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A GAP SERIES WITH GROWTH CONDITIONS AND ITS APPLICATIONS

SHINJI YAMASHITA

Abstract.

We shall give three applications of a specified Hadamard gap series $h(z) = \sum_{k=1}^{\infty} z^{n_k}$ satisfying some growth conditions on h and h' in $D = \{|z| < 1\}$. A typical one is the proof of the existence of a quasiconformal homeomorphism f from $\{|z| \leqq \infty\}$ onto itself, holomorphic in D, which is "smooth" on $\bar{D}$, yet not semiconformal at any point of ∂D.

1. Introduction.

Let $F(r)$ be a continuous and strictly increasing function of r, $0 \leqq r < 1$, such that $F(0) = 1$ and $F(r) \to +\infty$ as $r \to 1$. For each $q > e$ we can construct an Hadamard gap series $h(z) = \sum_{k=1}^{\infty} z^{n_k}$ $(n_k/n_{k-1} \geqq q,\ k \geqq 2)$ such that

$$(1.1) \qquad |h(z)| \leqq F(|z|) \quad \text{for } |z| < 1;$$

$$(1.2) \qquad \liminf_{k \to \infty} [\min \{(1-|z|)|h'(z)|\ ;\ |z| = 1 - n_k^{-1}\}] \geqq e^{-1} - q^{-1};$$

$$(1.3) \qquad \sup_{|z|<1} (1-|z|^2)|h'(z)| \leqq 2(1+q^{-1}).$$

After the proof of the existence of h in Section 2, three applications will be proposed. First we construct a univalent (injective) holomorphic function in $D = \{|z| < 1\}$ nowhere semiconformal on $\partial D = \{|z| = 1\}$, which improves our former example (see [9]). We show that this function also proposes more information than A. J. Lohwater, G. Piranian, and W. Rudin's [5] concerning the Bloch–Nevanlinna conjecture.

The third application is on minimal surfaces in the space R^3. We construct a minimal surface whose "derivatives" satisfy a "good" growth condition, yet whose Gauss map is "bad". This improves our former result (see [10, Theorem 2]).

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2. The construction of h and its properties.

To choose n_k inductively, we pick up the increasing sequence $\{r_k\}$ with $F(r_k)=k+1$ $(k\geqq 0)$; obviously, $r_0=0$, $r_k<r_{k+1}<1$ $(k\geqq 0)$, and $r_k\to 1$ as $k\to\infty$. The sequence $\{n_k\}$ should then satisfy for $k\geqq 1$,

$$r_k^{n_k} \leqq 2^{-k}; \tag{2.1}$$

for $k\geqq 2$,

$$n_k > q(n_1+\dots+n_{k-1}); \tag{2.2}$$

and for $k\geqq 2$,

$$n_k(1-n_j^{-1})^{n_k} \leqq 2^{-k+1} \quad \text{for all } j,\ 1\leqq j<k. \tag{2.3}$$

First choose $n_1\geqq 2$ such that (2.1) for $k=1$ holds. Next, since $r_2<1$ and since

$$n(1-n_1^{-1})^n \to 0 \quad \text{as } n\to\infty,$$

there exists n_2 such that (2.1)~(2.3) for $k=2$ are true. Suppose that $n_1,\dots,n_{k-1}$ $(k\geqq 3)$ are chosen. Since for $j=1,\dots,k-1$,

$$n(1-n_j^{-1})^n \to 0 \quad \text{as } n\to\infty,$$

there exists n_k such that (2.1)~(2.3) for k are valid.

To prove (1.1) we note that, for each r, $0\leqq r<1$, there exists $j\geqq 1$ such that $r_{j-1}\leqq r<r_j$. Then, in view of (2.1) we have

$$\max_{|z|=r}|h(z)| = h(r) \leqq \sum_{k=1}^{\infty} r_j^{n_k} \leqq \sum_{k=1}^{j-1} r_j^{n_k}+\sum_{k=j}^{\infty} r_k^{n_k}$$

$$\leqq j-1+\sum_{k=j}^{\infty} 2^{-k} = j-1+2^{-j+1} \leqq j = F(r_{j-1}) \leqq F(r).$$

To prove (1.2) we set $R_j=1-n_j^{-1}$ $(j\geqq 1)$. Then, for $j\geqq 2$, we obtain, in view of (2.2),

$$a \equiv \sum_{k=1}^{j-1} n_k R_j^{n_k} \leqq \left(\sum_{k=1}^{j-1} n_k\right) R_j^{n_1} \leqq q^{-1} n_j R_j^{n_1},$$

and, in view of (2.3), we have

$$b \equiv \sum_{k=j+1}^{\infty} n_k R_j^{n_k} \leqq \sum_{k=j+1}^{\infty} 2^{-k+1} = 2^{-j+1}.$$

Consequently, for $|z|=R_j$ $(j\geqq 2)$ we have:

$$|zh'(z)| = \left|\sum_{k=1}^{\infty} n_k z^{n_k}\right| \geqq n_j R_j^{n_j} - a - b \geqq n_j(R_j^{n_j} - q^{-1}R_j^{n_1}) - 2^{-j+1}.$$

Therefore,

$$R_j \min_{|z|=R_j} (1-|z|)|h'(z)| \geqq R_j^{n_j} - q^{-1}R_j^{n_1} - n_j^{-1}2^{-j+1}$$

for $j \geqq 2$. The right-hand side tends to $e^{-1} - q^{-1}$ as $j \to \infty$, so that (1.2) follows.

For the proof of (1.3) we set $S = \max\{k;\ n_k \leqq n\}$ for $n \geqq n_1$. Then, for $n \geqq n_2$, we obtain by (2.2),

$$\sum_{n_k \leqq n} n_k = n_S + (n_1 + \ldots + n_{S-1}) < n_S(1+q^{-1}) \leqq n(1+q^{-1});$$

this is also true for $n_1 \leqq n < n_2$. Therefore, for $0 \leqq r < 1$,

$$r(1-r)^{-1} \sum_{k=1}^{\infty} n_k r^{n_k - 1} = \sum_{k=1}^{\infty} n_k r^{n_k} \sum_{k=0}^{\infty} r^k = \sum_{n=n_1}^{\infty} \left(\sum_{n_k \leqq n} n_k\right) r^n$$

$$\leqq (1+q^{-1}) \sum_{n=n_1}^{\infty} nr^n \leqq (1+q^{-1}) \sum_{n=1}^{\infty} nr^n = (1+q^{-1})r(1-r)^{-2}.$$

We thus obtain

$$(1-r) \sum_{k=1}^{\infty} n_k r^{n_k - 1} \leqq 1 + q^{-1},$$

which yields (1.3) on setting $|z| = r$.

LEMMA 1. *Neither* Re h *nor* Im h *has finite angular limit at any point of* ∂D.

A mapping μ from D into $\mathbf{C}^* = \{|z| \leqq \infty\}$ is said to have an angular limit $\mu(\zeta) \in \mathbf{C}^*$ at $\zeta \in \partial D$ if $\mu(z) \to \mu(\zeta)$ as $z \to \zeta$ within each triangular domain at ζ, namely, the interior of a triangle whose vertices are ζ and two points of D. We use the same terminology for μ from D into the two-point compactification $\{r;\ -\infty \leqq r \leqq +\infty\}$ of the real axis. "Finite $\mu(\zeta)$" means "$|\mu(\zeta)| \neq \infty$" in both cases.

Lemma 1 is an immediate consequence of [8, Theorem 8, p. 124], together with (1.2).

Particularly, *h has no finite angular limit*. This fact alone, follows from the result of K. G. Binmore [3, Corollary 1, p. 215] that h has no finite asymptotic value.

The other boundary properties of h will be described in Lemma 2 in Section 4.

3. Nowhere semiconformal functions.

Let f be a function holomorphic and univalent in D, and suppose that f has the angular limit $f(\zeta) \neq \infty$ at $\zeta \in \partial D$. Then f is said to be conformal at ζ if the function $\arg\{(f(z)-f(\zeta))/(z-\zeta)\}$ of z has a finite angular limit at ζ, while f is said to be semiconformal at ζ if the function $\{f(z)-f(\zeta)\}/\{(z-\zeta)f'(z)\}$ of z has the angular limit one at ζ; see [8, Introduction], for example. If f is conformal at ζ, then f is semiconformal at ζ; the converse is false. In [9] we constructed a quasiconformal homeomorphism f from $\mathbf{C}^*$ onto $\mathbf{C}^*$ such that, $\infty \notin f(\bar{D})(\bar{D}=D\cup\partial D)$, f is holomorphic in D, and f is not semiconformal at any point of ∂D. In the present section we shall improve this; see Corollary 1 below.

THEOREM 1. *Let* $0<A<1$, *and let* φ *be a continuous and strictly decreasing function for* $0\leqq\varrho_0<r<1$ *such that* $\varphi(r) \to 0$ *as* $r \to 1$. *Then we can construct a function* f *holomorphic in* D *satisfying the following*:

$$\sup_{z\in D}(1-|z|^2)|f''(z)/f'(z)| \leqq A; \tag{3.1}$$

$$\lim_{|z|\to 1}\varphi(|z|)|f'(z)| = 0; \tag{3.2}$$

$$\limsup_{r\to 1-0}(1-r)|f''(r\zeta)/f'(r\zeta)| > A/(3e) \quad \textit{for each } \zeta \in \partial D\,. \tag{3.3}$$

COROLLARY 1. *Given* φ *of Theorem* 1 *we can construct a quasiconformal homeomorphism* f_Q *from* $\mathbf{C}^*$ *onto* $\mathbf{C}^*$, *such that* $\infty \notin f_Q(\bar{D})$, f_Q *is holomorphic in* D, f_Q *is not semiconformal at any point of* ∂D, *and further* f_Q *satisfies the growth condition* (3.2) *in* D.

COROLLARY 2. *There exists a nonrectifiable quasicircle* $\{f(w);\, w \in \partial D\}$ *in* $\mathbf{C}=\{|z|<\infty\}$ [7, p. 286] *having no tangent at any point, yet satisfying*

$$\sup_{|w_1-w_2|\leqq t}|f(w_1)-f(w_2)| = o(t\log 1/t) \quad \textit{as } t \to +0\,. \tag{3.4}$$

It follows from (3.4) that, given $\varepsilon>0$ we may choose δ, $0<\delta<1$, such that

$$|f(w_1)-f(w_2)| \leqq \varepsilon|w_1-w_2|\log\frac{1}{|w_1-w_2|}$$

for $w_1, w_2 \in \partial D$ with $|w_1-w_2|<\delta$.

PROOF OF THEOREM 1. There exists ϱ_1, $\varrho_0<\varrho_1<1$, such that $\varphi(\varrho_1)<1$. We define

$$\begin{aligned}\Phi(r) &= \big(\varphi(\varrho_1)-1\big)\varrho_1^{-1}r+1, & 0 \leqq r \leqq \varrho_1;\\ &= \varphi(r), & \varrho_1 < r < 1\,.\end{aligned}$$

We fix $q>e$ so large that

(3.5) $c(e^{-1}-q^{-1}) > A/(3e)$, where $c = A2^{-1}(1+q^{-1})^{-1}$.

Then the function

$$F_1(r) = 1-(2c)^{-1}\log\Phi(r), \quad 0 \leqq r < 1\,,$$

is continuous and increasing with $F_1(0)=1$ and $F_1(r)\to\infty$ as $r\to 1$.

For $F=F_1$ and q we consider h and we set

$$f(z) = \int_0^z \exp\big(ch(w)\big)dw, \quad z\in D\,,$$

so that $\log f'=ch$ in D. In particular, we have

(3.6) $$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{with } a_n \geqq 0 \ (n\geqq 1)\,.$$

Now, (3.1) follows from (1.3). It follows from (1.1) that

$$\log|f'(z)| \leqq c-2^{-1}\log\Phi(|z|)\,,$$

whence

$$\Phi(|z|)|f'(z)| \leqq e^c\Phi(|z|)^{1/2}\,.$$

We thus obtain (3.2). Finally, (3.3) is a consequence of (1.2), together with (3.5).

Proof of Corollary 1. Let f be as in Theorem 1. First it follows from (3.1) that f can be extended to $\mathbf{C}^*$ so that the resulting function f_Q is quasiconformal; see [2, Theorems 3.1, 4.1, Corollary 4.1 and formula (4.2)]. After a calculation for the normalized f, $\{f-f(0)\}/f'(0)$, we obtain

$$\begin{aligned}f_Q(z) &= f(z), & z\in\bar{D};\\ &= f(1/\bar{z})+(z-1/\bar{z})f'(1/\bar{z}), & z\in\mathbf{C}^*\setminus\bar{D}\,,\end{aligned}$$

where we use the same notation f for the extension to $\bar{D}$. We note that f_Q is $(1+A)/(1-A)$-quasiconformal.

Particularly, f is continuous and univalent on $\bar{D}$. Next, f is semiconformal at $\zeta\in\partial D$ if and only if $(\zeta-z)f''(z)/f'(z)$, or, equivalently, $(1-|z|^2)|f''(z)/f'(z)|$ has the angular limit zero at ζ; see for example, [8, Theorem 1, p. 120 et ff.]. Consequently, f is not semiconformal at any point of ∂D by (3.3).

PROOF OF COROLLARY 2. Let $\varphi(r)=\{-\log(1-r)\}^{-1}$ for $1/2 \leqq r<1$, and apply Theorem 1 to φ to obtain f with the aid of h, namely, $h=c^{-1}\log f'$. Then, by the obvious "small *oh*" modification of [4, Theorem 5.2, p. 76], together with (3.2), f on ∂D satisfies (3.4). If our quasicircle is rectifiable, then $f' \in H^1$ [7, Lemma 10.7, p. 319], whence f' has finite angular limit at almost every point of ∂D, and the limits are nonzero almost everywhere [7, Theorem 10.14, p. 325]. Then h has finite angular limit at a.e. point of ∂D; this contradicts the property of h described in Section 2. Since f is nowhere semiconformal on ∂D, f is nowhere conformal on ∂D, so that our quasicircle has no tangent at any point by E. Lindelöf's theorem [7, Theorem 10.4, p. 302].

4. Further properties of f in Theorem 1.

Lohwater, Piranian, and Rudin [5, Theorem], constructed, for a suitably increasing sequence $\{m_p\}$ of natural numbers, a function of z,

$$f_0(z) = \int_0^z \exp\left\{(1/2)\sum_{p=1}^{\infty} w^{m_p}\right\}dw = \sum_{n=1}^{\infty} b_n z^n \quad (b_n \geqq 0, n \geqq 0),$$

which is continuous and univalent on $\bar{D}$, holomorphic in D, and for a.e. point $\zeta \in \partial D$,

$$\begin{aligned}(4.1)\qquad \infty &= \limsup_{r\to 1-0} |f_0'(r\zeta)| = 1/\liminf_{r\to 1-0} |f_0'(r\zeta)| \\ &= \limsup_{r\to 1-0} \arg f_0'(r\zeta) = -\liminf_{r\to 1-0} \arg f_0'(r\zeta);\end{aligned}$$

further, $\sum b_n < \infty$. We shall show that our f in Theorem 1, where some properties are described in Corollary 1, satisfies more detailed limit conditions than f_0.

For each $\zeta \in \partial D$ we let $\Lambda(\zeta)$ be the set of all continuous curves λ: $z=z(t) \in D$, $0 \leqq t < 1$, such that $z(t) \to \zeta$ as $t \to 1$, and λ has a chord of ∂D ending at ζ, as a tangent at ζ. We shall show

THEOREM 2. *For our f in Theorem* 1, *expressed by* (3.6) *we have* $\sum a_n < \infty$ *and the following*: *For almost every $\zeta \in \partial D$ and for all $\lambda \in \Lambda(\zeta)$, we have, as* $z \to \zeta$, $z \in \lambda$,

$$\begin{aligned}(4.2)\qquad \infty &= \limsup |f'(z)| = 1/\liminf |f'(z)| \\ &= \limsup \arg f'(z) = -\liminf \arg f'(z).\end{aligned}$$

Lohwater, Piranian, and Rudin's proof of (4.1) is not available for our function. They do not refer to the growth condition (3.2).

As to the conditions (4.2) let us consider the corresponding ones for $h=c^{-1}\log f'$.

LEMMA 2. *Let h be as in Section 1. Then for almost every $\zeta \in \partial D$ and for all $\lambda \in \Lambda(\zeta)$, we have, as $z \to \zeta$, $z \in \lambda$,*

$$\begin{aligned}(4.3)\qquad \infty &= \limsup \operatorname{Re} h(z) = -\liminf \operatorname{Re} h(z)\\ &= \limsup \operatorname{Im} h(z) = -\liminf \operatorname{Im} h(z).\end{aligned}$$

For the proof, we first let E be the set of $\zeta \in \partial D$ where $\limsup \operatorname{Re} h(z) < \infty$ as $z \to \zeta$, $z \in \lambda$, for a $\lambda \in \Lambda(\zeta)$. Then we can find $m>0$, and t_0, $0<t_0<1$, such that

$$(4.4)\qquad \operatorname{Re} h(z(t)) < m \quad \text{for } t_0<t<1 ,$$

where λ: $z=z(t)$, $0\leqq t<1$. On the other hand, it follows from (1.3) that

$$|h(z_1)-h(z_2)| \leqq 2(1+q^{-1})\tanh^{-1}\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|, \qquad z_1, z_2 \in D .$$

Then, for each w in the Apollonius disk

$$\Delta(t) = \left\{\left|\frac{w-z(t)}{1-\bar{z}(t)w}\right| < \tanh\left[2^{-1}(1+q^{-1})^{-1}\right]\right\},$$

we have $|h(w)-h(z(t))|\leqq 1$, $t_0<t<1$, which, combined with (4.4), yields

$$\operatorname{Re} h(w) \leqq m+1 \quad \text{for all } w \in \Delta(t) \ \ (t_0<t<1) .$$

Now, as $t \to 1$, the disks $\Delta(t)$ sweep a domain which contains a triangular domain $T(\zeta)$ at ζ. Then, $\overline{h(T(\zeta))} \neq \mathbf{C}^*$, or, ζ is not a Plessner point of h [7, p. 323]. It then follows from the Plessner theorem [7, Theorem 10.13, p. 324] that h has a finite angular limit at a.e. point of E. Remembering the property of h, we observe that E must be of measure zero, or, the assertion for $\operatorname{Re} h$ holds.

By the similar arguments we observe that the remaining three conditions in (4.3) hold for a.e. point $\zeta \in \partial D$ and all $\lambda \in \Lambda(\zeta)$.

PROOF OF THEOREM 2. The boundary properties of f' immediately follow from Lemma 2. It remains to be proved that $\sum a_n < \infty$. Suppose the contrary that for each $K>0$ there exists a natural number N such that $s_n=a_1+\ldots+a_n>K$ for all $n>N$. We let $0\leqq r<1$ to obtain

$$f(r) = (1-r)\sum_{n=1}^{\infty} s_n r^n \geqq (1-r)\sum_{n=N+1}^{\infty} s_n r^n \geqq K r^{N+1},$$

whence $f(1) = \lim_{r\to 1} f(r) \geqq K$; this is a contradiction.

REMARK. The series $\sum a_n \zeta^n$ absolutely converges on ∂D. By the celebrated Abel's continuity theorem on complex power series we have

$$f(\zeta) = \lim_{r\to 1-0} f(r\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n \quad \text{on } \partial D .$$

5. Minimal surfaces.

A nonconstant map $x: D \to \mathsf{R}^3$ is called a minimal surface if each component x_k of $x = (x_1, x_2, x_3)$ is harmonic and

$$\sum_{k=1}^{3} (\partial x_k/\partial w)^2 \equiv 0 \quad \text{in } D ,$$

where $2\partial/\partial w = \partial/\partial u - i\partial/\partial v$, $w = u + iv \in D$. Suppose that $\partial x_1/\partial w \not\equiv i\partial x_2/\partial w$. Then the meromorphic function

$$G = (\partial x_3/\partial w)/(\partial x_1/\partial w - i\partial x_2/\partial w)$$

is called the Gauss map of x. See [6] for basic facts on minimal surfaces.

In [10, Theorem 2] we observed that, for each α, $0 < \alpha < 1$, there exists a minimal surface x which is α-Lipschitz continuous on $\bar{D}$, yet $|G|$ has no angular limit at almost every point of ∂D, and the range of values $R(G, \zeta) = \mathsf{C}\setminus\{0\}$, at each $\zeta \in \partial D$. Here, $\gamma \in R(G, \zeta)$ if and only if $G(z_n) = \gamma$ for each z_n of a certain sequence $\{z_n\}$ with $z_n \to \zeta$. This is a consequence of the following on setting $\varphi(r) = (1-r)^{1-\alpha}$, $0 \leqq r < 1$; see [4, Theorem 3.1, p. 74].

THEOREM 3. *Let φ be a continuous and strictly decreasing function for $0 \leqq \varrho_0 < r < 1$ such that $\varphi(r) \to 0$ as $r \to 1$. Then there exists a minimal surface $x: D \to \mathsf{R}^3$ such that*

$$\lim_{|w|\to 1} \varphi(|w|) \sum_{k=1}^{3} |\partial x_k(w)/\partial w| = 0 . \tag{5.1}$$

Furthermore, at almost every $\zeta \in \partial D$ the modulus $|G|$ of the Gauss map G of x has no finite angular limit, and at each $\zeta \in \partial D$, $R(G, \zeta) = \mathsf{C}\setminus\{0\}$.

Thus, for $\varphi(r) = \{-\log(1-r)\}^{-1}$, $1/2 \leqq r < 1$, a similar consideration as in the proof of Corollary 2 is possible. In this case, x admits a continuous extension to $\bar{D}$, and satisfies on ∂D,

$$\sup_{|w_1-w_2|\leqq t} |x(w_1)-x(w_2)| = o(t\log 1/t) \quad \text{as } t \to +0 .$$

The set of $\zeta \in \partial D$, where G has the angular limit ∞ is of measure zero.

PROOF OF THEOREM 3. Fix $\beta>1$, and choose ϱ_2, $\varrho_0<\varrho_2<1$, such that $\varphi(\varrho_2)<e^{-2\beta}$. Set

$$\begin{aligned} F_2(r) &= 1-\{(2\beta)^{-1}\log\varphi(\varrho_2)+1\}\varrho_2^{-1}r, & 0 \leqq r \leqq \varrho_2; \\ &= -(2\beta)^{-1}\log\varphi(r), & \varrho_2 < r < 1 . \end{aligned}$$

For $F=F_2$ and $q>4$ we construct h. Let $x_k=\operatorname{Re} f_k (k=1,2,3)$, where

$$\begin{aligned} f_1' &= (1/2)(1-e^{2h}) , \\ f_2' &= (i/2)(1+e^{2h}) , \\ f_3' &= e^h , \end{aligned} \tag{5.2}$$

in D. Then, $x\colon D \to \mathrm{R}^3$ with the Gauss map $G=e^h$ is the requested. Actually, (5.1) follows from

$$|e^{2h(w)}| \leqq e^{2|h(w)|} \leqq \varphi(|w|)^{-1/\beta} \quad \text{for } \varrho_2 < |w| < 1 . \tag{5.3}$$

Apparently, the set of points on ∂D where $|G|=\exp(\operatorname{Re} h)$ has the angular limit zero is of measure zero. The set of points on ∂D where $|G|$ has the nonzero and finite angular limit is empty by Lemma 1. Finally, by J. M. Anderson's theorem [1, Theorem 2, p. 248], $R(h,\zeta)=\mathrm{C}$ at each $\zeta \in \partial D$, so that $R(G,\zeta)=\mathrm{C}\setminus\{0\}$ at each $\zeta \in \partial D$.

REMARK. By the similar argument as in the proof of (4.2) in Theorem 2, we have, for almost every $\zeta \in \partial D$,

$$\limsup_{\substack{z\to\zeta \\ z\in\lambda}} |G(z)| = \infty = 1/\liminf_{\substack{z\to\zeta \\ z\in\lambda}} |G(z)| \quad \text{for all } \lambda \in \Lambda(\zeta) .$$

Since $f_3'=e^h$ never vanishes, our minimal surface is regular. If

$$\int_{\varrho_0}^1 \varphi(r)^{-2/\beta}\, dr < \infty ,$$

then the area of our minimal surface is finite because

$$\text{Area} = \iint_D \sum_{k=1}^3 (\partial x_k/\partial u)^2\, dudv \leqq \iint_D \sum_{k=1}^3 |f_k'|^2\, dudv < \infty$$

by (5.2) and (5.3). For example, we let $\beta>2$ for

$$\varphi(r) = \{-\log(1-r)\}^{-1}, \quad 1/2 \leqq r < 1 .$$

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DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
FUKASAWA, SETAGAYA, TOKYO 158
JAPAN