MATH. SCAND. 60 (1987), 5-8

NORMAL RINGS AND LOCAL IDEALS

M. M. PARMENTER and P. N. STEWART

Methods from non-standard analysis are used in [2] to show that an ideal of $C(X)$ is local if and only if it is an intersection of pseudoprime ideals, and the case when all ideals of $C(X)$ are local is considered in [2, Theorem 6.2]. In this paper we give ring theoretic proofs which show that these results are true in a much more general setting.

Recall that a right ideal I of a ring R is *local* if $x \in R, \{x_1,\ldots,x_n\} \subseteq I$ and

$$(x-x_1)R(x-x_2)R\ldots(x-x_{n-1})R(x-x_n) = 0$$

imply that $x \in I$, while an ideal P of R is *pseudoprime* if P contains a prime ideal of R. A ring R is *normal* if $x, y \in R$ and $xRy=0$ imply that $\operatorname{ann}(x)+\operatorname{ann}(y)=R$. Here

$$\operatorname{ann}(x) = \{z \in R : zRx=0\},$$

but since normal rings are clearly semiprime,

$$\operatorname{ann}(x) = \{z \in R : xRz=0\}$$

also. Unless otherwise stated, the word "ideal" will refer to a two-sided ideal throughout this work.

We will prove the following.

Let R be an associative ring with identity in which the product of any two finitely generated ideals is finitely generated.

THEOREM 1. *An ideal of R is local if and only if it is an intersection of pseudoprime ideals.*

THEOREM 2. *If R is normal, then every right ideal of R is local, and if R is commutative and every principal ideal of R local, then R is normal.*

Notice that all commutative rings as well as all noetherian noncommutative rings satisfy the above condition on finitely generated ideals.

If I is a local ideal of the ring R, then I contains all nilpotent ideals of R. Also, the sum NR of the nilpotent ideals of R is local. Thus NR is the unique smallest local ideal of R.

Received June 6, 1985; in revised form October 26, 1985.

Of course, every pseudoprime ideal of a ring R contains $\beta(R)$ (the intersection of the prime ideals of R), and so Theorem 1 can not hold for a ring R in which $NR \neq \beta(R)$. An example of such a ring is given in [7, Chapter 8, Lemma 3.6].

PROOF OF THEOREM 1.

One direction is clear. For the other, let I be a local ideal and suppose that $x \notin I$. Let Z be the collection of all ideals of R which do not contain a product of the form $R(x-x_1)R(x-x_2)R\ldots R(x-x_n)R$ where $x_1,\ldots,x_n \in I$. Since I is local and $x \notin I$, $\{0\} \in X$, and since the ideals $R(x-x_1)R(x-x_2)R\ldots R(x-x_n)R$, being products of finitely generated ideals, are finitely generated, we may apply Zorn's lemma to obtain an ideal Q maximal in Z. The ideal Q is prime, $P=I+Q$ is pseudoprime and $x \notin P$.

PROOF OF THEOREM 2. Assume that R is normal. We begin by showing that for $a_1, a_2, \ldots, a_{n-1}, a_n \in R$,

$$\operatorname{ann}(a_1Ra_2\ldots a_{n-1}Ra_n) = \operatorname{ann}(a_1)+\operatorname{ann}(a_2)+\ldots+\operatorname{ann}(a_{n-1})+\operatorname{ann}(a_n).$$

The result is clear for $n=1$, so we assume that $n>1$. Let $w \in \operatorname{ann}(a_1Ra_2\ldots a_{n-1}Ra_n)$ and let $u_1,\ldots,u_k$ be generators for the ideal $RwRa_1Ra_2\ldots a_{n-1}R$, which is finitely generated because it is a product of finitely generated ideals. Since R is normal, $\operatorname{ann}(u_i)+\operatorname{ann}(a_n)=R$ for all $i=1,\ldots,k$ and so for each i we may choose $v_i \in \operatorname{ann}(u_i)$ and $b_i \in \operatorname{ann}(a_n)$ such that $v_i+b_i=1$. Since $(v_1+b_1)\ldots(v_k+b_k)=1$ there is a $\bar{b} \in \operatorname{ann}(a_n)$ such that $\bar{v}+\bar{b}=1$ where $\bar{v}=v_1\ldots v_k$. Thus

$$w = \bar{v}w+\bar{b}w \in \operatorname{ann}(a_1Ra_2\ldots Ra_{n-1})+\operatorname{ann}(a_n).$$

Thus, by induction,

$$w \in \operatorname{ann}(a_1)+\operatorname{ann}(a_2)+\ldots+\operatorname{ann}(a_{n-1})+\operatorname{ann}(a_n).$$

Of course,

$$\operatorname{ann}(a_1)+\operatorname{ann}(a_2)+\ldots+\operatorname{ann}(a_{n-1})+\operatorname{ann}(a_n) \subseteqq \operatorname{ann}(a_1Ra_2\ldots a_{n-1}Ra_n),$$

and so we conclude that

$$\operatorname{ann}(a_1Ra_2\ldots a_{n-1}Ra_n) = \operatorname{ann}(a_1)+\operatorname{ann}(a_2)+\ldots+\operatorname{ann}(a_{n-1})+\operatorname{ann}(a_n).$$

Let I be a right ideal of R and suppose that $x_1,\ldots,x_k \in I$, $x \in R$ and

$$(x-x_1)R(x-x_2)R\ldots(x-x_{k-1})R(x-x_k) = 0 .$$

From the above paragraph we see that

$$\operatorname{ann}(x-x_1)+\ldots+\operatorname{ann}(x-x_k) = R ,$$

so there are $e_i \in \operatorname{ann}(x-x_i)$ such that $1=e_1+\ldots+e_k$. Hence

$$x = xe_1+\ldots+xe_k = x_1e_1+\ldots+x_ke_k \in I$$

which shows that I is local.

Now assume that R is commutative and that every principal ideal of R is local. Let $a,b \in R$ and suppose that $aRb=0$. Then $(a-0)R(a-(a-b))=0$ and since the ideal generated by $a-b$ is local, $a=(a-b)x$ for some $x \in R$. Hence $a-ax=-bx$. Because $\{0\}$ is a local ideal, R is semiprime and so $aRb=0$ implies that $aR \cap bR=(0)$. Thus $a(1-x)=0=bx$ from which we see that

$$1 = (1-x)+x \in \operatorname{ann}(a)+\operatorname{ann}(b) .$$

This shows that R is normal and the proof is complete.

To see that commutativity cannot be omitted from the hypothesis of the second part of Theorem 2, we consider the following example constructed by Goodearl [5, page 44].

Let V be a vector space having countable dimension over a field F, $Q=\operatorname{End}_F(V)$ and

$$J = \{x \in Q : \dim(xV)<\infty\} .$$

Then Q and J are von Neumann regular and J is the unique proper two-sided ideal of Q.

Set

$$R = \{(x,y) \in Q\times Q : x-y \in J\} .$$

Then R is a subring of $Q\times Q$, $J\times J$ is an ideal of R and $R/(J\times J)\cong Q/J$. Hence R is von Neumann regular and therefore satisfies the condition that every principal ideal (in fact, every one-sided ideal) is local. Moreover, since R has only three proper two-sided ideals it is clear that a product of finitely generated ideals of R is again finitely generated. However, R is not normal because if $x \in J\times 0$ and $y \in 0\times J$, then $xRy=0$ but $\operatorname{ann}(x)+\operatorname{ann}(y)=J\times J$.

Using different techniques, W. H. Cornish [1] has obtained a variation of the first part of Theorem 2. Also, a version of Theorem 2 for Archimedian f-algebras is given in [6, Proposition 6.3].

Various conditions on X are known which imply that $C(X)$ is normal; for example, see [3, 14.26 and 14.27] in conjunction with [4, 6.2]. More generally, we note that if R is normal, then so is the polynomial ring $R[x]$ and the ring of $n\times n$ matrices with entries from R. Also, if KG is a semiprime group ring over a field K, then KG is normal. To see this, suppose that $aKGb=0$. From [8, Corollary 5.6] there is a central idempotent $e \in \operatorname{ann}(a)$ such that $b=be$. Since e is central, $1-e \in \operatorname{ann}(b)$ and hence $1 \in \operatorname{ann}(a)+\operatorname{ann}(b)$.

ACKNOWLEDGEMENT. This work was supported in part by Natural Sciences and Engineering Research Council of Canada Grants A8775 and A8789.

REFERENCES

1. W. H. Cornish, private communication.
2. J. C. Dyre, *Non-standard characterizations of ideals in $C(X)$*, Math. Scand. 50 (1982), 44–54.
3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, Toronto, London, 1960.
4. L. Gillman and C. W. Kohls, *Convex and pseudoprime ideals in rings of continuous functions*, Math. Z. 72 (1960), 399–409.
5. K. R. Goodearl, *Von Neumann Regular Rings*, (Monographs Stud. Math. 4), Pitman Publishing, Ltd., London, San Francisco, 1979.
6. C. B. Huijsmans and B. de Pagter, *Ideal theory in f-algebras*, Trans. Amer. Math. Soc. 269 (1982), 225–245.
7. D. S. Passman, *The Algebraic Structure of Group Rings*, John Wiley and Sons, New York, London, 1977.
8. M. Smith, *Group algebras*, J. Algebra 18 (1971), 477–499.

DEPARTMENT OF MATHEMATICS AND STATISTICS
MEMORIAL UNIVERSITY OF NEWFOUNDLAND
ST. JOHN'S, NEWFOUNDLAND
CANADA A1B 3X7

AND

DEPARTMENT OF MATHEMATICS, STATISTICS AND
COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA B3H 4H8