numbers with $\kappa+2\tau<\min\{\varrho,1\}$ and $\kappa<\tau$. By Lemma 1.3.3 we can suppose that $p \in C^\infty(\mathbf{C}^n\setminus\{0\})$ and that p satisfies (1.3.1) and (1.3.2). Set

$X = \{z \in \mathbf{C}^n ;\ 1/2 \leqq |z| \leqq 4\}, \quad Y = \{z \in \mathbf{C}^n ;\ 1 \leqq |z| \leqq 2\}, \quad \text{and} \quad s_j = 2^j$.

For every j we choose by induction an increasing sequence $\{Z_{jk}\}$ of finite subsets of Y such that for every k the set Z_{jk} is maximal among the subsets W of Y satisfying:

i) $|z-w| > 8s_j^{-\tau}$ if $z, w \in W$ and $z \neq w$.
ii) the distance from W to the boundary of Y is $> 4s_j^{-\tau}$.
iii) for all $z \in W$ and all $\varepsilon \in (0, 2^{-k}]$

$$\left(\int_{|w| \leqq 1} T_{s_j} p(z+\varepsilon w)\, d\lambda(w) \Big/ \int_{|w| \leqq 1} d\lambda\right) \leqq T_{s_j} p(z) + \varepsilon \int_X |T_{s_j} p|\, d\lambda . \tag{1.3.9}$$

By Lemma 1.3.4 the sets $\{Z_{jk}\}$ are non-empty if k is sufficiently large and i) implies that Z_{jk} does not depend on k for large k. We denote the largest set by Y_j and we order the elements of $\bigcup_{j=0}^{\infty} s_j Y_j$ in a sequence $\{z_k\}$.

For every k we define U_k, V_k, and W_k by

$$U_k = \{z \in \mathbf{C}^n ;\ \tfrac{1}{2}|z_k|^{1-\tau} \leqq |z - z_k| \leqq |z_k|^{1-\tau}\} ,$$
$$V_k = \{z \in \mathbf{C}^n ;\ |z - z_k| \leqq |z_k|^{1-\tau}\} ,$$

and

$$W_k = \{z \in \mathbf{C}^n ;\ |z - z_k| \leqq 2|z_k|^{1-\tau}\} .$$

We have $U_k \subset V_k \subset W_k$. By i) and ii) the balls W_k are disjoint, and if $s_{j_k} \leqq |z_k| \leqq s_{j_k+1}$, then

$$W_k \subset \{z \in \mathbf{C}^n ;\ s_{j_k} \leqq |z| \leqq s_{j_k+1}\} .$$

We set

$$h_k(z) = p(z_k) + 2 \sum_l \partial p(z_k)/\partial z_l (z_l - z_{kl}) +$$
$$+ \sum_{l,m} \partial^2 p(z_k)/\partial z_l \partial z_m (z_l - z_{kl})(z_m - z_{km}) .$$

Then Taylor's formula gives

$$p(z) - \operatorname{Re}(h_k(z)) = \sum_{l,m} \partial^2 p(z_k)/\partial z_l \partial \bar{z}_m (z_l - z_{kl})\overline{(z_m - z_{km})} + R_k(z)$$

where

$$|R_k(z)| \leqq C \sup |D^\beta p(w)| |z - z_k|^3$$

and the supremum is taken over all β with $|\beta|=3$ and all w on the line segment between z and z_k. By (1.3.1) and the fact that $|w|<2|z|$ for all $z, w \in W_k$ and $k \geqq 1$, we get

$$(1.3.10) \qquad |R_k(z)| \leqq C'(\log(1+|z|))^M |z|^{\varrho-3}|z-z_k|^3, \quad z \in W_k,$$

where C' is a positive constant and $M=2n^2+3$. By (1.3.2) we get

$$\begin{aligned} &p(z)-\operatorname{Re}(h_k(z)) \\ &\qquad \geqq |z|^{\varrho-\kappa-2}|z-z_k|^2\big(1-C'(\log(1+|z|))^M|z|^{\kappa-1}|z-z_k|\big), \end{aligned}$$

for $z \in W_k$. Since $|z-z_k| \leqq 2|z|^{1-\tau}$ for $z \in W_k$ and $\kappa<\tau$, the last factor can be estimated from below by

$$1-2C'(\log(1+|z|))^M|z|^{\kappa-\tau} \geqq 1/2, \quad z \in W_k,\ k \geqq k_0,$$

where k_0 is some positive integer. Hence

$$(1.3.11) \qquad \operatorname{Re}(h_k(z)) \leqq p(z)-|z|^{\varrho-\kappa-2}|z-z_k|^2/2,$$

for $z \in W_k$, $k \geqq k_0$. In U_k we have $|z-z_k| \geqq |z|^{1-\tau}/4$, so

$$(1.3.12) \qquad \operatorname{Re}(h_k(z)) \leqq p(z)-|z|^{\varrho-\kappa-2\tau}/32, \quad z \in U_k,\ k \geqq k_0.$$

Let $\chi \in C_0^\infty(\mathbf{C}^n)$ with $\chi(z)=0$ if $|z| \geqq 1$, and $\chi(z)=1$ if $|z| \leqq 1/2$, and set

$$\chi_k(z) = \chi\big((z-z_k)|z_k|^{\tau-1}\big), \qquad z \in \mathbf{C}^n,$$

and

$$\psi_k(z) = \chi\big((z-z_k)|z_k|^{\tau-1}/2\big), \qquad z \in \mathbf{C}^n.$$

Then $\operatorname{supp}\psi_k \subset W_k$, $\operatorname{supp}\chi_k \subset V_k$, $\operatorname{supp}\bar\partial\chi_k \subset U_k$, and $\psi_k=1$ in $\operatorname{supp}\chi_k$. Since $0<\tau<1$, we have a bound on $|\bar\partial\chi_k|$ independent of k. We are going to choose f of the form

$$(1.3.13) \qquad f = \sum \chi_k e^{h_k} - v.$$

Then v has to satisfy

$$(1.3.14) \qquad \bar\partial v = \sum \bar\partial\chi_k e^{h_k} = g,$$

where the last equality is a definition. The second order partial derivatives of $z \mapsto |z|^{\varrho-\kappa-2\tau}(\sum\psi_k)$ are $O(|z|^{\varrho-\kappa-2})$ as $|z| \to \infty$ and the Levi form of p can be estimated from below by $|z|^{\varrho-\kappa-2}|w|^2$. If c is a sufficiently small positive constant, then it follows that the function ψ defined by

$$(1.3.15) \qquad \psi(z) = p(z)-c|z|^{\varrho-\kappa-2\tau}\big(\textstyle\sum \psi_k(z)\big), \quad z \in \mathbf{C}^n,$$

is plurisubharmonic in $\mathbf{C}^n$. By (1.3.12) and (1.3.14) we have

$$|g(z)| \leqq Ce^{\psi(z)}.$$

Hence

$$\int |g|^2(1+|z|^2)^{-n-1}e^{-2\psi}\,d\lambda < \infty .$$

Theorem 4.4.2 in Hörmander [10] now gives that there exists a solution v to (1.3.14) with

$$\int |v|^2(1+|z|^2)^{-n-3}e^{-2\psi}\,d\lambda < \infty . \tag{1.3.16}$$

We define f by (1.3.13). In order to prove that $T_t(p-\log|f|) \to 0$ in L^1_{loc} it is sufficient to show that $T_{s_j}(p-\log|f|) \to 0$ as $j \to \infty$, where $s_j=2^j$. In fact, if $t_k \to \infty$ then there exist $j_k \to \infty$ such that t_k/s_{j_k} has a convergent subsequence. Hence

$$T_{t_k}(p-\log|f|) = T_{t_k/s_{j_k}}\big(T_{s_{j_k}}(p-\log|f|)\big)$$

has a subsequence converging to zero.

In order to prove that $T_{s_j}(p-\log|f|) \to 0$ as $j \to \infty$ it is sufficient to show that every subsequence of $\{T_{s_j}(p-\log|f|)\}$ has a subsequence converging to zero. By Proposition 1.1.1 every subsequence of $\{s_j\}$ has a subsequence $\{t_k\}$ such that $T_{t_k}p \to q \in L(p)$ and $T_{t_k}\log|f| \to r \in L(\log|f|)$. It is sufficient to show that $q=r$.

By (1.3.11), (1.3.13), and (1.3.16) the function f satisfies

$$\int |f|^2(1+|z|^2)^{-\nu}e^{-2p}\,d\lambda < \infty ,$$

where $\nu=n+3$. Lemma 1.3.5 gives

$$|f(z)| \leqq C(1+|z|)^{\nu}\exp\Big(\sup_{w\in B} p(z+w)\Big), \quad z \in \mathbf{C}^n . \tag{1.3.17}$$

Hence

$$\overline{\lim_{k\to\infty}}\, T_{t_k}\log|f(z)| \leqq \overline{\lim_{k\to\infty}}\sup_{w\in B}\big(T_{t_k}p(z+w/t_k)\big), \quad z \in \mathbf{C}^n .$$

Since $T_{t_k}p \to q$, we have for every compact subset K of $\mathbf{C}^n$

$$\overline{\lim_{k\to\infty}}\sup_K T_{t_k}p \leqq \sup_K q .$$

(See e.g. Hörmander [11, Theorem 4.1.9].) If we combine the last two inequalities, we get

$$\overline{\lim_{k\to\infty}}\, T_{t_k}\log|f(z)| \leqq \overline{\lim_{w\to z}}\, q(w) = q(z) \quad z \in \mathbf{C}^n .$$

Thus

$$r(z) = \overline{\lim_{w\to z}}\,\overline{\lim_{k\to\infty}}\, T_{t_k}\log|f(w)| \leqq q(z), \quad z \in \mathbf{C}^n . \tag{1.3.18}$$

Next we prove that $r=q$ in Y. By (1.3.18) and the mean value property it is sufficient to show that $q(z)\leqq r(z)$ for almost all $z \in Y$ with respect to the Lebesgue measure. Let F_m denote the set of limit points of sequences $\{\zeta_k\}$ with $\zeta_k \in Z_{j_k m}$, where $t_k = s_{j_k}$. Then F_m is closed. Let K be a compact subset of the open set $Y \setminus F_m$. Then there exist positive numbers d and k_0 such that for $k>k_0$ the set $Z_{j_k m}$ has distance $>d$ to K. Choose $k>k_0$ such that $8t_k^{-\tau}<d$. Since $Z_{j_k m}$ is maximal among all subsets of Y satisfying i), ii), and iii), no point in K can satisfy (1.3.9) with $k=m$ and $s_j=t_k$. Lemma 1.3.4 gives that $\lambda(K)<C2^{-m}$, where C is a positive constant. Since K was arbitrarily chosen, $\lambda(Y\setminus F_m)<C2^{-m}$. Hence the set $Y\setminus \bigcup_m F_m = \bigcap_m Y\setminus F_m$ has measure zero.

This gives that in order to prove that $r=q$ in Y it is sufficient to show that $q(z)\leqq r(z)$ for all $z \in Y$ that are limit points of $\{\zeta_k\}$ with $\zeta_k \in Z_{j_k m}$ for some positive integer m. Let $\varepsilon \in (0, 2^{-m}]$. Then

$$\begin{aligned} q(z) &\leqq \int_{|w|\leqq 1} q(z+\varepsilon w)\,d\lambda(w)\Big/\int_{|w|\leqq 1} d\lambda \\ &= \lim \int_{|w|\leqq 1} q(\zeta_k+\varepsilon w)\,d\lambda(w)\Big/\int_{|w|\leqq 1} d\lambda , \end{aligned} \tag{1.3.19}$$

where the limit is taken over a subsequence of $\{\zeta_k\}$ converging to z. The equality holds, because the mean value is a continuous function of z. Since $T_{t_k}p \to q$ in L^1_{loc}, for every $\delta>0$ there exists k_δ such that if $k>k_\delta$

$$\begin{aligned} &\int_{|w|\leqq 1} q(\zeta_k+\varepsilon w)\,d\lambda(w)\Big/\int_{|w|\leqq 1} d\lambda \\ &\qquad \leqq \int_{|w|\leqq 1} T_{t_k}p(\zeta_k+\varepsilon w)\,d\lambda(w)\Big/\int_{|w|\leqq 1} d\lambda+\delta . \end{aligned} \tag{1.3.20}$$

Since $t_k=s_{j_k}$ and $\zeta_k \in Z_{j_k m}$, we get

$$\begin{aligned} &\left(\int_{|w|\leqq 1} T_{t_k}p(\zeta_k+\varepsilon w)\,d\lambda(w)\Big/\int_{|w|\leqq 1} d\lambda\right) \\ &\qquad \leqq T_{t_k}p(\zeta_k)+\varepsilon \int_X |T_{t_k}p|\,d\lambda . \end{aligned} \tag{1.3.21}$$

Since $\{T_t p;\ t \geqq 1\}$ is relatively compact, the integral in the right-hand side can be estimated by a positive constant C. The point $t_k \zeta_k$ is an element of the sequence $\{z_l\}$. We have

$$|p(z_l) - \log|f(z_l)|| = |\log|1 - v(z_l)e^{-p(z_l)}|| . \tag{1.3.22}$$

By (1.3.15) and (1.3.16) the function v satisfies the conditions in Lemma 1.3.5. Hence

$$|v(z)| \leqq C(1+|z|)^{n+3} \exp\left(\sup_{w \in B} \psi(z+w)\right) .$$

By Taylor's formula (1.3.1) and (1.3.15) there exist positive constants a, b, C, M and N such that

$$\begin{aligned} &|v(z_1)|\, e^{-p(z_l)} \\ &\quad \leqq C(1+|z_l|)^N \exp\left(a(\log|z_l|)^M |z_l|^{\varrho - 1} - b|z_l|^{\varrho - \kappa - 2\tau}\right). \end{aligned}$$

Since $\kappa + 2\tau < \min\{\varrho, 1\}$, the right-hand side converges to zero as $l \to \infty$ and (1.3.22) gives

$$p(z_l) - \log|f(z_l)| \to 0, \quad l \to \infty .$$

Hence

$$T_{t_k} p(\zeta_k) - T_{t_k} \log|f(\zeta_k)| \to 0, \quad k \to \infty . \tag{1.3.23}$$

We have $T_{t_k} \log|f| \to r$ in L^1_{loc} as $k \to \infty$. Hence

$$\overline{\lim_{k \to \infty}}\, T_{t_k} \log|f(\zeta_k)| \leqq r(z) . \tag{1.3.24}$$

If we combine (1.3.19), (1.3.20), (1.3.21), (1.3.23), and (1.3.24) we get

$$q(z) \leqq r(z) + C\varepsilon + \delta .$$

Since the numbers ε and δ were arbitrarily chosen, it follows that $q = r$ in Y.

It now follows that $q = r$ in $s_j Y$ for all integers j, for this is equivalent to $T_{s_j} r = T_{s_j} q$ in Y and we have $T_{t_k s_j} p \to T_{s_j} q$, and $T_{t_k s_j} \log|f| \to T_{s_j} r$ as $k \to \infty$. The set $\mathbf{C}^n \setminus \bigcup_j s_j Y$ has Lebesque measure zero, so $q = r$. This completes the proof.

1.4. A refined indicator theorem.

In this section we give a proof of the indicator theorem with a growth condition on the analytic function in the form of an L^2-estimate. In Chapter

Chapter 3 it will enable us to characterize the indicator functions of Fourier–Laplace transforms of distributions with compact support.

THEOREM 1.4.1. *Let ϱ be a positive real number and let M be a compact neighborhood of the origin in $\mathbf{C}^n$. Let p be a plurisubharmonic function in $\mathbf{C}^n$ and suppose that p is positively homogeneous of order ϱ. Then there exists an analytic function f in $\mathbf{C}^n$ with $i_f = p$ and*

$$\int |f|^2 (1+|z|^2)^{-\varrho-3n} \exp(-2p_M) d\lambda < \infty, \tag{1.4.1}$$

where $p_M(z) = \sup_{w \in M} p(z+w)$.

Our proof is similar to that of Kiselman [13] and Martineau [16], [17]. It is in two parts. First we prove that for every $z \in \mathbf{C}^n$ there exists an analytic function f satisfying (1.4.1) with $p(z) = i_f(z)$. This is done by induction on the dimension. Then we give a category argument in the Banach space of all analytic function satisfying (1.4.1) in order to prove the existence of a function f with $i_f = p$. The case when $n=1$ and p is harmonic in the complex plane outside a half line must be treated separately. It is the most tedious part of the proof and we start with it.

LEMMA 1.4.2. *Let p be a subharmonic function in $\mathbf{C}$ which is positively homogeneous of order $\varrho > 0$. Suppose that p is harmonic in $\mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_+)$ for some $\vartheta \in \mathbf{R}$. Then there exists an analytic function f in $\mathbf{C}$ such that*

$$|f(z)| \leqq Ce^{p(z)}, \quad z \in \mathbf{C}, \tag{1.4.2}$$

for some positive constant C, and for every $\varepsilon \in (0,1)$ there exists a positive constant C_ε such that

$$|f(z)| \geqq C_\varepsilon^{-1}(1+|z|)^{-N} e^{p(z)} \tag{1.4.3}$$

for all $z \in \mathbf{C}$ with $\vartheta + \varepsilon \leqq \arg z \leqq \vartheta + 2\pi - \varepsilon$. Here $\mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_+) \ni z \mapsto \arg z$ is defined so that $\vartheta < \arg z < \vartheta + 2\pi$, and $N > 0$.

PROOF. We begin by showing that there exists an analytic function f which satisfies (1.4.3) and

$$|f(z)| \leqq C(1+|z|)^N e^{p(z)} \quad z \in \mathbf{C}, \tag{1.4.2$'$}$$

for some positive N. By composing p with a rotation we can suppose that $\vartheta = -\pi$. We let

$$\mathbf{C} \setminus \overline{\mathbf{R}}_- \ni z \mapsto \operatorname{Log} z = \log|z| + i \arg z$$

be the branch of the logarithm which is real on R_+, and we define z^α by $z^\alpha = \exp(\alpha \operatorname{Log} z)$ for all $\alpha \in \mathsf{C}$. Since p is harmonic in $\mathsf{C} \setminus \overline{\mathsf{R}}_-$ and positively homogeneous of order ϱ, it is of the form $p(z) = \operatorname{Re}(az^\varrho)$ for some $a \in \mathsf{C}$. By Proposition 1.1.3, p is continuous, so the real parts of $a\exp(\pm i\varrho\pi)$ are equal. The absolute values of these numbers are equal, so they are either equal or complex conjugates. In the first case ϱ is an integer and f can be chosen as $f(z) = \exp(az^\varrho)$. Hence we can suppose that ϱ is not an integer. Then a is real.

Next we are going to derive a Riesz representation formula for p. We let k be the integer with $k < \varrho < k+1$ and let E be the Weierstrass primary factor defined by $E(z) = 1 - z$ if $k = 0$ and $E(z) = (1-z)\exp(z + z^2/2 + \ldots + z^k/k)$ if $k > 0$. Then

$$p(z) = \int_0^\infty \log|E(-z/s)|\, d\mu(s), \quad z \in \mathsf{C}\,, \tag{1.4.4}$$

where $d\mu(s) = \gamma\varrho s^{\varrho-1} ds$, with $\gamma = a\sin(\varrho\pi)/\pi$. Before we prove the formula we make some observations. We have

$$\log|E(-z/s)| = \log|1 + z/s| + \operatorname{Re}\left(\sum_1^k (-z/s)^l/l\right),$$

where the sum is omitted if $k = 0$. Hence the integrand is $O((\log s)s^{\varrho-k-1})$ as $s \to 0$. By Taylor's formula it is $O(s^{\varrho-k-2})$ as $s \to \infty$, and if $z \in \mathsf{R}_-$ it is $O(\log|z+s|)$ as $s \to -z$. Hence the integral in (1.4.4) is convergent and it defines a continuous function of z in C. We observe that

$$2\partial p/\partial z = \partial(az^\varrho + a\bar{z}^\varrho)/\partial z = a\varrho z^{\varrho-1}$$

in $\mathsf{C} \setminus \overline{\mathsf{R}}_-$ and

$$a\varrho\left((x+i0)^{\varrho-1} - (x-i0)^{\varrho-1}\right) = -\left(2ia\varrho\sin(\varrho\pi)\right)x_-^{\varrho-1},$$

so Theorem 3.1.12 in Hörmander [11] gives

$$\Delta p = 4\partial^2 p/\partial z\partial\bar{z} = \left(2a\varrho\sin(\varrho\pi)\right)x_-^{\varrho-1} \otimes \delta(y)\,.$$

Since p is subharmonic it follows that $\gamma = a\sin(\varrho\pi)/\pi > 0$. Furthermore

$$\int_0^\infty \varphi(-s)\, d\mu(s) = (2\pi)^{-1}\langle \Delta p, \varphi\rangle, \quad \varphi \in C_0(\mathsf{C})\,.$$

Now we turn to the derivation of (1.4.4). We have

$$\text{(1.4.5)}\qquad \frac{\partial}{\partial s}\log|E(-z/s)| = \frac{\partial}{\partial s}\operatorname{Re}\left(\operatorname{Log}(1+z/s)+\sum_1^k (-z/s)^l/l\right)$$
$$= \operatorname{Re}\big((-z/s)^{k+1}/(s+z)\big)$$
$$= (-s)^{-k-1}|z+s|^{-2}\operatorname{Re}\big(z^{k+1}(s+\bar{z})\big)\,.$$

A partial integration gives for $z \in \mathbf{C}\setminus\overline{\mathbf{R}}_-$

$$\text{(1.4.6)}\qquad \int_0^\infty \log|E(-z/s)|\,d\mu(s) = \gamma\operatorname{Re}\left((-1)^k z^{k+1}\int_0^\infty s^{\varrho-k-1}/(s+z)\,ds\right).$$

If z is a positive real number then a change of variables gives

$$\text{(1.4.7)}\qquad z^{k+1}\int_0^\infty s^{\varrho-k-1}/(s+z)\,ds = z^\varrho\int_0^\infty s^{\varrho-k-1}/(s+1)\,ds\,.$$

Both sides of this equality are analytic functions of z in $\mathbf{C}\setminus\mathbf{R}_-$, so the equality holds for $z \in \mathbf{C}\setminus\mathbf{R}_-$. By Hörmander [11, p. 86], the last integral is equal to

$$\pi/\sin\big((k+1-\varrho)\pi\big) = (-1)^k\pi/\sin(\varrho\pi)\,.$$

Hence (1.4.4) follows from (1.4.6) and (1.4.7).

We are going to show that f can be chosen as

$$\text{(1.4.8)}\qquad f(z) = e^{h(z)}\prod_{j=1}^\infty E(-z/t_j),\quad z \in \mathbf{C}\,,$$

where $\{t_j\}$ is the sequence of non-negative real numbers satisfying

$$\text{(1.4.9)}\qquad \gamma t_j^\varrho = j,\quad j \geqq 0\,,$$

and h is a polynomial to be chosen later. Since $\sum t_j^{-k-1} < \infty$, the product is convergent and it defines an entire analytic function. We have

$$\text{(1.4.10)}\qquad \int_{t_{j-1}}^{t_j} d\mu(s) = \gamma(t_j^\varrho - t_{j-1}^\varrho) = 1,\quad j \geqq 1\,,$$

and

$$(t_j - t_{j-1}) = (\varrho\gamma)^{-1}t_j^{1-\varrho}(1+O(1/j))\quad \text{as } j \to \infty\,.$$

Since $t_j/t_{j-1} \to 1$ as $j \to \infty$, this implies that there exists a positive constant α such that

$$\text{(1.4.11)}\qquad (t_j - t_{j-1}) \leqq \alpha s^{1-\varrho},\quad s \in [t_{j-1}, t_j],\ j \geqq 2\,.$$

In order to define the polynomial h and estimate $\log|f|-p$, we define f_j by

$$f_j(z) = \log|E(-z/t_j)|-\int_{t_{j-1}}^{t_j} \log|E(-z/s)|\,d\mu(s), \quad z \in \mathbf{C}.$$

The equality (1.4.10) gives

$$f_j(z) = \int_{t_{j-1}}^{t_j} (\log|E(-z/t_j)|-\log|E(-z/s)|)\,d\mu(s), \quad z \in \mathbf{C}.$$

We write f_j as a sum $f_j=g_j+h_j$, where

$$g_j(z) = \int_{t_{j-1}}^{t_j} (\log|1+z/t_j|-\log|1+z/s|)\,d\mu(s), \quad z \in \mathbf{C},$$

$h_j=0$ if $k=0$, and

$$h_j(z) = \int_{t_{j-1}}^{t_j} \operatorname{Re}\left(\sum_{l=1}^{k} [(-z/t_j)^l-(-z/s)^l]/l\right) d\mu(s), \quad z \in \mathbf{C},$$

if $k \geqq 1$. For $1 \leqq l \leqq k$ we define a_l by

$$a_l = -\sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} ((-1/t_j)^l-(-1/s)^l)/l\,d\mu(s).$$

The sum is convergent for the integrand is $O(s^{\varrho-l-1})$ as $s \to 0$, and the mean value theorem and (1.4.11) give

$$|t_j^{-l}-s^{-l}| \leqq C_1 s^{-\varrho-l}, \quad s \in [t_{j-1}, t_j],\ j \geqq 2,$$

which implies

$$\left|\int_{t_{j-1}}^{t_j} ((-t_j)^{-l}-(-s)^{-l})\,d\mu(s)\right| \leqq C_2(t_{j-1}^{-l}-t_j^{-l}) \leqq C_3 j^{-l/\varrho-1}$$

for $j \geqq 2$. We set $h=0$ if $k=0$, and

$$h(z) = \sum_{l=1}^{k} a_l z^l, \quad z \in \mathbf{C},$$

if $k \geqq 1$. We define f by (1.4.8). Then

$$\log|f(z)|-p(z) = \operatorname{Re}(h(z))+\sum_{j=1}^{\infty} f_j(z), \quad z \in \mathbf{C}.$$

We set

$$S_\varepsilon = \{z \in \mathbf{C}\ ;\ z=re^{i\varphi}, |\varphi| \leqq \pi-\varepsilon\}$$

and

$$T = \{z \in \mathbf{C}\ ;\ z=re^{i\varphi}, \pi-1 \leqq \varphi \leqq \pi+1\}.$$

By splitting the sum into two parts, we see that (1.4.2)′ and (1.4.3) hold if we can prove the following four inequalities

$$\sum_{t_{j-1}\geqq 2|z|} |f_j(z)| \leqq \log C, \quad \text{if } |z|\geqq R\,, \tag{1.4.12}$$

$$\left|\operatorname{Re}(h(z)) + \sum_{t_{j-1}<2|z|} h_j(z)\right| \leqq \log C, \quad \text{if } |z|\geqq R\,, \tag{1.4.13}$$

$$\sum_{t_{j-1}<2|z|} |g_j(z)| \leqq \log\big(C_\varepsilon(1+|z|)^N\big)\,, \tag{1.4.14}$$

for all $\varepsilon \in (0,1)$ and all $z \in S_\varepsilon$ with $|z|>R_\varepsilon$, and

$$\sum_{t_{j-1}<2|z|} g_j(z) \leqq \log\big(C(1+|z|)^N\big)\,, \tag{1.4.15}$$

for all $z \in T$ with $|z|\geqq R$. Here C, N, R, C_ε, and R_ε are positive constants.

The first three inequalities follow with an application of (1.4.11) and the mean value theorem. We set $R=\max\{2t_1,2\}$. If $t_{j-1}\geqq 2|z|$, then $|z+s|\geqq|z|$ for all $s\in[t_{j-1},t_j]$. Hence (1.4.5), (1.4.11), and the mean value theorem give

$$\big|\log|E(-z/t_j)|-\log|E(-z/s)|\big| \leqq C'|z|^k s^{-k-\varrho}, \quad s\in[t_{j-1},t_j]\,.$$

If we use the fact that

$$\big|\log|(1+z/t_j)/(1+z/s)|\big| \leqq C''|z||1+z/s|^{-1}(s^{-1}-t_j^{-1}) \quad \text{for } s\in[t_{j-1},t_j]\,,$$

then it follows from (1.4.11) and the mean value theorem that the factor $s^{-k-\varrho}$ in the left hand side can be replaced by $s^{-1-\varrho}$ if $k=0$. Hence the left hand side of (1.4.12) can be estimated by

$$C'''|z|^k \int_{2|z|}^{\infty} s^{-k-1}\,ds, \text{ if } k>0, \quad \text{and} \quad C''' \int_{2|z|}^{\infty} s^{-2}\,ds, \text{ if } k=0\,,$$

where C', C'', and C''' are positive constants. This gives (1.4.12). If $k\geqq 1$ then an application of the mean value theorem as above gives that the left-hand side of (1.4.13) can be estimated by

$$\sum_{l=1}^{k} |z|^l \sum_{t_{j-1}\geqq 2|z|} \int_{t_{j-1}}^{t_j} (|t_j^{-l}-s^{-l}|/l)\,d\mu(s) \leqq C' \sum_{l=1}^{k} |z|^l \int_{2|z|}^{\infty} s^{-l-1}\,ds\,,$$

and (1.4.13) follows. We have

$$g_1(z) = \log|1+z/t_1| + \int_0^{t_1} (\log s)\,d\mu(s) - \int_0^{t_1} \log|z+s|\,d\mu(s), \quad z\in \mathbf{C}\,.$$

If $|z|\geqq R$, then $|\log|z+s||\leqq\log(2|z|)$ for $s\in[0,t_1]$, so we conclude that

$$|g_1(z)| \leqq \log\big(C(1+|z|)\big), \quad \text{if } |z|\geqq R\,, \tag{1.4.16}$$

if C is chosen sufficiently large. If $z\in S_\varepsilon$, then $|z+s|\geqq(\sin\varepsilon)|z|$ for all $s\geqq 0$. If $R_\varepsilon\geqq(\sin\varepsilon)^{-1}$, then (1.4.11), (1.4.5), and the mean value theorem give

$$|\log|1+z/t_j|-\log|1+z/s|| \leqq C's^{-\varrho}\,,$$

for $s\in[t_{j-1},t_j]$, $|z|\geqq R_\varepsilon$, and $j\geqq 2$. Hence the sum of all the terms in the left-hand side of (1.4.14) except the first one can be estimated by

$$C''\int_{t_1}^{2|z|} s^{-1}\,ds = C''\log(2|z|/t_1)$$

for all $z\in S_\varepsilon$ with $|z|\geqq R_\varepsilon$. If we combine this estimate and (1.4.16), then (1.4.14) follows. Let $z\in T$. By (1.4.5) with $k=0$ it follows that the function $\mathbf{R}_+\ni s\mapsto\log|1+z/s|$ is decreasing when $s<-|z|^2/\mathrm{Re}\,z$ and increasing when $s>-|z|^2/\mathrm{Re}\,z$. We choose l such that $t_{l-1}<-|z|^2/\mathrm{Re}\,z\leqq t_l$ and m such that $t_{m-1}<2|z|\leqq t_m$. Then

$$\int_{t_{j-1}}^{t_j}\log|1+z/s|\,d\mu(s) \geqq \log|1+z/t_j|, \quad \text{if } j\leqq l-1\,,$$

and

$$\int_{t_{j-1}}^{t_j}\log|1+z/s|\,d\mu(s) \geqq \log|1+z/t_{j-1}|, \quad \text{if } j-1\geqq l\,.$$

These inequalities give that the left-hand side of (1.4.15) can be estimated from above by

$$\log|1+z/t_m|-\int_{t_{l-1}}^{t_l}\log|1+z/s|\,d\mu(s)\,. \tag{1.4.17}$$

Since $2|z|\leqq t_m$, the first term is bounded from above. We have

$$\begin{aligned} &-\int_{t_{l-1}}^{t_l}\log|1+z/s|\,d\mu(s) \\ &\qquad= \int_{t_{l-1}}^{t_l}(\log s)\,d\mu(s)-\int_{t_{l-1}}^{t_l}\log|z+s|\,d\mu(s)\,. \end{aligned} \tag{1.4.18}$$

Since $|z|<|\mathrm{Re}\,z|/\cos 1$ for $z\in T$ and $t_{l-1}<-|z|^2/\mathrm{Re}\,z\leqq t_l$, we can choose R such that $t_{l-1}\geqq 1$ if $|z|\geqq R$. Since $t_j/t_{j-1}\to 1$ and (1.4.10) holds, we conclude that the first term in the right-hand side of (1.4.18) can be estimated by $\log\big(C(1+|z|)\big)$ for some positive constant C. If $\varrho<1$, then

$$-\int_{t_{l-1}}^{t_l} \log|z+s|\,d\mu(s) \leqq -\gamma\varrho \int_{-1}^{1} \log|x|\,dx, \quad \text{if } |z| \geqq R .$$

If $\varrho > 1$, then (1.4.11) gives that the length of the interval $[t_{l-1}, t_l]$ is $\leqq \alpha t_l^{1-\varrho}$, so if $\delta = \min\{1, \alpha t_l^{1-\varrho}\}$, then

$$\int_{t_{l-1}}^{t_l} \log|z+s|\,d\mu(s) \leqq -\gamma\varrho t_l^{\varrho-1} \int_{-\delta}^{\delta} \log|x|\,dx \quad \text{if } |z| \geqq R .$$

The right-hand side can be estimated by $\log(C(1+|z|)^N)$ for some positive constants C and N. Hence we have proved (1.4.15).

We set

$$q(z) = \prod_{j=1}^{N} (1+z/t_j)$$

and replace f by f/q. Then (1.4.3) holds, and (1.4.2) follows from (1.4.2)′. This completes the proof.

Now we are able to treat the case $n=1$ in general.

LEMMA 1.4.3. *Let p be a subharmonic function in* C *and suppose that p is positively homogeneous of order $\varrho > 0$. Let $z_0 \in \mathbf{C} \setminus \{0\}$. Then there exists an analytic function f in* C *such that*

$$(1.4.20) \qquad |f(z)| \leqq C(1+|z|)^{\varrho+2} e^{p(z)} \quad z \in \mathbf{C} ,$$

and

$$(1.4.21) \qquad i_f(z_0) = \overline{\lim_{t\to\infty}}\, t^{-\varrho} \log|f(tz_0)| = p(z_0) .$$

PROOF. Let S be a maximal sector in $\mathbf{C} \setminus \{0\}$ containing z_0 such that $\Delta p = 0$ in the interior of S. Suppose first that $\mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_+) \subset \operatorname{int} S$ for some $\vartheta \in \mathbf{R}$. Then there exists an analytic function f satisfying the conditions in Lemma 1.4.2. We have $i_f = p$, for (1.4.3) gives that $i_f(z) = p(z)$ for all $z \in \mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_-)$ and the continuity gives that equality holds identically. Hence (1.4.20) and (1.4.21) are satisfied.

Suppose now that $\mathbf{C} \setminus S$ has an interior point $e^{i\vartheta}$. We let

$$\mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_+) \ni z \mapsto \operatorname{Log} z = \log|z| + i \arg z$$

be the branch of the logarithm with $\vartheta < \arg z < \vartheta + 2\pi$ and define z^ϱ by $z^\varrho = \exp(\varrho \operatorname{Log} z)$ for $z \in \mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_+)$. If S has a non empty interior, then there exists $a \in \mathbf{C}$ such that $p(z) = \operatorname{Re}(az^\varrho)$ in S. If the interior of S is empty then there exists $a \in \mathbf{C}$ such that $p(z_0) = \operatorname{Re}(az_0^\varrho)$, $p(z) \geqq \operatorname{Re}(az^\varrho)$ in a neighborhood of S with equality only in S, for Proposition 1.1.3 gives that p is a convex function of z^ϱ in a neighborhood of S.

The set S can be written as

$$S = \{z \in \mathbf{C} \setminus \{0\} \ ; \ \vartheta_1 \leqq \arg z \leqq \vartheta_2\} .$$

By Proposition 1.1.3 we can choose $\delta > 0$ such that the sector

$$T = \{z \in \mathbf{C} \setminus \{0\} \ ; \ \vartheta_1 - \delta \leqq \arg z \leqq \vartheta_2 + \delta\}$$

is contained in $\mathbf{C} \setminus (e^{i\vartheta}\overline{\mathbf{R}}_+)$ and p is a convex function of z^ϱ in the sectors

$$U = \{z \in \mathbf{C} \setminus \{0\} \ ; \ \vartheta_1 - \delta \leqq \arg z \leqq \vartheta_1 + \delta\}$$

and

$$V = \{z \in \mathbf{C} \setminus \{0\} \ ; \ \vartheta_2 - \delta \leqq \arg z \leqq \zeta_2 + \delta\} .$$

We let $\chi \in C^\infty(\mathbf{C})$ with $\operatorname{supp} \chi \subset T$, $0 \leqq \chi \leqq 1$, $\chi(z) = 1$ for all z with $\vartheta_1 - \delta/2 \leqq \arg z \leqq \vartheta_2 + \delta/2$ and $|z| \geqq 1$, and $\chi(tz) = \chi(z)$ if $t \geqq 1$ and $|z| \geqq 1$. We are going to show that it is possible to choose f of the form

$$f(z) = \chi(z) z^\nu e^{az^\varrho} - v(z) , \tag{1.4.22}$$

where ν is the integer with $1 + \varrho < \nu \leqq 2 + \varrho$. Then v has to satisfy

$$\partial v / \partial \bar{z} = \partial \chi / \partial \bar{z} z^\nu e^{az^\varrho} = g , \tag{1.4.23}$$

where the last equality is a definition. Since p is a convex function of z^ϱ in the sectors U and V, and S is a maximal sector where $p(z) = \operatorname{Re}(az^\varrho)$, it follows that there exists $\varepsilon > 0$ such that

$$\operatorname{Re}(az^\varrho) \leqq p(z) - \varepsilon |z|^\varrho$$

for all z with $\vartheta_1 - \delta \leqq \arg z \leqq \vartheta_1 - \delta/2$ or $\vartheta_2 + \delta/2 \leqq \arg z \leqq \vartheta_2 + \delta$. Since $\chi(z) = 1$ for all z with $\vartheta_1 - \delta/2 \leqq \arg z \leqq \vartheta_2 + \delta/2$ and $|z| \geqq 1$, and $|\partial \chi / \partial \bar{z}| = O(|z|^{-1})$ as $|z| \to \infty$, we have

$$\int |g|^2 e^{-2p} \, d\lambda < \infty . \tag{1.4.24}$$

Theorem 4.4.2 in Hörmander [10] gives that there exists a solution v of (1.4.23) with

$$\int |v|^2 (1 + |z|^2)^{-2} e^{-2p} \, d\lambda < \infty . \tag{1.4.25}$$

We define f by (1.4.22).

For $t > 0$ we let B_t denote the closed disk in $\mathbf{C}$ with center tz_0 and radius $t^{1-\varrho}$. Then v is analytic in B_t for all sufficiently large t. The mean value theorem and the Cauchy–Schwarz inequality give

$$(1.4.26)\quad |v(tz_0)| \leqq \pi^{-1}t^{-2+2\varrho}\int_{B_t}|v|\,d\lambda \leqq \pi^{-1/2}t^{-1+\varrho}\|v\|_{L^2(B_t)}\,.$$

By using the estimate (1.4.25) we conclude as in the proof of Theorem 1.3.5, that

$$(1.4.27)\qquad \|v\|_{L^2(B_t)} \leqq C_1(1+|tz_0|^2)\exp\left(\sup_{w\in B_t} p(w)\right).$$

By Proposition 1.1.3 p is a convex function of z^ϱ in a conic neighborhood of z_0, so it is locally Lipschitz continuous. We have $|z_0-w/t|\leqq t^{-\varrho}$ for $w \in B_t$, so

$$(1.4.28)\qquad \sup_{w\in B_t} p(w) = t^\varrho \sup_{w\in B_t} p(w/t) \leqq p(tz_0)+\gamma$$

for some positive constant γ. Now (1.4.26), (1.4.27), and (1.4.28) give

$$|v(tz_0)| \leqq C_2 t^{1+\varrho}\exp\bigl(p(tz_0)\bigr)$$

for some positive constant C_2. Hence

$$|z_0|^\nu t^\nu - C_2 t^{1+\varrho} \leqq |f(tz_0)|e^{-p(tz_0)} \leqq |z_0|^\nu t^\nu + C_2 t^{1+\varrho}\,.$$

Since $\nu>1+\varrho$, (1.4.21) follows. The estimate (1.4.20) follows if we apply (1.4.26) with v replaced by f. The proof is complete.

REMARK. If $\varrho=1$ then Lemma 1.4.3 is trivial. In fact, Proposition 1.1.3 ii) gives that $p(z)=\sup_{w\in K}\operatorname{Re}(z\bar{w})$ for $z \in \mathbf{C}$, where K is some compact convex subset of $\mathbf{C}$. Hence we can choose f as $f(z)=\exp(z\bar{w}_0)$, where $w_0 \in K$ satisfies $p(z_0)=\operatorname{Re}(z_0\bar{w}_0)$.

The next lemma is only a variant of Theorem 4.4.3 in Hörmander [10] and it is proved in an analogous way.

LEMMA 1.4.4. *Let p be a plurisubharmonic function in $\mathbf{C}^n$. Let Σ be a complex subspace of $\mathbf{C}^n$ of codimension k, and let M be a compact neighborhood of the origin in $\mathbf{C}^n$. For every analytic function h in Σ satisfying*

$$\int_\Sigma |h|^2(1+|z|^2)^{-N}\exp(-2p)\,d\sigma < \infty\,,$$

where N is a positive constant and $d\sigma$ denotes the Lebesgue measure in Σ, there exists an analytic function f in $\mathbf{C}^n$, such that $f=h$ in Σ and

$$\int |f|^2(1+|z|^2)^{-N-3k}\exp(-2p_M)\,d\lambda < \infty\,,$$

where $p_M(z)=\sup_{w\in M}p(z+w)$.

PROOF. Since $(p_L)_M \leqq p_{L+M}$ for all compact subsets M and L of $\mathbf{C}^n$, it is sufficient to prove the first assertion when Σ is a hyperplane and then iterate this special case k times. By composing h and p with a unitary linear map we can suppose that $\Sigma = \{z \in \mathbf{C}^n;\ z_n = 0\}$. Then h is an analytic function of $z' = (z_1, \ldots, z_{n-1})$ and can be considered as an analytic function in $\mathbf{C}^n$ independent of z_n. We choose $a > 0$ such that the polydisk $\{z \in \mathbf{C}^n;\ |z_j| \leqq a, j = 1, \ldots, n\}$ is contained in M. Since $p(z', 0) \leqq p_M(z', z_n)$ for all z with $|z_n| \leqq a$ it follows that

$$\int_{|z_n| \leqq a} |h|^2 (1+|z|^2)^{-N} \exp(-2p_M) d\lambda \leqq \pi a^2 \int_\Sigma |h|^2 (1+|z|^2)^{-N} \exp(-2p) d\sigma .$$

Let ψ be a continuous function in $\mathbf{C}$ which is equal to 1 in the disk $|z| \leqq a/2$, equal to 0 outside the disk $|z| \leqq a$, and a linear function of $|z|$ between $a/2$ and a. Then $|\partial\psi/\partial\bar{z}| \leqq a^{-1}$.

In the same way as in the proof of Theorem 4.4.3 in Hörmander [10] we can show that f can be chosen of the form

$$f(z) = \psi(z_n) h(z') - z_n v(z) ,$$

where v satisfies the equation $\bar\partial v = z_n^{-1} h(z') (\partial\psi(z_n)/\partial\bar{z})\, d\bar{z}_n$ and

$$\int |v|^2 (1+|z|^2)^{-N-2} \exp(-2p_M) d\lambda < \infty .$$

This completes the proof.

LEMMA 1.4.5. *Let M be a compact subset of $\mathbf{C}^n$. Let p be an upper semi-continuous function in $\mathbf{C}^n$ and suppose that p is positively homogeneous of order $\varrho > 0$. Define p_M by*

$$p_M(z) = \sup_{w \in M} p(z+w) \quad \text{for } z \in \mathbf{C}^n .$$

If f is analytic in $\mathbf{C}^n$ satisfying

$$\int |f|^2 (1+|z|^2)^{-N} \exp(-2p_M) d\lambda < \infty$$

for some $N > 0$, then f is of order $\leqq \varrho$ and $i_f \leqq p$.

PROOF. By Lemma 1.3.5, there exists a positive constant C such that

$$|f(z)| \leqq C(1+|z|)^N \exp\left(\sup_{w \in M+B} p(z+w) \right), \quad z \in \mathbf{C}^n ,$$

where B denotes the closed unit ball in $\mathbf{C}^n$. Hence f is of order $\leqq \varrho$. Since p is upper semi-continuous and homogeneous it follows that

$$\overline{\lim_{t\to\infty}}\, t^{-\varrho}\log|f(tz)| \leqq \overline{\lim_{t\to\infty}} \sup_{w\in M+B} p(z+w/t) \leqq p(z)\,.$$

Hence $i_f \leqq p$ and the proof is completed.

The last lemma we need can be found in Ronkin [20], Chapter 3:

LEMMA 1.4.6. *Let u be an upper semi-continuous function in an open subset U of $\mathbf{R}^N$ such that $\{x \in U;\, u(x) > -\infty\}$ is dense in U. Then there exists a countable subset S of U such that $u(x) > -\infty$ if $x \in S$, and any upper semi-continuous function v in U such that $v \leqq u$ with equality in S is identically equal to u.*

PROOF. Let $\{\varepsilon_j\}$ be a sequence of positive real numbers converging to zero. For every j we let $\{U_{jk}\}$ be an open covering of U such that the sets U_{jk} are relatively compact in U and have diameter $\leqq \varepsilon_j$. For every U_{jk} we choose a point $x_{jk} \in \bar{U}_{jk}$ such that

$$\sup_{x\in U_{jk}} u(x) < u(x_{jk})+\varepsilon_j, \quad j,k \geqq 1\,,$$

and set $S=\{x_{jk};\, j,k\geqq 1\}$. Then S is a countable dense subset of U and

$$u(x) = \overline{\lim_{S\ni y\to x}}\, u(y), \quad x \in U\,.$$

If v is upper semi-continuous in U, majorized by u in U and equal to u in S, then

$$v(x) = \overline{\lim_{y\to x}}\, v(y) \geqq \overline{\lim_{S\ni y\to x}}\, v(y) = \overline{\lim_{S\ni y\to x}}\, u(y) = u(x)$$

for all $x \in U$. This proves the assertion.

PROOF OF THEOREM 1.4.1. Let $z \in \mathbf{C}^n$ with $p(z) > -\infty$. We shall first show that there exists an analytic function f in $\mathbf{C}^n$ satisfying (1.4.1) and

$$p(z) \leqq i_f(z)\,. \tag{1.4.29}$$

The function $q(\tau)=p(\tau z)$ is subharmonic in $\mathbf{C}$ and positively homogeneous of order ϱ. By Lemma 1.4.3 there exists an analytic function g in $\mathbf{C}$ satisfying (1.4.20) and (1.4.21). We let Σ denote the complex line spanned by z and define the analytic function h in Σ by $h(\tau z)=g(\tau)$. Then Lemma 1.4.4 gives that h can be continued to an analytic function f in $\mathbf{C}^n$ satisfying (1.4.1). The equality (1.4.21) gives

$$p(z) = \overline{\lim_{t\to\infty}} t^{-\varrho} \log|f(tz)| \leqq i_f(z),$$

so (1.4.29) holds.

Since p is upper semi-continuous in $\mathbf{C}^n$, Lemma 1.4.6 gives that there exists a countable dense subset S of $\mathbf{C}^n$, such that any upper semi-continuous function in $\mathbf{C}^n$ majorized by p and equal to p in S is identically equal to p. We arrange the elements of S in a sequence $\{z_j\}$ such that every element in S appears infinitely many times in the sequence. Let $\{\varepsilon_j\}$ be a sequence of positive real numbers converging to zero and define the positively homogeneous functions p_j in $\mathbf{C}^n$ by

$$p_j(z) = \begin{cases} \min\{p(z), (p(z_j/|z_j|)-\varepsilon_j)|z|^\varrho\}, & z \in \Gamma_j, \\ p(z), & z \in \mathbf{C}^n \setminus \Gamma_j, \end{cases}$$

where Γ_j denotes the cone generated by the ball $\{z \in \mathbf{C}^n;\ |z-z_j|<\varepsilon_j\}$.

For every measurable function $q\colon \mathbf{C}^n \to \mathbf{R}$ we define the space B_q of all analytic functions f in $\mathbf{C}^n$ satisfying

$$\int |f|^2(1+|z|^2)^{-N}\exp(-2q_M)d\lambda < \infty,$$

where $N=\varrho+3n$ and $q_M(z)=\sup_{w\in M} q(z+w)$. If q is bounded from above in every compact subset of $\mathbf{C}^n$, it follows that B_q is a Banach space with the norm

$$f \mapsto \left(\int |f|^2(1+|z|^2)^{-N}\exp(-2q_M)d\lambda\right)^{1/2}.$$

Since p is upper semi-continuous, B_p and B_{p_j} are Banach spaces. Since $p_j \leqq p$ for all j, the injection $B_{p_j} \to B_p$ is continuous. By Lemma 1.4.5, we have $i_f \leqq q_j$ for all $f \in B_{p_j}$, where q_j denotes the least upper semi-continuous majorant of p_j. Since $q_j(z_j)=p_j(z_j)<p(z_j)$, the first part of the proof gives that $B_{p_j} \neq B_p$. By Banach's theorem, the spaces B_{p_j} are of the first category in B_p, and by Baire's theorem, $B_p \setminus \bigcup B_{p_j}$ is non-empty. For every $f \in B_p \setminus \bigcup B_{p_j}$ we have $i_f=p$. In fact, if $i_f \neq p$, there exists $w \in S$ such that $i_f(w)<p(w)$. Hence there exists $\varepsilon>0$ and a compact neighborhood V of w such that

$$i_f(z) < p(w)-\varepsilon, \quad z \in V.$$

By Hartog's theorem, there exists $t_0>0$ such that

$$t^{-\varrho}\log|f(tz)| < p(w)-\varepsilon/2, \quad z \in V,\ t>t_0.$$

Since w appears infinitely mamy times in the sequence $\{z_j\}$ and $\varepsilon_j \to 0$, there exists k such that $w = z_k$ and

$$t^{-\varrho} \log |f(tz)| < (p(z_k/|z_k|) - \varepsilon_k)|z|^\varrho$$

for all z with $|z - z_k| < \varepsilon_k$ and all $t > t_0$. This gives

$$|f(z)| \leqq C \exp(p_k(z)), \quad z \in \Gamma_k.$$

Since $f \in B_p$ and $p_k = p$ in $\mathbf{C}^n \setminus \Gamma_k$ it follows that $f \in B_{p_k}$. This is a contradiction. Hence $i_f = p$ and the theorem is proved.

Chapter 2. Fourier–Laplace transforms of hyperfunctions with compact support.

2.0 Introduction.

Let u be a hyperfunction in $\mathbf{R}^n$ with compact support. Then u can be represented by a unique analytic functional, which we also denote by u, with support contained in $\mathbf{R}^n$. We let K be the convex hull of the support of u and let H be the supporting function of K. In this chapter we are going to study the indicator function $i_{\hat{u}}$ of the Fourier–Laplace transform of u.

In section 2.1 we begin by showing that $i_{\hat{u}}(\zeta) \leqq H(\operatorname{Im} \zeta)$ for $\zeta \in \mathbf{C}^n$ with equality in $\mathbf{CR}^n$, the set of all complex vectors with proportional real and imaginary parts. This is a generalization of a theorem of Plancherel and Pólya [18]. We give two applications of the results of sections 1.2 and 1.3 to Fourier–Laplace transforms of hyperfunctions. The first result is a construction of a hyperfunction u with prescribed bounds on the limit set of $\log |\hat{u}|$. The second results shows that the Titchmarsh–Lions theorem of supports does not hold for hyperfunctions. It states that if K_1, K_2 and K_3 are compact convex subsets of $\mathbf{R}^n$ with $K_3 \subset K_1 + K_2$ and there exists a compact metric space A and surjective continuous functions $\varphi_1 : A \to K_1$ and $\varphi_2 : A \to K_2$ such that

$$K_3 = \operatorname{ch}\{\varphi_1(a) + \varphi_2(a) \; ; \; a \in A\},$$

then there exist hyperfunctions u_1 and u_2 with $K_1 = \operatorname{ch} \operatorname{supp} u_1$, $K_2 = \operatorname{ch} \operatorname{supp} u_2$, and $K_3 = \operatorname{ch} \operatorname{supp} u_1 * u_2$. This result is a generalization of a construction, given by Pólya [19, § 35], of an entire analytic function F in $\mathbf{C}$ of exponential type with indicator diagram equal to a prescribed compact convex set and the indicator diagram of $F(z)F(-z)$ equal to $\{0\}$.

The main result of the chapter is a description of the subset of all ζ in $\mathbf{C}^n$ where $i_{\hat{u}}(\zeta) = H(\operatorname{Im} \zeta)$ in terms of the analytic singularities of u at the supporting planes of K. As a preparation for that we prove in section 2.2 a Paley–Wiener type theorem for analytic singularities. It is due to Hörmander. It allows us to determine the convex hull of the analytic singular support of certain distributions in $\mathbf{R}^n$, that are analytic and exponentially decreasing in a conic neighborhood of infinity, by means of growth estimates of their Fourier–Laplace transforms.

Finally in section 2.3 we prove that if $\zeta \in \mathbf{C}^n$ with $\operatorname{Re}\zeta \neq 0$ and $\operatorname{Im}\zeta \neq 0$, then $i_{\hat{u}}(\zeta) = H(\operatorname{Im}\zeta)$ if and only if $(x, \operatorname{Re}\zeta) \in \mathrm{WF}_A(u)$ for some $x \in \partial K$ with $\langle x, \operatorname{Im}\zeta\rangle = H(\operatorname{Im}\zeta)$.

2.1. The Paley–Wiener theorem and the indicator function.

Let u be an analytic functional carried by a compact subset of $\mathbf{R}^n$. Then the support of u is well defined and it is the smallest compact subset of $\mathbf{R}^n$ which carries u. Let K be the convex hull of $\operatorname{supp} u$ and let H denote its supporting function. The Fourier–Laplace transform $\hat{u}$ of u is defined by $\hat{u}(\zeta) = u(\exp(-i\langle .,\zeta\rangle))$ for $\zeta \in \mathbf{C}^n$. It is an analytic function and for every $\varepsilon > 0$, there exists a positive constant C_ε such that

$$|\hat{u}(\zeta)| \leqq C_\varepsilon \exp\big(H(\operatorname{Im}) + \varepsilon|\zeta|\big), \quad \zeta \in \mathbf{C}^n . \tag{2.1.1}$$

We have an analogue of the Paley–Wiener theorem for analytic functionals. In fact, every analytic function in $\mathbf{C}^n$ which satisfies a growth estimate of the form (2.1.1) is the Fourier–Laplace transform of a unique analytic functional with support contained in K. (For a proof see Hörmander [11, Theorem 15.1.5].)

In the rest of this paper we are going to deal with analytic and plurisubharmonic functions of order one and of finite type, so we assume that $\varrho = 1$ in the definition of T_t, that is $T_t q(\zeta) = t^{-1} q(t\zeta)$ for $\zeta \in \mathbf{C}^n$. Since H is positively homogeneous of order one, every function p in the limit set of $\log|\hat{u}|$ satisfies

$$p(\zeta) \leqq H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}^n . \tag{2.1.2}$$

THEOREM 2.1.1. *Let u be an analytic functional carried by some compact subset of* $\mathbf{R}^n$. *Then* $\mathbf{R}^n \ni \eta \mapsto i_{\hat{u}}(i\eta)$ *is the supporting function of* $\operatorname{supp} u$.

PROOF. Set $K = \operatorname{ch} \operatorname{supp} u$ and let H be the supporting function of K. Then $i_{\hat{u}}(\zeta) \leqq H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n$. We define H' by $H'(\eta) = i_{\hat{u}}(i\eta)$ for $\eta \in \mathbf{R}^n$. By Proposition 1.1.4, H' is the supporting function of a compact convex subset K' of $\mathbf{R}^n$ and

$$i_{\hat{u}}(\zeta) \leqq H'(\operatorname{Im}\zeta) \leqq H(\operatorname{Im}\zeta) \quad \text{for all } \zeta \in \mathbf{C}^n.$$

Hence $K' \subset K$. Hartogs' theorem and the Paley–Wiener theorem now give $K = \mathrm{ch}\,\mathrm{supp}\, u \subset K'$. Hence $H' = H$ and the proof is complete.

We define P_H as the set of all plurisubharmonic functions p in $\mathbf{C}^n$ satisfying (2.1.2) and

$$p(\zeta) = H(\mathrm{Im}\,\zeta), \quad \zeta \in \mathbf{CR}^n . \tag{2.1.3}$$

The set $\mathbf{CR}^n = \{\zeta \in \mathbf{C}^n;\, \zeta = z\eta,\ z \in \mathbf{C},\ \eta \in \mathbf{R}^n\}$ consists of all complex vectors with proportional real and imaginary parts.

Let X be an open bounded subset of $\mathbf{R}^n$. Then the space of hyperfunctions $B(X)$ is defined as $B(X) = A'(\bar{X})/A'(\partial X)$, where $A'(Y)$ denotes the space of all analytic functionals with support in a compact subset Y of $\mathbf{R}^n$. The support of the class $u^\cdot \in B(X)$ of $u \in A'(X)$ is well defined as

$$\mathrm{supp}\, u^\cdot = X \cap \mathrm{supp}\, u .$$

If $\mathrm{supp}\, u$ is compact, then there exists a unique analytic functional v with support contained in X such that $u^\cdot = v^\cdot$. Hence the subspace of $B(X)$ consisting of all hyperfunctions with compact support can be identified with the space of all analytic functionals with support contained in X.

Proposition 1.1.4 and Theorem 2.1.1 now give:

THEOREM 2.1.2. *Let K be a compact convex subset of $\mathbf{R}^n$ and let H be its supporting function. Let u be a hyperfunction with* $\mathrm{ch}\,\mathrm{supp}\, u = K$. *Then $i_{\hat{u}} \in P_H$.*

In Chapter 3 we shall see that if u is a distribution with $\mathrm{ch}\,\mathrm{supp}\, u = K$, then the limit set of $\log|\hat{u}|$ is contained in P_H. The following theorem, which is a direct consequence of Theorem 1.2.1, Theorem 1.3.1, and the Paley–Wiener theorem, shows that hyperfunctions do not have this property in general:

THEOREM 2.1.3. *Let σ be a positive real number. Let M be a set of plurisubharmonic functions q in $\mathbf{C}^n$ with $q(0) = 0$ and $q(\zeta) \leqq \sigma|\mathrm{Im}\,\zeta|$, and suppose that M is compact and invariant under T_t for all $t > 0$.*

i) *There exists a hyperfunction u with compact support such that $M \subset L(\log|\hat{u}|) \subset N$, where*

$$N = \{\vartheta q_1 + (1-\vartheta) q_2 \ ;\ \vartheta \in [0,1],\ q_1, q_2 \in M\}$$

is the union of all line segments with endpoints in M.

ii) *If M is connected and all its elements are positively homogeneous of order one, then u can be chosen so that $M = L(\log|\hat{u}|)$.*

The Titchmarsh–Lions theorem of supports states that if u_1 and u_2 are distributions with compact support, then $\operatorname{ch}\operatorname{supp} u_1 * u_2 = \operatorname{ch}\operatorname{supp} u_1 + \operatorname{ch}\operatorname{supp} u_2$. The next proposition shows that hyperfunctions do not have this property in general:

PROPOSITION 2.1.4. *Let K_1, K_2, and K_3 be compact convex subsets of R^n with $K_3 \subset K_1 + K_2$. Suppose that A is a compact metric space and $\varphi_1 : A \to K_1$ and $\varphi_2 : A \to K_2$ are surjective continuous functions such that*

$$K_3 = \operatorname{ch}\{\varphi_1(a) + \varphi_2(a);\ a \in A\}.$$

Then there exist hyperfunctions u_1 and u_2 such that $\operatorname{ch}\operatorname{supp} u_1 = K_1$, $\operatorname{ch}\operatorname{supp} u_2 = K_2$, *and* $\operatorname{ch}\operatorname{supp} u_1 * u_2 = K_3$.

PROOF. Let H_j denote the supporting function of K_j for $j = 1, 2, 3$. For every $x \in \mathsf{R}^n$ we define the functional q_x on C^n by $q_x(\zeta) = \langle x, \operatorname{Im}\zeta\rangle$ for $\zeta \in \mathsf{C}^n$ and set

$$M = \{(q_{\varphi_1(a)}, q_{\varphi_2(a)});\ a \in A\} \subset (L^1_{\mathrm{loc}}(\mathsf{C}^n)^2).$$

Then M satisfies the conditions in Theorem 1.2.7 ii), so there exist plurisubharmonic functions p_1 and p_2 with $L(p_1, p_2) = M$. By Theorem 1.3.1 the functions p_1 and p_2 can be chosen of the form $p_1 = \log|f_1|$ and $p_2 = \log|f_2|$, where f_1 and f_2 are analytic functions in C^n. Since φ_j is surjective, Proposition 1.2.6 gives

$$L(\log|f_j|) = \{q_x\ ;\ x \in K_j\},$$

and Proposition 1.1.2 gives

$$i_{f_j}(\zeta) = \sup_{x \in K_j} q_x(\zeta) = \sup_{x \in K_j} \langle x, \operatorname{Im}\zeta\rangle = H_j(\operatorname{Im}\zeta) \quad \text{for } j = 1, 2.$$

Hartogs' theorem, the Paley–Wiener theorem, and Theorem 2.1.1 give that f_j is the Fourier–Laplace transform of a hyperfunction u_j with $\operatorname{ch}\operatorname{supp} u = K_j$ for $j = 1, 2$.

Set $u_3 = u_1 * u_2$. Then $\hat{u}_3 = \hat{u}_1 \hat{u}_2$, so Proposition 1.2.6 gives

$$L(\log|\hat{u}_3|) = L(\log|\hat{u}_1| + \log|\hat{u}_2|) = \{q_{\varphi_1(a)} + q_{\varphi_2(a)}\ ;\ a \in A\}.$$

Since $K_3 = \operatorname{ch}\{\varphi_1(a) + \varphi_2(a)\ ;\ a \in A\}$, Proposition 1.1.2 gives

$$i_{\hat{u}_3}(\zeta) = \sup_{a \in A} \langle \varphi_1(a) + \varphi_2(a), \operatorname{Im}\zeta\rangle = H_3(\operatorname{Im}\zeta).$$

Now Hartogs' theorem, the Paley–Wiener theorem, and Theorem 2.1.1 give $\operatorname{ch}\operatorname{supp} u_3 = K_3$. The proof is complete.

EXAMPLE. Let K be a compact convex subset of $\mathbf{R}^n$ and set $K_1=K$, $K_2=-K$, and $K_3=\{0\}$. If we choose $A=K$, $\varphi_1(x)=x$, and $\varphi_2(x)=-x$, then the proposition gives that there exist hyperfunctions u_1 and u_2 with $\operatorname{ch}\operatorname{supp} u_1=K$, $\operatorname{ch}\operatorname{supp} u_2=-K$, and $\operatorname{ch}\operatorname{supp} u_1 * u_2=\{0\}$.

EXAMPLE. Let $g: [a,b]\to[c,d]$ be a surjective continuous function. Set

$$K_1 = \{(t,0);\, t\in[a,b]\} \quad\text{and}\quad K_2 = \{(0,t);\, t\in[c,d]\}.$$

We set $A=[a,b]$, and $\varphi_1(t)=(t,0)$, $\varphi_2(t)=(0,g(t))$ for $t\in A$. Then the proposition gives that there exist hyperfunctions u_1 and u_2 with $\operatorname{ch}\operatorname{supp} u_1=K_1$, $\operatorname{ch}\operatorname{supp} u_2=K_2$, and $\operatorname{ch}\operatorname{supp} u_1 * u_2$ equal to the convex hull of the graph of g.

2.2. A Paley–Wiener type theorem for analytic singularities.

A well known theorem essentially due to Ehrenpreis (see Hörmander [11, Theorem 7.3.8]) allows one to determine the convex hull of the singular support of a distribution u with compact support by means of estimates of the Fourier–Laplace transform. We shall here prove an analogous result due to Hörmander for analytic singularities. Obviously we cannot assume u to have compact support then. Instead we shall assume analyticity and exponential decrease in an angular neighborhood of $\mathbf{R}^n$.

THEOREM 2.2.1. *Let Ω be an open convex subset of $\mathbf{R}^n$ with 0 on the boundary, let Γ be the convex cone generated by Ω, and let Γ° be the dual cone of Γ defined by*

$$\Gamma^\circ = \{\eta\in\mathbf{R}^n\,;\ \langle x,\eta\rangle \geqq 0,\ x\in\Gamma\}.$$

Suppose that for some compact convex subset K of $\mathbf{R}^n$

i) *u is analytic in $\mathbf{R}^n+i\Omega$.*
ii) *u is analytic in $\mathbf{R}^n\setminus K$.*
iii) *u is analytic in*

$$Z_\alpha = \{z\in\mathbf{C}^n\,;\ |\operatorname{Re} z| > \alpha_1|\operatorname{Im} z|,\ |\operatorname{Re} z|>\alpha_0\},$$

for some positive constants α_0 and α_1, and

$$|u(z)| \leqq C\exp(-\delta|\operatorname{Re} z|),\quad z\in Z_\alpha, \tag{2.2.1}$$

for some $\delta>0$.

Then the Fourier transform $\hat{u}$ of u is well defined by the formula

$$\hat{u}(\xi) = \int e^{-i\langle x+i\omega,\xi\rangle}u(x+i\omega)\,dx, \quad \xi \in \mathsf{R}^n,\ \omega \in \Omega\,, \tag{2.2.2}$$

thus independent of ω. *We have*

iv) $\hat{u}$ *can be continued to an analytic function in*

$$W_\beta = \{\zeta \in \mathsf{C}^n\ ;\ |\mathrm{Im}\,\zeta| < \beta_1|\mathrm{Re}\,\zeta|+\beta_0\}$$

for some positive constants β_0 *and* β_1.

v) *If* X *is a compact convex neighborhood of* K *and* H_X *denotes its supporting function, then there exists a positive constant* $\gamma_X \leqq \beta_1$ *such that for every* $\varepsilon > 0$

$$|\hat{u}(\zeta)| \leqq C_\varepsilon \exp\big(H_X(\mathrm{Im}\,\zeta)+\varepsilon|\mathrm{Re}\,\zeta|\big)\,, \tag{2.2.3}$$

for all $\zeta \in \mathsf{C}^n$ *with* $|\mathrm{Im}\,\zeta| \leqq \gamma_X|\mathrm{Re}\,\zeta|$.

vi) $\hat{u}$ *is exponentially decreasing in a conic neighborhood of* $\mathsf{R}^n \setminus \Gamma^\circ$ *in* C^n.

Conversely, let K *be a compact convex subset of* R^n, *let* Γ *be an open convex cone with* $0 \notin \Gamma$, *and suppose that* U *is a function satisfying* iv), v), *and* vi). *Then there exists an open convex set* Ω *in* R^n *generating* Γ *and a unique function* u *satisfying* i), ii), *and* iii) *for some positive constants* α_0, α_1, *and* δ *such that* $\hat{u} = U$.

PROOF. By iii) the integral (2.2.2) is convergent, and it is independent of ω. In fact, if $\tilde{\omega} \in \Omega$ then $\tau\omega+(1-\tau)\tilde{\omega} \in \Omega$ when $0 \leqq \tau \leqq 1$, and it follows from Cauchy's integral formula that the integral with ω replaced by $\tau\omega+(1-\tau)\tilde{\omega}$ is independent of τ.

In order to extend $\hat{u}$ analytically we must make further deformations of the integration contour in (2.2.2). We first observe that if $z = x+iy \in Z_\alpha$ and $\zeta \in \mathsf{C}^n$, then

$$|e^{-i\langle z,\zeta\rangle}u(z)| \leqq C\exp(\langle x,\mathrm{Im}\zeta\rangle+\langle y,\mathrm{Re}\zeta\rangle-\delta|x|)\,. \tag{2.2.4}$$

Let d be a non-negative C^1 function in R^n such that for *every* $x \in \mathsf{R}^n$, the function u is analytic at $x+iy$ when $|y| < d(x)$. By iii) and ii) we can choose d so that

$$d(x) = |x|/\alpha_1 \quad \text{when } |x| > \alpha_0+1;\ d(x) > 0 \text{ in } \complement K\,.$$

We choose an open set Y with $K \subset \bar{Y} \subset X$ and a bounded function φ with $0 \leqq \varphi \in C^\infty(\mathsf{R}^n)$, $\varphi = 0$ in Y, and $\varphi > 0$ in $\mathsf{R}^n \setminus \bar{Y}$. If $\varphi(x)+|\omega| < d(x)$ for $x \in \complement\bar{Y}$, then Stokes' formula gives

$$(2.2.5) \qquad \hat{u}(\xi) = \int_{\Sigma(\omega,\vartheta,\varphi)} \exp(-i\langle z,\xi\rangle)u(z)\,dz_1 \wedge \ldots \wedge dz_n, \qquad \xi \in \mathbf{R}^n ,$$

if $\omega \in \Omega$, $\vartheta \in \mathbf{R}^n$ is a unit vector with $\langle\vartheta,\xi\rangle < 0$, and $\Sigma(\omega,\vartheta,\varphi)$ is the chain defined by

$$\mathbf{R}^n \ni x \;\mapsto\; z = x+i(\omega+\varphi(x)\vartheta) \in \mathbf{C}^n .$$

Taking a sequence of such functions φ, we find that φ may be taken equal to $|x|/\alpha_1-|\omega|$ when $|x|>\alpha_0+1$. When $z \in \Sigma(\omega,\vartheta,\varphi)$ and $x=\operatorname{Re} z$, then (2.2.4) gives that $|e^{-i\langle z,\zeta\rangle}u(z)|$ can be estimated by

$$(2.2.6) \qquad C\exp(\langle x,\operatorname{Im}\zeta\rangle+\varphi(x)\langle\vartheta,\operatorname{Re}\zeta\rangle+\langle\omega,\operatorname{Re}\zeta\rangle-\delta|x|) ,$$

and it follows that (2.2.5) defines an analytic extension of $\hat{u}$ to

$$V_\vartheta = \{\zeta \in \mathbf{C}^n \; ; \; |\operatorname{Im}\zeta| < -\langle\vartheta,\operatorname{Re}\zeta\rangle/\alpha_1+\delta\} .$$

If we take $\beta_1 \leqq 1/\alpha_1$ and $\beta_0 \leqq \delta$, then iv) holds. In fact, ϑ is an arbitrary unit vector and the various analytic continuations agree in their common domain of definition since they agree in $\mathbf{R}^n$ and the domains are concentric balls for fixed $\operatorname{Re}\zeta$.

If ϑ is proportional to $-\operatorname{Re}\zeta$, then $\langle\vartheta,\operatorname{Re}\zeta\rangle = -|\operatorname{Re}\zeta|$. Since $\varphi>0$ in $\mathbf{R}^n\setminus\bar{Y}$ and $\varphi(x)=|x|/\alpha_1-|\omega|$ for $|x|>\alpha_0+1$, there exists a positive constant κ_X such that $\varphi(x)\geqq\kappa_X(|x|+1)$ for all $x \notin X$. If $\gamma_X\leqq\kappa_X$ is chosen such that $(-H_X(\eta))\gamma_X\leqq\kappa_X$ for all unit vectors η, then

$$(2.2.7) \qquad \langle x,\operatorname{Im}\zeta\rangle-H_X(\operatorname{Im}\zeta)-\varphi(x)|\operatorname{Re}\zeta| \leqq 0$$

for $x \notin X$ and $|\operatorname{Im}\zeta|<\gamma_X|\operatorname{Re}\zeta|$. When $x \in X$ we have $\langle x,\operatorname{Im}\zeta\rangle-H_X(\operatorname{Im}\zeta) \leqq 0$. If ω is chosen such that $|\omega|\leqq\varepsilon$, then the estimate v) follows from (2.2.6).

When $\vartheta \in \Omega$ and $|\vartheta|<1$ we can choose $\omega=0$ and φ so that $1\leqq\varphi(x) \leqq d(x)+1$, $\varphi(x)=1$ if $|x|\leqq\alpha_0$, and $\varphi(x)=d(x)$ if $|x|\geqq\alpha_0+1$. When $z \in \Sigma(0,\vartheta,\varphi)$ with $\operatorname{Re} z=x$ we obtain

$$|e^{-i\langle z,\zeta\rangle}u(z)| \leqq C\exp(\langle x,\operatorname{Im}\zeta\rangle+\varphi(x)\langle\vartheta,\operatorname{Re}\zeta\rangle-\delta|x|) .$$

Hence $\hat{u}$ is exponentially decreasing in

$$\{\zeta \in \mathbf{C}^n \; ; \; \langle x,\operatorname{Im}\zeta\rangle+\varphi(x)\langle\vartheta,\operatorname{Re}\zeta\rangle < 0 \text{ when } |x|\leqq\alpha_0+1\} .$$

The union when ϑ varies is a conic neighborhood of $\mathbf{R}^n\setminus\Gamma^\circ$ which proves vi).

Suppose now that U is a function satisfying iv), v), and vi). Let Ω_L be the set of all $y \in \mathbf{R}^n$ such that

$$\int |U(\xi)| e^{-\langle y,\xi\rangle} d\xi < \infty .$$

This is a convex set, for the exponential function is convex, and if $y \in \Gamma$, then $\varepsilon y \in \Omega_L$ if $\varepsilon > 0$ is sufficiently small. In fact, we can find a conic neighborhood V of Γ° such that $\langle y, \xi\rangle > c|y||\xi|$ for some $c > 0$ when $\xi \in V$. Then $U(\xi)$ is exponentially decreasing outside V, so $|U(\xi)|e^{-\langle \varepsilon y, \xi\rangle}$ is integrable in $\complement V$ if ε is small enough. It follows from v) that we have integrability in V for every $\varepsilon > 0$, which proves the claim. Hence the interior Ω of $\Omega_L \cap \Gamma$ is an open convex set generating Γ, 0 lies on the boundary of Ω, and

$$u(z) = (2\pi)^{-n} \int e^{i\langle z,\xi\rangle} U(\xi)\, d\xi \tag{2.2.8}$$

is analytic in $\mathsf{R}^n + i\Omega$. This proves i).

To prove ii) and iii) we have to make deformations of the integration contour in (2.2.8) as in the first part of the proof. Let $x \in \mathsf{R}^n \setminus K$ and let X be a compact convex neighborhood of K with supporting function H_X and $x \notin X$. By v) there exists a positive constant γ_X such that for every $\varepsilon > 0$, $|e^{i\langle z,\zeta\rangle} U(\zeta)|$ can be estimated by

$$C_\varepsilon \exp\bigl(-\langle \operatorname{Im} z, \xi\rangle - \langle \operatorname{Re} z, \eta\rangle + H_X(\eta) + \varepsilon|\xi|\bigr) , \tag{2.2.9}$$

for all $z \in \mathsf{C}^n$ and all $\zeta = \xi + i\eta \in \mathsf{C}^n$ with $|\eta| \leqq \gamma_X |\xi|$. We choose a unit vector $\vartheta \in \mathsf{R}^n$ and a positive constant κ such that $-\langle y, \vartheta\rangle + H_X(\vartheta) < -\kappa$ for all y in a neighborhood Y of x.

Set $\tilde{d}(\xi) = \beta_1|\xi| + \beta_0$. Then U is analytic at $\zeta = \xi + i\eta$ when $|\eta| < \tilde{d}(\xi)$. We choose $0 \leqq \varphi \in C^\infty(\mathsf{R}^n)$ with $\varphi(\xi) < \tilde{d}(\xi)$ for all $\xi \in \mathsf{R}^n$ and $\varphi(\xi) = \gamma_X|\xi|$ if $|\xi| \geqq 1$. We may assume $\gamma_X < \beta_1$. If $\zeta \in \Sigma(0, \vartheta, \varphi)$ with $\operatorname{Re}\zeta = \xi$ and $z \in \mathsf{C}^n$ with $\operatorname{Re} z \in Y$, then (2.2.9) gives that for every $\varepsilon > 0$

$$|e^{i\langle z,\zeta\rangle} U(\zeta)| \leqq C_\varepsilon \exp\bigl(-\langle \operatorname{Im} z, \xi\rangle - \kappa\varphi(\xi) + \varepsilon|\xi|\bigr) . \tag{2.2.10}$$

As in the first part of the proof Stokes' theorem gives

$$u(x) = (2\pi)^{-n} \int_{\Sigma(0,\vartheta,\varphi)} e^{i\langle x,\zeta\rangle} U(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n, \quad x \in Y ,$$

and (2.2.10) gives that u can be extended analytically to a neighborhood of x. This proves ii).

Now we choose $\omega \in \mathsf{R}^n$ with $|\omega| \leqq \beta_0$, $\vartheta \in \mathsf{R}^n$ with $|\vartheta| = 1$, and φ such that $0 \leqq \varphi(\xi) \leqq \beta_1|\xi|$ for $\xi \in \mathsf{R}^n$, and $\varphi(\xi) = \gamma_X|\xi|$ if $|\xi| \geqq 1$. If $\zeta \in \Sigma(\omega, \vartheta, \varphi)$, $\xi = \operatorname{Re}\zeta$, and $z \in \mathsf{C}^n$, then for every $\varepsilon > 0$ there exists a positive constant C_ε such that $|e^{i\langle z,\zeta\rangle} u(\zeta)|$ can be estimated by

$$\text{(2.2.11)}\quad C_\varepsilon \exp\bigl(-\langle \operatorname{Im} z,\xi\rangle - \langle \operatorname{Re} z,\omega\rangle - \varphi(\xi)[\langle \operatorname{Re} z,\vartheta\rangle - H_X(\vartheta)] + H_X(\omega) + \varepsilon|\xi|\bigr)$$

and it follows that u can be continued analytically to

$$\{z \in \mathbf{C}^n;\ |\operatorname{Im} z|/\gamma_X \ <\ \langle \operatorname{Re} z,\vartheta\rangle - H_X(\vartheta)\}.$$

The union over ϑ of these sets contains $\{z \in \mathbf{C}^n;\ |\operatorname{Im} z|/\gamma_X < |\operatorname{Re} z| - c\}$, where c is the maximum of H_X on the unit sphere. If $\alpha_1 > 1/\gamma_X$ and α_0 is large enough, then the set Z_α defined in iii) is contained in this set. If we choose ω and ϑ in the direction of $\operatorname{Re} z$ with $|\omega| = \beta_0$ and $|\vartheta| = 1$, then (2.2.1) follows from (2.2.11) with $\delta \leqq \beta_0$. This completes the proof.

REMARK. In the proof of iv) we have seen that the best choice of the constants β_0 and β_1 is $\beta_0 = \delta$ and $\beta_1 = 1/\alpha_1$. If the set X in v) contains the ball $\{x \in \mathbf{R}^n;\ |x| \geqq \alpha_0 + 1\}$, then H_X is non-negative, so we can replace the estimate (2.2.7) by

$$\langle x, \operatorname{Im} \zeta\rangle - \varphi(x)|\operatorname{Re} \zeta| \ \leqq\ \langle x, \operatorname{Im} \zeta\rangle - |\operatorname{Re} \zeta||\operatorname{Re} \zeta|$$

for $x \notin X$. This implies that γ_X can be chosen as $\gamma_X = \beta_1$.

If we add to v) the condition that $\gamma_X = \beta_1$ if X is a sufficiently large ball with center at the origin, then the proof of the converse gives that α_1 can be chosen with $\alpha_1 > 1/\beta_1$ and $\delta = \beta_0$.

2.3. The indicator function and the analytic wave front set.

Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H and let u be a hyperfunction with $\operatorname{ch} \operatorname{supp} u = K$. In section 2.1 we have seen that $i_{\hat u} \in P_H$, that is, $i_{\hat u}(\zeta) \leqq H(\operatorname{Im} \zeta)$ for all $\zeta \in \mathbf{C}^n$ with equality in $\mathbf{CR}^n$, the set of all complex vectors with proportional real and imaginary parts. The following theorem describes completely the subset of $\zeta \in \mathbf{C}^n$ where $i_{\hat u}(\zeta) = H(\operatorname{Im} \zeta)$ in terms of the analytic singularities of u at the supporting planes of K.

THEOREM 2.3.1. *Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H and let u be a hyperfunction with* ch supp $u = K$. *Let $\xi_0 + i\eta_0 \in \mathbf{C}^n$ with $\xi_0 \neq 0$ and $\eta_0 \neq 0$. Then*

$$i_{\hat u}(\xi_0 + i\eta_0) \ =\ H(\eta_0)$$

if and only if $(x, \xi_0) \in \mathrm{WF}_A(u)$ for some $x \in \partial K$ with $\langle x, \eta_0\rangle = H(\eta_0)$.

We observe that if $\xi_0 + i\eta_0 \in \mathbf{CR}^n$ with $\xi_0 \neq 0$, $\eta_0 \neq 0$, and $\langle x, \eta_0\rangle = H(\eta_0)$ for some $x \in \partial K \cap \operatorname{supp} u$, then (x, η_0) lies in the normal set

$N(\operatorname{supp} u)$ of the support of u. We have $N(\operatorname{supp} u) \subset \mathrm{WF}_A(u)$. Since ξ_0 and η_0 are proportional, this implies that $(x, \xi_0) \in \mathrm{WF}_A(u)$, so the theorem gives again that $i_{\hat{u}}(\xi_0 + i\eta_0) = H(\eta_0)$. (See Hörmander [11, Definition 8.5.7, Theorem 8.5.6, and the last part of section 9.3].)

Now we state some facts that we need in the proof. We are going to use the method developed in sections 8.4 and 9.3 of Hörmander [11] for determining the analytic singularities of hyperfunctions. Let u be a hyperfunction with $\operatorname{ch} \operatorname{supp} u = K$ and let U be the analytic function in

$$Z = \{z \in \mathbf{C}^n \ ; \ |\operatorname{Im} z|^2 < 1 + |\operatorname{Re} z - t|^2, \ t \in \operatorname{supp} u\}$$

defined by $U(z) = \kappa * u(z) = u(\kappa(z - \,.\,))$, where κ is the analytic function in

$$W = \{z \in \mathbf{C}^n \ ; \ |\operatorname{Im} z|^2 < 1 + |\operatorname{Re} z|^2\}$$

satisfying

$$\kappa(z) = (2\pi)^{-n} \int e^{i\langle z, \xi\rangle} / I(\xi)\, d\xi, \quad z \in W,$$

and

$$I(\zeta) = \int_{|\omega| = 1} e^{-\langle \omega, \zeta\rangle}\, d\omega, \quad \zeta \in \mathbf{C}^n .$$

For every closed cone $\Gamma \subset W$ there exist positive constants C and c such that

$$\text{(2.3.1)} \qquad |\kappa(z)| \leqq Ce^{-c|z|}, \quad z \in \Gamma .$$

The function I can be written as $I(\zeta) = I_0(\langle \zeta, \zeta\rangle^{1/2})$, where I_0 is an even entire function of one variable. For every $\alpha > 0$

$$\text{(2.3.2)} \qquad I(\zeta) = (2\pi)^{(n-1)/2} e^{\langle \zeta, \zeta\rangle^{1/2}} \langle \zeta, \zeta\rangle^{-(n-1)/4} (1 + 0(|\zeta|^{-1}))$$

as $|\zeta| \to \infty$ and $|\arg(\langle \zeta, \zeta\rangle^{1/2})| \leqq \pi/2 - \alpha$. Here the square root is taken positive on the positive real axis.

For $\omega \in S^{n-1}$ we set $U_\omega(z) = U(z + i\omega)$. Then U_ω is analytic in $\mathbf{R}^n + i\Omega_\omega$, where

$$\Omega_\omega = \{y \in \mathbf{R}^n \ ; \ |y + \omega| < 1\} .$$

The set Ω_ω is open, convex, and has 0 on its boundary. The function U is analytic at $x + i\omega$ if and only if $(x, -\omega) \notin \mathrm{WF}_A(u)$, so U_ω is analytic in $\mathbf{R}^n \setminus K_\omega$, where

$$K_\omega = \operatorname{ch}\{x \in K \ ; \ (x, -\omega) \in \mathrm{WF}_A(u)\} .$$

Hence U_ω satisfies the conditions i) and ii) in Theorem 2.2.1, with Ω replaced by Ω_ω and K by K_ω. The condition iii) is also satisfied. In fact, u is an analytic functional carried by K, so

$$|U_\omega(z)| \leqq C_X \sup_{t\in X} |\kappa(z+i\omega-t)|, \quad z \in (\{-i\omega\}+Z) ,$$

where X is a compact neighborhood of K. If α_0 and α_1 are sufficiently large, then $Z_\alpha \subset (\{-i\omega\}+Z) \cap W$ and (2.2.1) follows from (2.3.1).

The zeroes of the function I_0 have absolute value >1 and are located on the imaginary axis. Hence the function $\zeta \mapsto 1/I(\zeta)$ is analytic at every ζ with $\langle\zeta,\zeta\rangle \notin (-\infty, -1]$. We have

$$\hat{U}_\omega(\zeta) = \hat{u}(\zeta)e^{-\langle\omega,\zeta\rangle}/I(\zeta) \tag{2.3.3}$$

for $\zeta \in \mathbf{C}^n$ with $\langle\zeta,\zeta\rangle \notin (-\infty, -1]$. (A proof of these facts can be found in sections 8.4 and 9.3 of Hörmander [11].)

The main work in the proof of Theorem 2.3.1 is done in:

LEMMA 2.3.2. *Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H and let u be a hyperfunction with* ch supp $u=K$. *Let ξ_0 and η_0 be two non-zero vectors in $\mathbf{R}^n$ and set $\omega = -\xi_0/|\xi_0|$. If $i_{\hat{u}}(\xi_0+i\eta_0)<H(\eta_0)$, then there exists a compact convex subset K' of the half space $\{x\in \mathbf{R}^n;\ \langle x,\eta_0\rangle < H(\eta_0)\}$ with supporting function H', and a positive constant γ such that for every $\varepsilon>0$*

$$|\hat{U}_\omega(\zeta)| \leqq C_\varepsilon \exp\big(H'(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Re}\zeta|\big) \tag{2.3.4}$$

for all $\zeta \in \mathbf{C}^n$ with $|\operatorname{Im}\zeta| \leqq \gamma|\operatorname{Re}\zeta|$.

PROOF. Since $i_{\hat{u}}$ is upper semi-continuous, there exist neighborhoods V_1 and V_2 of ξ_0 and η_0 respectively, and positive constants β and δ such that

$$i_{\hat{u}}(\xi+w\eta)-\operatorname{Im} wH(\eta) < -\beta$$

for $\xi \in V_1, \eta \in V_2$, and all w in the upper half plane satisfying $|w-i|/|w+i| \leqq \delta$. Since the function $w \mapsto (w-i)/(w+i)$ maps the upper half plane onto the unit disk and $i_{\hat{u}}(\xi+w\eta)-\operatorname{Im} wH(\eta) \leqq 0$, Hadamard's three circles theorem gives

$$i_{\hat{u}}(\xi+w\eta)-\operatorname{Im} wH(\eta) \leqq -(\beta/\log\delta)\log(|w-i|/|w+i|) ,$$

for $\xi \in V_1, \eta \in V_2$, and all w in the upper half plane with $|w-i|/|w+i| \geqq \delta$. From the estimate

$$\log(|w-i|^2/|w+i|^2) \leqq (|w-i|^2-|w+i|^2)/|w+i|^2 ,$$

which follows from $1+x \leqq e^x$ where x is the right hand side, we conclude that there exists a positive constant σ such that

$$(2.3.5) \qquad i_{\hat{u}}(\xi+w\eta) \leqq \operatorname{Im} wH(\eta)-\sigma \operatorname{Im} w/(1+|w|^2)$$

for $\xi \in V_1, \eta \in V_2$, and w in the upper half plane. We choose closed conic neighborhoods Γ_1 and Γ_2 of ξ_0 and η_0, and positive constants τ_1 and τ_2 such that $\xi/\tau_1|\xi| \in V_1$ for $\xi \in \Gamma_1 \setminus \{0\}$ and $\eta/\tau_2|\eta| \in V$ for $\eta \in \Gamma_2 \setminus \{0\}$. We apply (2.3.5) with ξ replaced by $\xi/\tau_1|\xi|$ and η replaced by $\eta/\tau_2|\eta|$. If we choose $w=i(\tau_2|\eta|/\tau_1|\xi|)$, then the homogeneity gives that there exists $c>0$ such that

$$(2.3.6) \qquad i_{\hat{u}}(\zeta) \leqq H(\operatorname{Im}\zeta)-c(|\operatorname{Im}\zeta||\operatorname{Re}\zeta|^2/|\zeta|^2)$$

for all $\zeta \in \mathbf{C}^n \setminus \{0\}$ with $\operatorname{Re}\zeta \in \Gamma_1$ and $\operatorname{Im}\zeta \in \Gamma_2$. Now (2.3.3), (2.3.2), (2.3.6), and Hartogs' theorem give

$$(2.3.7)\ |\hat{U}_\omega(\zeta)| \leqq C_\varepsilon \exp\big(H(\operatorname{Im}\zeta)-\langle\omega,\operatorname{Re}\zeta\rangle-\operatorname{Re}(\langle\zeta,\zeta\rangle^{1/2})+\varepsilon|\zeta|\big)$$

for all $\zeta \in \mathbf{C}^n$ with $|\arg(\langle\zeta,\zeta\rangle^{1/2})| \leqq \pi/4$, and

$$(2.3.8) \qquad |\hat{U}_\omega(\zeta)| \leqq C_\varepsilon \exp\big(H(\operatorname{Im}\zeta)-c(|\operatorname{Im}\zeta||\operatorname{Re}\zeta|^2/|\zeta|^2)- \\ -\langle\omega,\operatorname{Re}\zeta\rangle-\operatorname{Re}(\langle\zeta,\zeta\rangle^{1/2})+\varepsilon|\zeta|\big)\ ,$$

if ζ also satisfies $\operatorname{Re}\zeta \in \Gamma_1$ and $\operatorname{Im}\zeta \in \Gamma_2$. If $x+iy=\langle\zeta,\zeta\rangle^{1/2}$, then $x^2-y^2 =|\xi|^2-|\eta|^2$ so if $|\eta|<|\xi|$, then $-x \leqq -(|\xi|^2-|\eta|^2)^{1/2}=-|\xi|+0(|\eta|^2/|\xi|)$. We have that $-\langle\omega,\xi\rangle \leqq |\xi|$ for all $\xi \in \mathbf{R}^n$, so (2.3.7) gives that there exist positive constants α and γ such that

$$(2.3.9) \qquad |\hat{U}_\omega(\zeta)| \leqq C_\varepsilon \exp\big(H(\operatorname{Im}\zeta)+\alpha|\operatorname{Im}\zeta|+\varepsilon|\operatorname{Re}\zeta|\big)$$

for all $\zeta \in \mathbf{C}^n$ with $|\operatorname{Im}\zeta|<\gamma|\operatorname{Re}\zeta|$. Since $\omega=-\xi_0/|\xi_0|$ and Γ_1 is a neighborhood of ξ_0, there exists a positive constant $a<1$ such that $-\langle\omega,\xi\rangle<a|\xi|$ for $\xi \in \mathbf{R}^n \setminus \Gamma_1$, so (2.3.7) and (2.3.8) give that $\tau>0$ and γ can be chosen such that

$$(2.3.10) \qquad |\hat{U}_\omega(\zeta)| \leqq C_\varepsilon \exp\big(H(\operatorname{Im}\zeta)-\tau|\operatorname{Im}\zeta|+\varepsilon|\operatorname{Re}\zeta|\big)$$

for all $\zeta \in \mathbf{C}^n$ with $\operatorname{Im}\zeta \in \Gamma_2$ and $|\operatorname{Im}\zeta| \leqq \gamma|\operatorname{Re}\zeta|$. By composing u with a translation followed by a linear map, we can suppose that $\eta_0=(1,0,\ldots,0)$ and $H(\eta_0)=0$. Furthermore we can suppose that $\Gamma_2=\{(\eta_1,\eta');\ |\eta'| \leqq c\eta_1\}$

for some $c>0$. If $0<a<b$ and H' is the supporting function of the set $K'=[-b,-a]\times\{x'\in \mathbf{R}^{n-1};\ |x'|\leqq b\}$, then

$$H'(\eta) = \begin{cases} -a\eta_1+b|\eta'|, & \eta_1\geqq 0\,, \\ -b\eta_1+b|\eta'|, & \eta_1<0\,. \end{cases}$$

If a is sufficiently small and b is sufficiently large, it follows that $H'(\eta)\geqq H(\eta)-\tau|\eta|$ for all $\eta\in\Gamma_2$ and $H'(\eta)\geqq H(\eta)+\alpha|\eta|$ for all $\eta\in \mathbf{R}^n\setminus\Gamma_2$. The inequality (2.3.4) now follows from (2.3.9) amd (2.3.10). The proof is complete.

PROOF OF THEOREM 2.3.1. Set $L=\{x\in K;\ \langle x,\eta_0\rangle=H(\eta_0)\}$ and $\omega=-\xi_0/|\xi_0|$. Then the theorem is equivalent to the following statement:

$$i_{\hat u}(\xi_0+i\eta_0) < H(\eta_0) \Leftrightarrow L\times\{\xi_0\}\cap \mathrm{WF}_A(u) = \varnothing\,.$$

Suppose first that $i_{\hat u}(\xi_0+i\eta_0)<H(\eta_0)$. Then U_ω satisfies the conditions in Lemma 2.3.2. Since U_ω also satisfies the conditions in Theorem 2.2.1, it follows that U_ω is analytic in $\mathbf{R}^n\setminus K'$. Since $L\cap K'=\varnothing$, U is analytic at every point $x+i\omega$ with $x\in L$. Hence $L\times\{\xi_0\}\cap \mathrm{WF}_A(u)=\varnothing$.

Suppose conversely that $L\times\{\xi_0\}\cap \mathrm{WF}_A(u)=\varnothing$. Then U is analytic at $x+i\omega$ for all $x\in L$, so the set $K_\omega=\mathrm{ch}\{x\in K;\ (x,-\omega)\in \mathrm{WF}_A(u)\}$ is contained in the half space $V=\{x\in \mathbf{R}^n;\ \langle x,\eta_0\rangle<H(\eta_0)\}$. If X is a compact convex neighborhood of K_ω contained in V, then $\hat U_\omega$ satisfies (2.2.3). By (2.3.3)

$$\hat u(\zeta) = \hat U_\omega(\zeta)I(\zeta)e^{\langle\omega,\zeta\rangle}$$

so (2.3.2) gives that for every $\varepsilon>0$

$$\text{(2.3.11)}\quad |\hat u(\zeta)| \leqq C_\varepsilon \exp\big(H_X(\mathrm{Im}\,\zeta)+\mathrm{Re}\,(\langle\zeta,\zeta\rangle^{1/2})+\langle\omega,\mathrm{Re}\,\zeta\rangle+\varepsilon|\mathrm{Re}\,\zeta|\big)$$

for all $\zeta\in\mathbf{C}^n$ with $|\mathrm{Im}\,\zeta|\leqq\gamma_X|\mathrm{Re}\,\zeta|$. It is sufficient to show that $i_{\hat u}(\xi_0+i\delta\eta_0)<H(\delta\eta_0)$ for some $\delta>0$, for then the maximum principle applied to the non-positive subharmonic function $w\mapsto i_{\hat u}(\xi_0+w\eta_0)-\mathrm{Im}\,wH(\eta_0)$ in the upper half plane gives that $i_{\hat u}(\xi_0+i\eta_0)<H(\eta_0)$. If δ is chosen so that $\delta|\eta_0|\leqq\gamma_X|\xi_0|$, then (2.3.11) gives

$$i_{\hat u}(\xi_0+i\delta\eta_0) \leqq H_X(\delta\eta_0)+\mathrm{Re}\,(\langle\xi_0+i\delta\eta_0,\xi_0+i\delta\eta_0\rangle^{1/2})-|\xi_0|\,.$$

If $x+iy=(\langle\xi_0+i\delta\eta_0,\xi_0+i\delta\eta_0\rangle^{1/2})$, then

$$(x^2+y^2)^2 = (|\xi_0|^2-\delta^2|\eta_0|^2)^2+4\delta^2\langle\xi_0,\eta_0\rangle^2\,,$$

so $x=|\xi_0|+O(\delta^2)$ as $\delta\to 0$. Since $H_X(\eta_0)<H(\eta_0)$ it follows that $i_{\hat u}(\xi_0+i\delta\eta_0)<H(\delta\eta_0)$ if δ is sufficiently small. The proof is complete.

Chapter 3. Fourier–Laplace transforms of distributions with compact support.

3.0 Introduction.

Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H. Let u be a distribution with $\operatorname{ch}\operatorname{supp} u = K$. In this chapter we study the limit set of $\log|\hat{u}|$ and the indicator function $i_{\hat{u}}$ of the Fourier–Laplace transform $\hat{u}$ of u. If $n=1$, then the theorem of Ahlfors and Heins [1] gives

$$T_t \log|\hat{u}| \to H(\mathrm{Im}\,.) \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{C}) \text{ as } t \to \infty\,.$$

We have no analogue of this result if $n>1$, for Vauthier [22] has constructed a distribution u with support equal to the closed unit ball such that $T_t \log|\hat{u}|$ is not convergent in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$. (See Corollary 3.3.2 below.) However, there is a certain regularity in the growth of $\hat{u}$ near $\mathbf{C}\mathbf{R}^n$ for Hörmander [8] has proved that if $\eta \in \mathbf{R}^n$ and L is a compact subset of $\mathbf{C}$, then

$$\int_L |(T_t \log|\hat{u}|)(z\eta + \zeta/t) - H(\mathrm{Im}\, z\eta)|\, d\lambda(z) \to 0$$

as $t \to \infty$ for almost all $\zeta \in \mathbf{C}^n$ with respect to the Lebesgue measure. Furthermore, Vauthier [21] has proved that the function $(z,\eta) \mapsto (T_t \log|\hat{u}|)(z\eta)$ converges to $(z,\eta) \mapsto H(\mathrm{Im}\, z\eta)$ in $L^1_{\mathrm{loc}}(\mathbf{C} \times S^{n-1}; d\lambda \otimes d\sigma)$ as $t \to \infty$, where $d\lambda$ denotes the Lebesgue measure in $\mathbf{C}$ and $d\sigma$ is the Euclidean area element of S^{n-1}.

In section 3.1 we begin by showing that the set P_H consisting of all plurisubharmonic functions p satisfying $p(\zeta) \leq H(\mathrm{Im}\,\zeta)$ for $\zeta \in \mathbf{C}^n$ with equality in $\mathbf{C}\mathbf{R}^n$ is a compact convex subset of $L^1_{\mathrm{loc}}(\mathbf{C}^n)$, and that it is invariant under T_t for $t>0$. Hörmander's theorem mentioned above gives that the limit set of $\log|\hat{u}|$ is contained in P_H. We give a variant of the indicator theorem, which states that every positively homogeneous plurisubharmonic function p satisfying a growth estimate of the form $p(\zeta) \leq \sigma|\mathrm{Im}\,\zeta|$ for $\zeta \in \mathbf{C}^n$ is the indicator function of the Fourier–Laplace transform of some distribution u with $\operatorname{ch}\operatorname{supp} u = K$. This result was proved by Wiegerinck [24] in the special case when p is Hölder continuous on the unit sphere in $\mathbf{C}^n$. Next we use the maximum principle to describe a subset of $\mathbf{C}^n$ where all the functions in P_H are equal. It enables us to conclude that $T_t \log|\hat{u}| \to H(\mathrm{Im}\,.)$ in L^1_{loc} as $t \to \infty$, when K is a polyhedron. This has also been proved by Wiegerinck [23]. From Theorem 2.3.1 we conclude that $i_{\hat{u}}(\zeta) < H(\mathrm{Im}\,\zeta)$ for $\zeta \notin \mathbf{C}\mathbf{R}^n$, if u is the characteristic function of K and K has a non-empty interior and an analytic boundary, and we prove that if the characteristic function of K has this property and ∂K is a C^1 manifold, then ∂K is analytic. We end the section by proving that if K has a non-empty interior and a C^2 boundary then there exists a function u with $\operatorname{ch}\operatorname{supp} u = K$ and $i_{\hat{u}}(\zeta) < H(\operatorname{Im}\zeta)$ for $\zeta \notin \mathbf{CR}^n$.

As we mentioned above, the limit set of $\log|\hat{u}|$ is contained in P_H if $\operatorname{ch}\operatorname{supp} u = K$. A natural question to ask is if any subset M of P_H, which is compact, connected, and invariant under T_t for $t > 0$, is the limit set of $\log|\hat{u}|$ for some distribution u with $\operatorname{ch}\operatorname{supp} u = K$. We deal with this problem in sections 3.2 and 3.3. We begin section 3.2 by defining an operator which regularizes plurisubharmonic functions in $\mathbf{C}^n \setminus \mathbf{CR}^n$, preserves inequalities of the form

$$p(\zeta) \leqq \log\left(C(1+|\zeta|)^N\right) + c|\operatorname{Im}\zeta|, \quad \zeta \in \mathbf{C}^n, \tag{3.0.1}$$

and maps P_H to another set of the same type. Then we show that if K has a non-empty interior and a C^∞ boundary with strictly positive curvature, then P_H contains a function which is positively homogeneous, infinitely differentiable in $\mathbf{C}^n \setminus \mathbf{R}^n$, and strictly plurisubharmonic in $\mathbf{C}^n \setminus \mathbf{R}^n$. We end the section by constructing a plurisubharmonic function in P_H with limit set consisting of homogeneous functions that are equal outside a closed cone $\Gamma \subset (\mathbf{C}^n \setminus \mathbf{CR}^n) \cup \{0\}$.

In section 3.3 we prove a theorem on the asymptotic approximation of plurisubharmonic functions p by Fourier–Laplace transforms of distributions u with compact support. We assume that the limit set of p is contained in P_H, that p satisfies (3.0.1) and certain regularity conditions in $\mathbf{C}^n \setminus \mathbf{CR}^n$, and we conclude that there exists a distribution u with $\operatorname{ch}\operatorname{supp} u = K$ and $T_t p - T_t \log|\hat{u}| \to 0$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ as $t \to \infty$.

In section 3.4 we prove that if K has a non-empty interior and a C^∞ boundary with strictly positive curvature, then there exists a distribution u with $\operatorname{ch}\operatorname{supp} u = K$ and $i_{\hat{u}}$ discontinuous. The existence of discontinuous indicator functions of several variables was first proved by Lelong [14].

3.1. The indicator function and the analytic wave front set.

Let u be a distribution in $\mathbf{R}^n$ with compact support. Let K be the convex hull of the support of u and let H denote the supporting function of K. Then the Fourier transform $\hat{u}$ of u can be continued to an analytic function in $\mathbf{C}^n$. The continuation is called the Fourier–Laplace transform of u and it is given by $\hat{u}(\zeta) = \langle u, \exp(-i\langle .\,, \zeta\rangle)\rangle$ for $\zeta \in \mathbf{C}^n$. We have

$$|\hat{u}(\zeta)| \leqq C(1+|\zeta|)^N \exp\left(H(\operatorname{Im}\zeta)\right), \quad \zeta \in \mathbf{C}^n, \tag{3.1.1}$$

for some positive constants C and N. Conversely, the Paley–Wiener–Schwartz theorem states that every analytic function in $\mathbf{C}^n$ satisfying a growth estimate of the form (3.1.1) is the Fourier–Laplace transform of a distribution with support contained in K. (For a proof see Hörmander [11, Theorem 7.3.1].) The constant N can be chosen as the order of u. In the special case when u is an integrable function we can choose $N=0$ and $C=\|u\|_{L^1}$.

It is clear from (3.1.1) that every function p in the limit set of $\log|\hat{u}|$ satisfies

$$(3.1.2) \qquad p(\zeta) \leqq H(\operatorname{Im}\zeta) \quad \zeta \in \mathbf{C}^n .$$

As in the previous chapter we let P_H denote the set of all plurisubharmonic functions p in $\mathbf{C}^n$ satisfying (3.1.2) and

$$(3.1.3) \qquad p(\zeta) = H(\operatorname{Im}\zeta) \quad \zeta \in \mathbf{CR}^n .$$

PROPOSITION 3.1.1. *Let H be a supporting function in* $\mathbf{R}^n$. *Then P_H is a compact convex subset of $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ which is invariant under T_t for all $t>0$.*

PROOF. It is clear that P_H is convex and invariant. Let $\{p_j\}$ be a sequence in P_H. Since $p_j(\zeta)\leqq H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n$ with equality in $\mathbf{CR}^n$, Theorem 4.1.9 in Hörmander [11] gives that there exists a subsequence $\{p_{j_k}\}$ of $\{p_j\}$ converging to a plurisubharmonic function p. Since p is the least upper semi-continuous majorant of $\overline{\lim}_{k\to\infty} p_{j_k}$, it follows that $p(\zeta)\leqq H(\operatorname{Im}\zeta)$ for $\zeta\in\mathbf{C}^n$ and $p(\zeta)\geqq H(\operatorname{Im}\zeta)$ for $\zeta\in\mathbf{CR}^n$. This completes the proof.

In section 2.1 we have seen that $i_{\hat{u}}\in P_H$ if u is a hyperfunction with $\operatorname{ch}\operatorname{supp} u=K$. In view of Proposition 1.1.2 the following proposition is an improvement of this result when u is a distribution:

PROPOSITION 3.1.2. *Let K be a compact convex subset of* $\mathbf{R}^n$ *and let H be its supporting function. If u is a distribution in* $\mathbf{R}^n$ *with* $\operatorname{ch}\operatorname{supp} u=K$, *then the limit set of* $\log|\hat{u}|$ *is contained in P_H.*

PROOF. Let $p \in L(\log|\hat{u}|)$ and suppose that $T_{t_j}\log|\hat{u}| \to p$ in L^1_{loc} where $t_j \to \infty$. Let $\eta \in \mathbf{R}^n$ and $z \in \mathbf{C}$ and suppose that $p(z\eta)<H(\operatorname{Im} z\eta)$. Since p is upper semi-continuous and H is continuous, there exists a compact neighborhood L of $z\eta$ in $\mathbf{C}^n$ and $\varepsilon>0$ such that $p(\zeta)-H(\operatorname{Im}\zeta)< -\varepsilon$ for all $\zeta \in L$. By Hörmander [11, Theorem 4.1.9] we have

$$(3.1.4) \qquad ((T_{t_j}\log|\hat{u}|)(\zeta)-H(\operatorname{Im}\zeta)) < -\varepsilon/2 \quad \zeta \in L ,$$

for all sufficiently large j.

If u is a measure then Theorem 16.2.4 and Lemma 16.3.1 in Hörmander [11], give that for every compact subset M of $\mathbf{C}$

$$(3.1.5) \qquad \int_M |(T_t \log|\hat{u}|)(z\eta + \zeta/t) - H(\operatorname{Im} z\eta)|\, d\lambda(z) \to 0$$

as $t \to \infty$ for almost all $\zeta \in \mathbf{C}^n$. If we regularize u by convolving it with a C_0^∞-function and use the theorem of supports, we get that (3.1.5) holds for all distributions u with $\operatorname{ch} \operatorname{supp} u = K$. This contradicts (3.1.4). Hence $p(z\eta) = H(\operatorname{Im} z\eta)$ and the proposition is proved.

Corollary 3.1.3. *Let K be a compact convex subset of $\mathbf{R}^n$ and let H denote its supporting function. Let Γ be an open cone in $\mathbf{C}^n$ and suppose that $p(\zeta) = H(\operatorname{Im}\zeta)$ for all $p \in P_H$ and $\zeta \in \Gamma$. Then $T_t \log|\hat{u}| \to H(\operatorname{Im} .)$ in $L^1_{\text{loc}}(\Gamma)$ as $t \to \infty$ for every distribution u with* $\operatorname{ch}\operatorname{supp} u = K$.

Now we prove a variant of the indicator theorem for Fourier–Laplace transforms of distributions with compact support:

Theorem 3.1.4. *Let p be a plurisubharmonic function in $\mathbf{C}^n$. Suppose that p is positively homogeneous of order one and $p(\zeta) \leqq \sigma|\operatorname{Im}\zeta|$ for $\zeta \in \mathbf{C}^n$, where σ is a positive constant. Then there exists a distrubtion u in $\mathbf{R}^n$ with compact support such that $i_{\hat{u}} = p$.*

Proof. By Theorem 1.4.1, there exists an analytic function f in $\mathbf{C}^n$ such that $i_f = p$ and (1.4.1) is satisfied. By Lemma 1.3.5 there exists a compact subset L of $\mathbf{C}^n$ such that

$$|f(\zeta)| \leqq C(1+|\zeta|)^{3n+1} \exp\left(\sup_{w \in L} p(\zeta + w)\right), \qquad \zeta \in \mathbf{C}^n .$$

We have $\sup_{w \in L} p(\zeta + w) \leqq \sigma|\operatorname{Im}\zeta| + a$, where $a = \sup_{w \in L} \sigma|\operatorname{Im}\zeta|$. The proof is now completed by the Paley–Wiener–Schwartz theorem.

Remark. Proposition 1.1.4 gives that $p \in P_H$, where H is given by $H(\eta) = p(i\eta)$ for $\eta \in \mathbf{R}^n$. Theorem 2.1.1 gives that H is the supporting function of $\operatorname{supp} u$.

With the aid of the maximum principle we are now able to describe a subset of $\mathbf{C}^n$ containing $\mathbf{C}\mathbf{R}^n$, where all the functions in P_H take the same value. For every $x \in \partial K$ we let $N_x = \{\eta \in \mathbf{R}^n;\ \langle x, \eta\rangle = H(\eta)\}$ denote the set of all outer normals to K at x and V_x denote the subspace of $\mathbf{R}^n$ spanned by N_x.

Proposition 3.1.5. *Let K be a compact convex subset of $\mathbf{R}^n$ and let H be its supporting function. Let $p \in P_H$. Then*

$$(3.1.6) \qquad p(\zeta) = H(\operatorname{Im}\zeta), \quad \zeta \in V_x + iN_x, \quad x \in \partial K.$$

If $\zeta_0 = \xi_0 + i\eta_0 \in \mathbf{C}^n$ *and* $p(\zeta_0) = H(\eta_0)$, *then* $p(\xi_0 + w\eta_0) = H(\operatorname{Im} w\eta_0)$ *for all* w *in the upper half plane. If* η_0 *lies in the interior of* N_x *in the relative topology in* V_x *for some* $x \in \partial K$, *then* $p(\zeta) = H(\operatorname{Im}\zeta)$ *for all* $\zeta \in \{\xi_0\} + V_x + iN_x$.

Proof. The set N_x is a convex cone which spans the subspace V_x of $\mathbf{R}^n$, so it has a non-empty interior U_x in the relative topology in V_x. The function $\zeta \mapsto p(\zeta) - H(\operatorname{Im}\zeta) = p(\zeta) - \langle x, \operatorname{Im}\zeta\rangle$ is plurisubharmonic in the open set $V_x + iU_x$ of the complex subspace $V_x + iV_x$ of $\mathbf{C}^n$. We have $p(\zeta) - H(\operatorname{Im}\zeta) \leqq 0$ for all $\zeta \in \mathbf{C}^n$, and equality holds for $\zeta = \eta + i\eta$ with $\eta \in U_x$. By the maximum principle the equality holds in $V_x + iU_x$ for all $x \in \partial K$. Since p is upper semi-continous and H is continuous, the set

$$\{\zeta \in \mathbf{C}^n;\ p(\zeta) = H(\operatorname{Im}\zeta)\} = \{\zeta \in \mathbf{C}^n;\ p(\zeta) \geqq H(\operatorname{Im}\zeta)\}$$

is closed. The closure of $V_x + iU_x$ is $V_x + iN_x$, so (3.1.6) holds. The other statements are proved in a similar way.

Corollary 3.1.6. *Let* K *be a polyhedron in* $\mathbf{R}^n$ *and let* H *be its supporting function. For every distribution* u *in* $\mathbf{R}^n$ *with* $\operatorname{ch}\operatorname{supp} u = K$ *we have* $T_t \log|\hat{u}| \to H(\operatorname{Im}\,.)$ *in* $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ *as* $t \to \infty$.

Proof. We have $V_x = \mathbf{R}^n$ for every extreme point x of K and $\mathbf{R}^n = \cup N_x$, where the union is taken over all extreme points of K. By (3.1.6) we have $P_H = \{H(\operatorname{Im}.)\}$, so the corollary follows from Corollary 3.1.3.

Example. The characteristic function of the euclidean unit ball K in $\mathbf{R}^n$ is an example of a distribution u with $\operatorname{ch}\operatorname{supp} u = K$ and $i_{\hat{u}}(\zeta) < H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$, so $\mathbf{C}\mathbf{R}^n$ is the largest subset of $\mathbf{C}^n$ where all the elements of P_H are equal. We have

$$\hat{u}(\zeta) = \int_K e^{-i\langle x,\zeta\rangle}\,dx, \quad \zeta \in \mathbf{C}^n.$$

Since u is invariant under orthogonal transformations we get that $\hat{u}(\zeta) = \hat{u}_0(\langle\zeta,\zeta\rangle^{1/2})$, where u_0 is the function given by

$$u_0(x) = \begin{cases} c_n(1-x^2)^{(n-1)/2} & x \in [-1,1], \\ 0 & x \in \mathbf{R}\setminus[-1,1], \end{cases}$$

with c_n equal to the volume of the unit ball in $\mathbf{R}^{n-1}$. (The choice of the square root is irrelevant for $\hat{u}_0$ is an even function.) Since $i_{\hat{u}_0}$ is equal to the supporting function of the unit interval, that is $i_{\hat{u}_0}(\tau) = |\operatorname{Im}\tau|$ for $\tau \in \mathbf{C}$, we have

$$(3.1.7)\qquad i_{\hat{u}}(\zeta) = |\mathrm{Im}\,(\langle\zeta,\zeta\rangle^{1/2})|, \quad \zeta \in \mathbf{C}^n .$$

If $\zeta=\xi+i\eta \in \mathbf{C}^n\setminus \mathbf{C}\mathbf{R}^n$, then ξ and η are linearly independent. The Pragmén–Lindelöf principle gives that $i_{\hat{u}}(\xi+w\eta)<\mathrm{Im}\,w|\eta|$ for all $w \in \mathbf{C}$ with $\mathrm{Im}\,w>0$, because $i_{\hat{u}}(\xi+w\eta)=0$ for all real w, $\lim_{t\to\infty} i_{\hat{u}}(\xi+it\eta)/t=|\eta|$ and $i_{\hat{u}}(\xi+w\eta)$ is not identically equal to $\mathrm{Im}\,w|\eta|$. Hence $i_{\hat{u}}(\zeta)<H(\mathrm{Im}\,\zeta)$.

That $i_{\hat{u}}(\zeta)<H(\mathrm{Im}\,\zeta)$ when $\zeta \notin \mathbf{C}\mathbf{R}^n$ is no surprise, for it follows from Theorem 2.3.1 and the fact that $\mathrm{WF}_A(u)$ is the conormal bundle of the unit sphere. More generally, let K be a compact convex subset of $\mathbf{R}^n$ with non-empty interior and an analytic boundary and let u denote the characteristic function of K. Then $\mathrm{WF}_A(u)$ is equal to the conormal bundle of ∂K, so Theorem 2.3.1 gives that $i_{\hat{u}}(\zeta)<H(\mathrm{Im}\,\zeta)$ for $\zeta \in \mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n$. We have a converse of this result:

PROPOSITION 3.1.7. *Let K be a compact convex subset of* $\mathbf{R}^n$ *with a non-empty interior and a C^1 boundary. Let u denote the characteristic function of K and suppose that* $\mathrm{WF}_A(u)$ *is equal to the conormal bundle of ∂K. Then the boundary of K is analytic.*

PROOF. Let $x_0 \in \partial K$. By translating K and then rotating it we can suppose that $x_0=0$ and that $(1,0,\dots,0)$ is the outer unit normal to K at 0. There exists an open neighborhood U of 0 in $\mathbf{R}^n$ of the form $U=(-\alpha,\alpha)\times U'$ such that $x=(x_1,x')\in U\cap K$ is equivalent to $x_1\leqq f(x')$, for some $f\in C^1(U')$. In U we have $u=\theta\big(f(x')-x_1\big)$, where θ is the Heaviside function.

We choose β with $0<\beta<\alpha$ and an open subset V' of $\mathbf{R}^{n-1}$ with $0 \in V'$, $\overline{V}'\subset U'$, and $|f|\leqq\beta$ in V', and set $V=(-\beta,\beta)\times V'$. We choose $\varphi \in C_0^\infty(U)$ with $\varphi=1$ in V and define the distribution v in $\mathbf{R}^n$ by $v=-\varphi x_1\partial_1 u$. Finally we define the distribution w in $\mathbf{R}^{n-1}$ by $w=\int v\,dx_1$. If $\psi \in C_0^\infty(V')$, then

$$\langle w,\psi\rangle = \langle v,1\otimes\psi\rangle = \int \theta\big(f(x')-x_1\big)\partial_1\big(x_1\varphi(x)\big)\psi(x')dx$$

$$= \int\bigg(\int_{-\infty}^{f(x')} \partial_1\big(x_1\varphi(x)\big)dx_1\bigg)\psi(x')dx' = \int f\psi\,dx' .$$

Hence $w=f$ in V'. By Theorem 8.5.4 in Hörmander [11]

$$\mathrm{WF}_A(w) \subset \{(x',\xi')\ ;\ (x_1,x',0,\xi')\in \mathrm{WF}_A(v) \text{ for some } x_1\} .$$

In V the wave front set $\mathrm{WF}_A(v)$ is equal to the conormal bundle of ∂K and $\big(1,-f'(x')\big)$ lies in the conormal direction for $x=(x_1,x')\in\partial K\cap V$. Hence $\mathrm{WF}_A(w)|_{V'}$ is empty, that is, $w=f$ is analytic in V'. The proof is completed.

We have seen that if K has a non-empty interior and an analytic boundary then $\mathbf{C}\mathbf{R}^n$ is the largest subset of $\mathbf{C}^n$ where all the functions in P_H are equal. This even holds if K has a C^2 boundary:

PROPOSITION 3..8. *Let K be a compact convex subset of $\mathbf{R}^n$ with a non-empty interior and a C^2 boundary and let H denote its supporting function. Then there exists a distribution u in $\mathbf{R}^n$ with* $\operatorname{ch}\operatorname{supp} u = K$ *and* $i_{\hat{u}}(\zeta) < H(\operatorname{Im}\zeta)$ *for all* $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$.

PROOF. Since $\operatorname{int} K \neq \varnothing$ and K is a C^2 manifold there exists $r > 0$ such that for every $x \in \partial K$ there exists a closed ball B_x contained in K with radius r and $\partial K \cap B_x = \{x\}$. Set

$$K_1 = \{x \in K \ ; \ d(x, \partial K) \geqq r\} \quad \text{and} \quad K_2 = \{x \in \mathbf{R}^n \ ; \ |x| \leqq r\} .$$

Then K_1 and K_2 are compact convex sets and $K = K_1 + K_2$. If H_1 and H_2 denote the supporting functions of K_1 and K_2, respectively, then $H = H_1 + H_2$. We let u_1 and u_2 denote the characteristic functions of K_1 and K_2, respectively, and set $u = u_1 * u_2$. Then $\operatorname{supp} u = K$ and $i_{\hat{u}} \leqq i_{\hat{u}_1} + i_{\hat{u}_2}$. By the example above $i_{\hat{u}_2}(\zeta) < H_2(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$. Hence $i_{\hat{u}}(\zeta) < H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$ and the proposition is proved.

3.2. Plurisubharmonic functions with prescribed limit sets contained in P_H

Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H. In this section and the next one we deal with the problem of constructing a distribution u with $\operatorname{ch}\operatorname{supp} u = K$ and the limit set of $\log|\hat{u}|$ equal to a prescribed subset M of P_H. We use similar methods as in sections 1.2 and 1.3. First we construct a plurisubharmonic function p with $L(p) = M$, and then we construct an analytic function f in $\mathbf{C}^n$ with $T_t p - T_t \log|f| \to 0$ and

$$\int |f|^2 (1 + |\zeta|)^{-\nu} e^{-2p} \, d\lambda < \infty$$

for some $\nu > 0$. In order to be able to apply the Paley–Wiener–Schwartz theorem we have to assume that

$$p(\zeta) \leqq \log\bigl(C(1 + |\zeta|)^N\bigr) + c|\operatorname{Im}\zeta|, \quad \zeta \in \mathbf{C}^n , \tag{3.2.1}$$

for some positive constants C, N, and c. We are not able to modify the methods of section 1.2 so that p satisfies (3.2.1) if M is a general subset of P_H satisfying the conditions in Theorem 1.2.1, but we only give a weak analogue of Theorem 1.2.1 ii). One of the main reasons is that the regularization operator R_δ defined in section 1.2 violates inequalities of the form (3.2.1). We shall replace it by another one S_δ which only gives us regular functions in $\mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$ but preserves (3.2.1) and maps P_H to another set of the same type.

Let $0 \leqq \alpha \in C_0^\infty(\mathbf{R}^{n^2})$ with $\int \alpha s\lambda = 1$, where $d\lambda$ denotes the Lebesgue measure in $\mathbf{R}^{n^2}$. We identify $\mathbf{R}^{n^2}$ with the space of all real $n \times n$ matrices, let I denote the identity matrix and set $\alpha_\delta(A) = \delta^{-n^2}\alpha((A-I)/\delta)$ for $\delta > 0$. If q is plurisubharmonic in $\mathbf{C}^n$ we define $S_\delta q$ by

$$S_\delta q(\zeta) = \int q(A\zeta)\alpha_\delta(A)\, d\lambda(A), \quad \zeta \in \mathbf{C}^n . \tag{3.2.2}$$

If H is a supporting function in $\mathbf{R}^n$, then we define the supporting function H_δ by

$$H_\delta(\eta) = \int H(A\eta)\alpha_\delta(A)\, d\lambda(A), \quad \eta \in \mathbf{R}^n . \tag{3.2.3}$$

LEMMA 3.2.1. *Let H be a supporting function in $\mathbf{R}^n$ and let q be a plurisubharmonic function in $\mathbf{C}^n$. Define $S_\delta q$ by* (3.2.2) *and H_δ by* (3.2.3). *Then:*

i) *$S_\delta q$ is plurisubharmonic in $\mathbf{C}^n$. If q satisfies* (3.2.1), *then*

$$S_\delta q(\zeta) \leqq \log\left(C_\delta(1+|\zeta|)^N\right) + c_\delta|\operatorname{Im}\zeta|, \quad \zeta \in \mathbf{C}^n , \tag{3.2.4}$$

for some positive constants C_δ and c_δ. If $q \in P_H$, then $S_\delta q \in P_{H_\delta}$.

ii) *we have*

$$S_\delta q - q \to 0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbf{C}^n)$$

as $\delta \to 0$. The convergence is uniform for $q \in P_H$.

iii) *$S_\delta q \in C^\infty(\mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n)$ and $H_\delta \in C^\infty(\mathbf{R}^n \setminus \{0\})$. For every multi-index β and every closed cone Γ with $\Gamma \subset (\mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n) \cup \{0\}$, there exists a positive constant $C_{\Gamma,\beta}$ such that*

$$|D^\beta S_\delta q(\zeta)| \leqq C_{\beta,\Gamma}\delta^{-n^2-|\beta|}|\zeta|^{1-|\beta|}$$

for $\zeta \in \Gamma$, and $\delta \in (0,1)$, where $D = (\partial/\partial\zeta, \partial/\partial\bar{\zeta})$. The inequality holds uniformly for $q \in P_H$.

iv) *$H_\delta \to H$ uniformly in every compact subset of $\mathbf{R}^n$ and $S_\delta q(\zeta) \to q(\zeta)$ for all $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$ as $\delta \to 0$.*

PROOF. i) Since $\alpha \geqq 0$, $S_\delta q$ is plurisubharmonic in $\mathbf{C}^n$. If (3.2.1) is satisfied, then

$$S_\delta q(\zeta) \leqq \int \big(\log(C(1+|\zeta+\delta A\zeta|)^N)+c|\mathrm{Im}\,(\zeta+\delta A\zeta)|\big)\alpha(A)\,d\lambda(A)$$

for all $\zeta \in \mathsf{C}^n$, so (3.2.4) holds. If $q \in P_H$, then

$$S_\delta q(\zeta) \leqq \int H(A\,\mathrm{Im}\,\zeta)\alpha_\delta(A)\,d\lambda(A) = H_\delta(\mathrm{Im}\,\zeta), \quad \zeta \in \mathsf{C}^n\,,$$

with equality in CR^n, so $S_\delta q \in P_{H_\delta}$.

ii) In the proof of Lemma 1.2.1 i) we only assumed that M is compact. The first statement follows as there if we take $M=\{q\}$ and the second one follows if we take $M=P_H$.

iii) If $\eta_0 \in \mathsf{R}^n\setminus\{0\}$, then the linear mapping $\mathsf{R}^{n^2} \ni A \mapsto A\eta_0 \in \mathsf{R}^n$ is surjective. The fact that $H_\delta \in C^\infty(\mathsf{R}^n\setminus\{0\})$ follows from the proof of Lemma 1.2.2 ii) with C replaced by R and z_0 by η_0. If $\zeta_0 \in \mathsf{C}^n\setminus\mathsf{CR}^n$, then $\mathrm{Re}\,\zeta_0$ and $\mathrm{Im}\,\zeta_0$ are linearly independent and the linear mapping $\mathsf{R}^{n^2} \ni A \mapsto A\zeta_0 \in \mathsf{C}^n$ is surjective. Hence there exists a linear mapping $L\colon \mathsf{R}^{n^2} \to \mathsf{R}^{n^2-2n}$ which is bijective from $\{A;\, A\zeta_0=0\}$. Then

$$\mathsf{R}^{n^2} \ni A \mapsto \big(A\zeta_0, L(A)\big) \in \mathsf{C}^n \oplus \mathsf{R}^{n^2-2n} \cong \mathsf{R}^{n^2}$$

is a bijection. With obvious modifications we can now continue as in the proof of Lemma 1.2.2 ii).

iv) The first statement is obvious. The convergence in $\mathsf{C}^n\setminus\mathsf{CR}^n$ follows in the same way as in the proof of Lemma 1.2.2 iii). The proof is completed.

Let K denote the unit ball in R^n and let H be its supporting function. Then $H(\eta)=|\eta|$. Let q be the function in C^n defined by

$$q(\zeta) = \Big(\big((\mathrm{Im}\,(\langle\zeta,\zeta\rangle^{1/2})^2+|\mathrm{Im}\,\zeta|^2\big)/2\Big)^{1/2}, \quad \zeta \in \mathsf{C}^n\,. \tag{3.2.5}$$

From the example after Corollary 3.1.6 we know that $\zeta \mapsto |\mathrm{Im}\,(\langle\zeta,\zeta\rangle^{1/2})|$ is the indicator function of the Fourier–Laplace transform of the characteristic function of K, so $q(\zeta)\leqq H(\mathrm{Im}\,\zeta)$ for $\zeta \in \mathsf{C}^n$ with equality in CR^n and strict inequality in $\mathsf{C}^n\setminus\mathsf{CR}^n$. We observe that

$$|\mathrm{Im}\zeta|^2 = \big(|\zeta|^2-\mathrm{Re}\,(\langle\zeta,\zeta\rangle)\big)/2, \quad \zeta \in \mathsf{C}^n$$

and

$$\big(\mathrm{Im}\,(v^{1/2})\big)^2 = \big(|v|-\mathrm{Re}\,v\big)/2, \quad v \in \mathsf{C}\,,$$

so

$$q(\zeta) = \big(|\zeta|^2+|\langle\zeta,\zeta\rangle|-2\,\mathrm{Re}(\langle\zeta,\zeta\rangle)\big)^{1/2}/2, \quad \zeta \in \mathsf{C}^n. \tag{3.2.6}$$

We have $|\langle\zeta,\zeta\rangle|-\mathrm{Re}(\langle\zeta,\zeta\rangle) \geqq 0$ and

$$|\zeta|^2 = |\mathrm{Re}\,\zeta|^2+|\mathrm{Im}\,\zeta|^2 > |\mathrm{Re}\,\zeta|^2-|\mathrm{Im}\,\zeta|^2 = \mathrm{Re}(\langle\zeta,\zeta\rangle)$$

for all $\zeta \in \mathsf{C}^n\setminus\mathsf{R}^n$. Hence q is infinitely differentiable in $\mathsf{C}^n\setminus(\mathsf{R}^n \cup M)$, where

$$M = \{\zeta \in \mathsf{C}^n \; ; \; \langle\zeta,\zeta\rangle=0\} .$$

Wiegerinck [24] has proved that q is strictly plurisubharmonic in $\mathsf{C}^n\setminus(\mathsf{CR}^n \cup M)$. By regularizing the function q with a similar method as in Lemma 3.2.1 we get strict plurisubharmonicity in $\mathsf{C}^n\setminus\mathsf{CR}^n$:

LEMMA 3.2.2. *Let $n>1$ and let H denote the supporting function of the closed unit ball in R^n. Then there exists a positively homogeneous function p in P_H which is infinitely differentiable in $\mathsf{C}^n\setminus\mathsf{R}^n$ and strictly plurisubharmonic in $\mathsf{C}^n\setminus\mathsf{CR}^n$. Furthermore $p(\zeta)<H(\mathrm{Im}\,\zeta)$ for $\zeta\in\mathsf{C}^n\setminus\mathsf{CR}^n$.*

PROOF. Let α be defined as before Lemma 3.2.1 and suppose that

$$\mathrm{supp}\,\alpha\subset N = \{A=(a_{jk}) \in \mathsf{R}^n \; ; \; \det A \geqq 1\}$$

and $\alpha(AO)=\alpha(A)$ for all orthogonal matrices O and all $A \in \mathsf{R}^{n^2}$. It is obvious that there exists a function α satisfying these conditions for we can always choose $0\leqq\beta \in C_0^\infty(\mathsf{R}^{n^2})$ with $\beta \neq 0$ and $\mathrm{supp}\,\beta\subset N$ and then define α by $\alpha(A)=c\int\beta(AO)\,d\pi(O)$, where we integrate over the orthogonal group $O(n)$ with respect to the Haar measure $d\pi$, and choose the constant c such that $\int\alpha\,d\lambda=1$. The set N is invariant under the mapping $A\mapsto AO$ for all orthogonal matrices, so α has the stated properties. Set

$$H'(\eta) = \int H(A\eta)\alpha(A)\,d\lambda(A), \qquad \eta \in \mathsf{R}^n .$$

The absolute value of the determinant of the mapping $A\mapsto AO$ is equal to one for all orthogonal matrices, so the invariance of α gives that H' is orthogonally invariant. Hence $H=cH'$, for some positive constant c. We set

$$p(\zeta) = c\int q(A\zeta)\alpha(A)\,d\lambda(A), \qquad \zeta \in \mathsf{C}^n ,$$

where q is defined by (3.2.5). Then $p \in P_H$. From the proof of Lemma 3.2.1 ii) it follows that p is infinitely differentiable in $\mathsf{C}^n\setminus\mathsf{CR}^n$. If $\zeta_0 \in \mathsf{CR}^n\setminus\mathsf{R}^n$, then we can write ζ_0 as $\zeta_0=e^{i\vartheta}\xi$ for some $\xi \in \mathsf{R}^n\setminus\{0\}$ and $\vartheta \in (0,\pi)$. We have $\mathrm{Im}\,A\zeta_0=\sin\vartheta A\xi$. Since all the matrices in $\mathrm{supp}\,\alpha$ are invertible, it follows that $|A\xi|\geqq a|\xi|$ for $A\in\mathrm{supp}\,\alpha$, where $a>0$. *Hence*

$A\zeta \in \mathbf{C}^n \setminus (\mathbf{R}^n \cup M)$ for all $A \in \operatorname{supp}\alpha$ and all ζ in a neighborhood of ζ_0. Since $q \in C^\infty(\mathbf{C}^n \setminus (\mathbf{R}^n \cup M))$ we get $p \in C^\infty(\mathbf{C}^n \setminus \mathbf{R}^n)$.

Let $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$. Since $\mathbf{R}^{n^2} \ni A \mapsto A\zeta_0 \in \mathbf{C}^n$ is surjective there exists a matrix B in the interior of the support of α such that

$$B\zeta_0 \notin M = \{\zeta \in \mathbf{C}^n\ ;\ \langle \zeta, \zeta \rangle = 0\}\ .$$

We write α as a sum of two non-negative functions $\alpha_1, \alpha_2 \in C_0^\infty(\mathbf{R}^{n^2})$ with $\alpha_1 = \alpha$ in a neighborhood of B, and $A\zeta_0 \notin M$ for $A \in \operatorname{supp}\alpha_1$. Then $\sum \partial^2 p(\zeta_0)/\partial\zeta_j \partial\bar{\zeta}_k w_j \bar{w}_k$ can be written as a sum of two non-negative terms and one of them is

$$\int \sum (\partial^2 q/\partial\zeta_j \partial\bar{\zeta}_k)(Aw)_j \overline{(Aw)}_k \alpha_1(A)\, d\lambda(A)\ .$$

By Wiegerinck [24], q is strictly plurisubharmonic in $\mathbf{C}^n \setminus (\mathbf{C}\mathbf{R}^n \cup M)$. Hence p is strictly plurisubharmonic at ζ_0. The last statement follows from the fact that $q(A\zeta) < H(A \operatorname{Im}\zeta)$ for all $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$ and $A \in \operatorname{supp}\alpha$. The proof is complete.

Proposition 3.2.3. *Let $n > 1$. Let K be a compact convex subset of* $\mathbf{R}^n$ *with supporting function H. Suppose that K has a non-empty interior and that ∂K is a C^∞ manifold with strictly positive curvature. Then there exists a positively homogeneous function p in P_H which is infinitely differentiable in* $\mathbf{C}^n \setminus \mathbf{R}^n$ *and strictly plurisubharmonic in* $\mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$. *Furthermore $p(\zeta) < H(\operatorname{Im}\zeta)$ for $\zeta \in \mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n$.*

Proof. By the proof of Proposition 3.1.8 we can write K as a sum $K = K_1 + K_2$, where K_2 is the closed unit ball in $\mathbf{R}^n$ with center at the origin and radius r for some $r > 0$, and K_1 consists of all points in K with distance $\geq r$ to $\complement K$. We set $p(\zeta) = H_1(\operatorname{Im}\zeta) + r\varphi(\zeta)$ for $\zeta \in \mathbf{C}^n$, where φ is a function satisfying the conditions in Lemma 3.2.2 and H_1 is the supporting function of K_1. The proposition follows if we can show that $H_1 \in C^\infty(\mathbf{R}^n \setminus \{0\})$ if r is sufficiently small.

If r is small enough, then K_1 has a non-empty interior and the mapping $\partial K \to \partial K_1$, $x \mapsto x - r\nu(x)$ is infinitely differentiable and bijective, where $\nu(x)$ is the outer normal to ∂K at x. Hence ∂K_1 is a C^∞ manifold. If r is sufficiently small, then it follows that K_1 has a strictly positive curvature.

If $\eta \in \mathbf{R}^n \setminus \{0\}$, then

$$H_1(\eta) = \sup_{x \in \partial K_1} \langle x, \eta \rangle = \langle x(\eta), \eta \rangle\ ,$$

where $x(\eta)$ is uniquely determined by the condition $\nu_1(x) = \eta/|\eta|$ and $\nu_1(x)$

is the outer unit normal of ∂K_1 at x. The Gauss mapping $\partial K_1 \ni x \mapsto \nu_1(x) \in S^{n-1}$ is a diffeomorphism. Hence $H_1 \in C^\infty(\mathsf{R}^n \setminus \{0\})$. This completes the proof.

The main result of this section is the following variant of Theorem 1.2.1 ii):

THEOREM 3.2.4. *Let H be a supporting function in R^n. Let M be a connected subset of P_H consisting of homogeneous functions and suppose that there exists a closed cone Γ contained in $(\mathsf{C}^n \setminus \mathsf{C}\mathsf{R}^n) \cup \{0\}$ such that all the functions in M are equal in $\mathsf{C}^n \setminus \Gamma$.*

i) *If all functions in M are infinitely differentiable and strictly plurisubharmonic in a neighborhood of $\Gamma \setminus \{0\}$, then there exists a function $p \in P_H$ with $L(p) = \bar{M}$.*

ii) *Let γ be a positive continuous function on $\bar{\mathsf{R}}_+$ with $\gamma(r) \to 0$ as $r \to \infty$. Suppose that all the functions q in M are infinitely differentiable in $\mathsf{C}^n \setminus \mathsf{C}\mathsf{R}^n$ and strictly plurisubharmonic there, and that there exists a uniform lower bound for their Levi forms of the form $c|\zeta|^{-1}|w|^2$ for $\zeta \in \Gamma \setminus \{0\}$ and $w \in \mathsf{C}^n$. Then the function p can be chosen of class C^∞ and strictly plurisubharmonic in $\mathsf{C}^n \setminus \mathsf{C}\mathsf{R}^n$, and for every closed cone $\Lambda \subset (\mathsf{C}^n \setminus \mathsf{C}\mathsf{R}^n) \cup \{0\}$ there exists a positive constant R_Λ such that*

$$\sum \partial^2 p(\zeta)/\partial\zeta_l \partial\bar{\zeta}_m w_l \bar{w}_m \geqq \gamma(|\zeta|)|\zeta|^{-1}|w|^2 \tag{3.2.7}$$

for all $\zeta \in \Lambda$ with $|\zeta| \geqq R_\Lambda$ and $w \in \mathsf{C}^n$. Suppose that there exists a positive constant C such that $|D^\beta q(\zeta)| \leqq C|\zeta|^{1-|\beta|}$ for all $q \in M$, $\zeta \in \Gamma \setminus \{0\}$, and all multi-indices β with $1 \leqq |\beta| \leqq 3$. Then there exists a positive constant C_Λ such that

$$|D^\beta p(\zeta)| \leqq C_\Lambda |\zeta|^{1-|\beta|}, \tag{3.2.8}$$

for all $\zeta \in \Lambda \setminus \{0\}$ and β with $1 \leqq |\beta| \leqq 3$.

PROOF. We only have to make a slight modification of the construction in section 1.2. By Lemma 1.2.5 and Proposition 3.1.1 we can choose a sequence $\{q_k\}$ in M such that its elements form a dense subset of M, every element appears infinitely many times in the sequence, and $q_k - q_{k+1} \to 0$ in $L^1_{\mathrm{loc}}(\mathsf{C}^n)$ as $k \to \infty$. Let $\{\varphi_k\}$ be the partition of unity constructed before Lemma 1.2.3. We are going to show that if the sequence $\{\beta_k\}$, used for defining $\{\varphi_k\}$, tends sufficiently fast to infinity as $k \to \infty$, then the function p defined by

$$p = \sum \varphi_k q_k$$

is plurisubharmonic in $\mathbf{C}^n$. The fact that $L(p)=\bar{M}$ follows as in the proof of Theorem 1.2.1.

Since all the functions are equal in $\mathbf{C}^n\setminus\Gamma$, p is plurisubharmonic there. If ii) is satisfied, then p is positively homogeneous and strictly plurisubharmonic in $\mathbf{C}^n\setminus(\Gamma\cup\mathbf{C}\mathbf{R}^n)$. Hence it is sufficient to prove that p is plurisubharmonic in Γ and that (3.2.7) and (3.2.8) hold with $\Lambda=\Gamma$. In a neighborhood of the set $\Gamma\cap\{z\in\mathbf{C}^n;\ \beta_{k-1}\leqq|z|\leqq\beta_k\}$ we have $p=\varphi_{k-1}q_{k-1}+\varphi_k q_k$ and the Levi form of p is equal to

$$\varphi_{k-1}\sum\partial^2 q_{k-1}/\partial\zeta_l\partial\bar{\zeta}_m w_l\bar{w}_m+\varphi_k\sum\partial^2 q_k/\partial\zeta_l\partial\bar{\zeta}_m w_l\bar{w}_m+$$
$$2\,\mathrm{Re}\big(\langle\partial\varphi_k/\partial\zeta,w\rangle\langle\partial(q_k-q_{k-1})/\partial\bar{\zeta},\bar{w}\rangle+$$
$$+\sum\partial^2\varphi_k/\partial\zeta_l\partial\bar{\zeta}_m w_l\bar{w}_m(q_k-q_{k-1})\,.$$

Since q_k is positively homogeneous of order one there exist positive constants c_k and C_k such that

$$\sum\partial^2 q_k/\partial\zeta_l\partial\bar{\zeta}_m w_l\bar{w}_m\ \geqq\ c_k|\zeta|^{-1}|w|^2$$

for $\zeta\in\Gamma\setminus\{0\}$ and $w\in\mathbf{C}^n$, and

$$|\partial(q_k-q_{k-1})/\partial\zeta|+|\zeta|^{-1}|q_k-q_{k-1}|\ \leqq\ C_k$$

for $\zeta\in\Gamma\setminus\{0\}$. By (1.2.6) there exists a lower bound for the Levi form of p in $\Gamma\cap\{\zeta\in\mathbf{C}^n;\ \beta_{k-1}\leqq|\zeta|\leqq\beta_k\}$ of the form

$$(c_k-C_k'(\log\sigma_k)^{-1})|\zeta|^{-1}|w|^2$$

where C_k' is a positive constant. We choose β_k such that

$$(c_k-C_k'(\log\sigma_k)^{-1})\ \geqq\ c_k/2$$

for all $k\geqq 1$. Then p is plurisubharmonic and i) holds.

If the conditions in ii) are satisfied, then we can take $c_k=c$. We choose R_Γ so large that $\gamma(r)\leqq c/2$ for $r\geqq R_\Gamma$. Then (3.2.7) holds. The last statement follows from (1.2.6). The proof is complete.

It is not difficult to find sets that satisfy the conditions in the theorem:

THEOREM 3.2.5. *Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H. Suppose that K has a non-empty interior and that ∂K is a C^∞ manifold with strictly positive curvature. Let k be a positive integer. Then there exist positively homogeneous functions $q_1,\dots,q_k$ with $q_l\neq q_m$ if $l\neq m$, such that the convex hull of $\{q_1,\dots,q_k\}$ satisfies all conditions in Theorem* 3.2.4.

PROOF. Let p be a function satisfying the conditions in Proposition 3.2.3. For $j=1,\ldots,k$ we choose a real function χ_j in $C^\infty(\mathbf{C}^n\setminus\{0\})$ such that χ_j is positively homogeneous of order zero, $\operatorname{supp}\chi_j\subset(\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n)\cup\{0\}$, and $\chi_l\neq\chi_m$ if $l\neq m$. We let $\varepsilon>0$ and set $q_j=(1-\varepsilon\chi_j)p$. Then q_j is infinitely differentiable in $\mathbf{C}^n\setminus\mathbf{R}^n$, positively homogeneous of order one, and $q_l\neq q_m$ if $l\neq m$. Since p is strictly plurisubharmonic in $\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n$ and $p(\zeta)<H(\operatorname{Im}\zeta)$ for $\zeta\in\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n$, it follows that $q_j\in P_H$ and q_j is strictly plurisubharmonic in $\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n$, if ε is sufficiently small. If we take Γ as the union of the supports of χ_j, then $M=\operatorname{ch}\{q_1,\ldots,q_k\}$ satisfies all the conditions in Theorem 3.2.4. The proof is complete.

3.3. Asymptotic approximation of plurisubharmonic functions by Fourier–Laplace transforms of distributions.

Let K be a compact convex subset of $\mathbf{R}^n$ and let H denote its supporting function. In this section we continue our study of the problem of constructing a distribution u with $\operatorname{ch}\operatorname{supp}u=K$ and the limit set of $\log|\hat{u}|$ contained in a prescribed subset of P_H, by proving an approximation theorem similar to that in section 1.3. We are not able to prove the theorem for a general plurisubharmonic function p satisfying the necessary growth conditions, for we have not been able to modify Lemma 1.3.3 by using the regularization operator studied in section 3.2.

THEOREM 3.3.1. *Let K be a compact convex subset of $\mathbf{R}^n$ and let H be its supporting function. Let p be a plurisubharmonic function in $\mathbf{C}^n$ with limit set contained in P_H. Suppose that*

i) *there exist positive constants C, N, and c such that*

$$p(\zeta)\leqq\log\big(C(1+|\zeta|)^N\big)+c|\operatorname{Im}\zeta|,\quad \zeta\in\mathbf{C}^n. \tag{3.3.1}$$

ii) *p is infinitely differentiable and strictly plurisubharmonic in $\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n$. There exists a positive number $\kappa<1/3$ such that for every closed cone Γ contained in $(\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n)\cup\{0\}$ there exists a positive number R_Γ such that*

$$\sum\partial^2p(\zeta)/\partial\zeta_j\partial\bar{\zeta}_kw_j\bar{w}_k\geqq|\zeta|^{-1-\kappa}|w|^2 \tag{3.3.2}$$

for all $\zeta\in\Gamma$ with $|\zeta|\geqq R_\Gamma$ and $w\in\mathbf{C}^n$.

iii) *for every closed cone Λ contained in $(\mathbf{C}^n\setminus\mathbf{C}\mathbf{R}^n)\cup\{0\}$ there exists a positive constant C_Λ such that*

$$|D^\beta p(\zeta)|\leqq C_\Lambda(\log(2+|\zeta|))^\mu|\zeta|^{1-|\beta|}, \tag{3.3.3}$$

for all $\zeta\in\Lambda\setminus\{0\}$ and every multi-index β with $1\leqq|\beta|\leqq3$. Here $D=(\partial/\partial\zeta,\partial/\partial\bar{\zeta})$ and μ is a positive constant.

Then there exists a distribution u with $\operatorname{ch}\operatorname{supp} u = K$ *such that* $T_t p - T_t \log|\hat{u}| \to 0$ *in* $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ *as* $t \to \infty$.

If we combine Theorem 3.2.4, Theorem 3.2.5, and Theorem 3.3.1 we can give an example of a distribution with a prescribed limit set:

COROLLARY 3.3.2. *Let K be a compact convex subset of $\mathbf{R}^n$ with supporting function H. Suppose that K has a non-empty interior and that ∂K is a C^∞ manifold with strictly positive curvature. Let k be a positive integer. Then there exist positively homogeneous functions $q_1, \ldots, q_k$ with $q_l \neq q_m$ if $l \neq m$, and a distribution u such that* $\operatorname{ch}\operatorname{supp} u = K$ *and the limit set of* $\log|\hat{u}|$ *is equal to the convex hull of* $\{q_1, \ldots, q_k\}$.

PROOF OF THEOREM 3.3.1. We choose a positive constant τ such that $\kappa + 2\tau < 1$ and $\kappa < \tau$, and let the notation be the same as in the proof of Theorem 1.3.1. We are going to show that the function f can be constructed as before with $\{z_k\}$ replaced by a subsequence.

Let $\{\Gamma_j\}$ be an increasing sequence of closed cones with $\bigcup \Gamma_j = (\mathbf{C}^n \setminus \mathbf{C}\mathbf{R}^n) \cup \{0\}$. By ii) we can choose an increasing sequence $\{R_j\}$ of positive constants such that (3.3.2) holds for $\zeta \in \Gamma_j$ and $|\zeta| \geqq R_j$. We set $\Lambda_j = \{\zeta \in \Gamma_j;\ |\zeta| \geqq R_j\}$. There exists a positive constant C_j such that (1.3.10) holds with $C' = C_j$ if $W_k \subset \Gamma_j$. We can replace R_j by a larger number such that (1.3.11) holds if $W_k \subset \Lambda_j$. If we replace $\{z_k\}$ by the subsequence consisting of all z_k with $W_k \subset \bigcup \Lambda_j$, then the analytic function f can be constructed as in the proof of Theorem 1.3.1. By (1.3.17), (3.3.1), and the Paley–Wiener theorem it follows that f is the Fourier–Laplace transform of a distribution u with compact support.

The fact that $T_t p - T_t \log|\hat{u}| \to 0$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ as $t \to \infty$ follows in the same way as in the proof of Theorem 1.3.1. We only have to observe that $\mathbf{C}\mathbf{R}^n$ has Lebesgue measure zero, and that if $z \in Y \setminus \mathbf{C}\mathbf{R}^n$ is a limit point of $\{\zeta_k\}$ with $\zeta_k \in Z_{j_k m}$, then $t_k \zeta_k$ is an element in the sequence $\{z_l\}$ if k is sufficiently large. Since $L(p) = L(\log|\hat{u}|) \subset P_H$, we have $\operatorname{ch}\operatorname{supp} u = K$. This completes the proof.

3.4. Discontinuous indicator functions.

We have seen in section 1.1 that every positively homogeneous subharmonic function in the complex plane is continuous. An analogous result does not hold for plurisubharmonic functions as Lelong [14] has proved. The following theorem combined with Theorem 3.1.4 shows that for every $n > 1$ there exists a distribution u in $\mathbf{R}^n$ with $i_{\hat{u}}$ discontinuous.

THEOREM 3.4.1. *Let $n>1$ and let H be the supporting function of a compact convex subset of $\mathbf{R}^n$ with non-empty interior and C^∞ boundary with strictly positive curvature. Then there exists a discontinuous function in P_H which is positively homogeneous.*

PROOF. By writing K as a sum of a closed ball and a compact convex set as in the proof of Proposition 3.2.3, we can suppose that K is the closed unit ball. Let $\zeta_0 \in \mathbf{C}^n \setminus \mathbf{CR}^n$. We begin by constructing a positively homogeneous plurisubharmonic function r in a conic neighborhood of ζ_0 which is discontinuous at ζ_0. Let $\{z_k\}$ be a sequence of non-zero complex numbers converging to zero. Let $\sum a_k$ be a convergent series with positive terms such that $\sum a_k \log|z_k| > -\infty$ and define q by

$$q(z) = \sum a_k \log|z-z_k|, \quad z \in \mathbf{C}.$$

Then q is subharmonic in $\mathbf{C}$, $q(0)=\sum a_k \log|z_k| > -\infty$ and $\underline{\lim}_{z\to 0} q(z) = -\infty$. We choose j and k with $\zeta_{0j} \neq 0$ and $k \neq j$ and set

$$r(\zeta) = |\zeta_j| \exp\big(q(\zeta_k/\zeta_j - \zeta_{0k}/\zeta_{0j}) - q(0)\big).$$

Then r is plurisubharmonic in $\{\zeta \in \mathbf{C}^n;\ \zeta_j \neq 0\}$, positively homogeneous of order one and discontinuous at ζ_0.

Let p be the function constructed in the proof of Lemma 3.2.2. Then p is positive in $\mathbf{C}^n \setminus \mathbf{CR}^n$. Set $\Gamma = \{\zeta \in \mathbf{C}^n;\ \zeta_j \neq 0\} \cap (\mathbf{C}^n \setminus \mathbf{CR}^n)$ and choose an open conic neighborhood Λ of ζ_0 with $\bar{\Lambda} \setminus \{0\} \subset \Gamma$. We choose $\kappa \in C^\infty(\mathbf{C}^n \setminus \{0\})$ which is positively homogeneous of order zero, with $\operatorname{supp}\kappa \subset \Gamma \cup \{0\}$, $0 \leq \kappa \leq 1$, $\kappa = 0$ in a neighborhood of ζ_0, and $\kappa = 1$ in a neighborhood of $\partial\Lambda \setminus \{0\}$. Let δ and ε be positive real numbers and set

$$s(\zeta) = \begin{cases} \max\{(1-\delta\kappa(\zeta))p(\zeta) + \varepsilon r(\zeta), p(\zeta)\}, & \zeta \in \Lambda, \\ p(\zeta) & \zeta \in \mathbf{C}^n \setminus \bar{\Lambda}. \end{cases}$$

Since p is strictly plurisubharmonic in $\mathbf{C}^n \setminus \mathbf{CR}^n$ and κ has its support there, the number δ can be chosen such that $\zeta \mapsto (1-\delta\kappa(\zeta))p(\zeta)$ is plurisubharmonic in $\mathbf{C}^n$. In a neighborhood of $\partial\Lambda \setminus \{0\}$ in $\mathbf{C}^n \setminus \mathbf{CR}^n$ we have $\kappa = 1$ so ε can be chosen such that $(1-\delta\kappa(\zeta))p(\zeta) + \varepsilon r(\zeta) < p(\zeta)$ for all ζ in a neighborhood of $\partial\Lambda$ and $(1-\delta\kappa(\zeta))p(\zeta) + \varepsilon r(\zeta) < H(\operatorname{Im}\zeta)$ in Λ. Then $s \in P_H$. Since $\kappa = 0$ in a neighborhood of ζ_0 we have $s = \max\{p + \varepsilon r, p\}$ there so

$$s(\zeta_0) = p(\zeta_0) + \varepsilon|\zeta_{0j}|, \quad \underline{\lim}_{w \to \zeta_0} s(w) = p(\zeta_0)$$

Hence s is discontinuous at ζ_0. The proof is completed.

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DEPARTMENT OF MATHEMATICS
LUND UNIVERSITY
BOX 118
S-221 00 LUND
SWEDEN

Present address:
DEPARTMENT OF MATHEMATICS
SCIENCE INSTITUTE
UNIVERSITY OF ICELAND
DUNHAGA 3
IS-107 REYKJAVÍK
ICELAND