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ON THE INSTABILITY OF HAUSDORFF CONTENT

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Let E be a subset of Euclidean space R^n and α a non-negative real number. We study all coverings of E with a countable number of open balls B_j with radii r_j and define the α-dimensional Hausdorff content $H_\alpha(E)$ as

$$\inf \sum r_j^\alpha$$

for all such coverings. A property, which holds for all points on $E \setminus E_1$ with $H_\alpha(E_1)=0$, is said to hold H_α-a.e. on E. Denote by $B(x,\delta)$ the open ball $\{y;\ |y-x|<\delta\}$.

O'Farrell has conjectured in [7] that an "instability" result might hold for the content H_α. This paper is devoted to prove the following "instability" theorem:

THEOREM. *Let E be a set in R^n and α and β constants such that $0<\alpha<\beta$. Then H_β-a.e. on R^n one of the following relations holds:*

$$\limsup_{\delta \to 0} \frac{H_\alpha(E \cap B(x,\delta))}{\delta^\alpha} \geqq \frac{1}{6^\alpha}$$

or

$$\lim_{\delta \to 0} \frac{H_\alpha(E \cap B(x,\delta))}{\delta^\beta} = 0 .$$

Similar theorems are true for the Lebesgue measure m (see Stein [8]), the analytic capacity (see Vituškin [9]) and the Riesz capacities (see Fernström [5]).

The main tool to prove the theorem is to use the fractional maximal function to define a capacity, which is equivalent to Hausdorff content. It is then possible to use the technique used in Fernström [5] for Riesz capacities to prove our theorem. The idea of using the fractional maximal function to define an equivalent capacity to Hausdorff content can be found in Adams [1].

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Proof of Theorem. The proof of the theorem will be split in a series of lemmas.

Since $H_\alpha(E)=0$ for all sets E when $\alpha>n$, the theorem is trivial when $b>n$. We shall therefore in the rest of the paper always assume that $0<\alpha\leqq n$ and $0<\beta\leqq n$.

A set function C on R^n is said to be subadditiv if $C(E)\leqq\sum_{i=1}^{\infty} C(E_i)$, where $E=\bigcup_{i=1}^{\infty} E_i$.

The set function C is increasing if

$$E_1 \subset E_2 \;\Rightarrow\; C(E_1) \leqq C(E_2)\,.$$

Let C_α denote a subadditiv, increasing set function on R^n such that

$$C_\alpha\big(B(x,\delta)\big) = A(C_\alpha)\delta^\alpha\,,$$

where $A(C_\alpha)$ is independent of x and δ. It is easy to see that H_α is subadditiv, increasing and $H_\alpha\big(B(x,\delta)\big)=\delta^\alpha$.

We begin with a Vitali covering lemma, which is proved as lemma 1.6 in Stein [8].

Lemma 1. *Let E be a subset of R^n which is covered by the union of a family of balls $\{B_\gamma\}$, of bounded diameter. Let $\varepsilon>0$. Then from this family of balls we can select a disjount subsequence $\{B_j\}$ so that*

$$C_\alpha(E) \leqq (3+\varepsilon)^\alpha \sum C_\alpha(B_j)\,.$$

If $f\in L^1(\mathsf{R}^n)$ we define the fractional maximal function $M_\alpha f$ as

$$M_\alpha f(x) = \sup_{\delta>0}\delta^{-\alpha}\int_{B(x,\delta)} |f(y)|\,dy\,.$$

Lemma 2. *Let $f\in L^1(\mathsf{R}^n)$ and $d>0$. Then*

$$C_\alpha\big(\{x\,;\; M_\alpha f(x)>d\}\big) \leqq \frac{3^\alpha}{d}A(C_\alpha)\|f\|_1, \quad \textit{where } \|f\|_1 = \int_{\mathsf{R}^n}|f(y)|\,dy\,.$$

Proof. Set $E=\{x;\, M_\alpha f(x)>d\}$.

For every x, $x\in E$, there is a $\delta(x)$, $\delta(x)>0$, so that

$$\delta(x)^{-\alpha}\int_{B(x,\delta(x))}|f(y)|\,dy > d\,.$$

This gives $\delta(x)^\alpha<1/d\|f\|_1$.

Thus $\delta(x)$ are uniformly bounded for $x\in E$. We also have

$$E \subset \bigcup_{x\in E} B\big(x,\delta(x)\big)\,.$$

Let $\varepsilon > 0$. From Lemma 1 we see that there is a disjoint sequence of balls $B(x_j, \delta(x_j))$ so that

$$C_\alpha(E) \leqq (3+\varepsilon)^\alpha \sum_j C_\alpha\big(B(x_j, \delta(x_j))\big) = (3+\varepsilon)^\alpha \sum_j A(C_\alpha)\delta(x_j)^\alpha$$

$$< (3+\varepsilon)^\alpha A(C_\alpha) \sum_j \frac{1}{d} \int_{B(x_j, \delta(x_j))} |f(y)|\, dy \leqq (3+\varepsilon)^\alpha A(C_\alpha) \frac{1}{d} \|f\|_1 .$$

Since ε, $\varepsilon > 0$, is arbitrary the lemma follows.

We need the following lemma which is stated for Riesz capacities in Bagby–Ziemer [2].

LEMMA 3. *Let $f \in L^1(\mathrm{R}^n)$. Then*

i) *if* $0 < \alpha < n$

$$\lim_{\delta \to 0} \delta^{-\alpha} \int_{B(x,\delta)} |f(y)|\, dy = 0 \quad C_\alpha\text{-a.e. } on \ \mathrm{R}^n$$

ii)
$$\lim_{\delta \to 0} \delta^{-n} \int_{B(x,\delta)} |f(y) - f(x)|\, dy = 0 \quad C_n\text{-a.e. } on \ \mathrm{R}^n .$$

PROOF. The case $\alpha = n$ is proved in Stein [8]. We give the proof for $0 < \alpha < n$. Set

$$\Omega f(x) = \limsup_{\substack{\delta \to 0 \\ \circ}} \delta^{-\alpha} \int_{B(x,\delta)} |f(y)|\, dy .$$

If g is a continuous function with compact support, it is easy to see that

$$\Omega g(x) \equiv 0 .$$

Let $\varepsilon > 0$. There is a continuous function with compact support so that

$$f = g + h \quad \text{and} \quad \|h\|_1 < \varepsilon .$$

This gives

$$\Omega f(x) \leqq \Omega g(x) + \Omega h(x) = \Omega h(x) \leqq M_\alpha h(x) .$$

Let n be a positive integer. Lemma 2 now gives

$$C_\alpha(\{x \ ; \ \Omega f(x) > 1/n\}) \leqq C_\alpha(\{x \ ; \ M_\alpha h(x) > 1/n\})$$
$$\leqq n 3^\alpha A(C_\alpha) \|h\|_1 < n 3^\alpha A(C_\alpha) \varepsilon .$$

Thus $C_\alpha(\{x; \Omega f(x) > 1/n\}) = 0$.

The subadditivity of C_α now gives

$$C_\alpha(\{x\ ;\ \Omega f(x)>0\}) \leqq C_\alpha\left(\bigcup_{n=1}^{\infty} (\{x\ ;\ \Omega f(x)>1/n\})\right)$$

$$\leqq \sum_{n=1}^{\infty} C_\alpha(\{x\ ;\ \Omega f(x)>1/n\}) = 0\ ,$$

which proves the lemma.

The following lemma is the crucial step in the proof of the theorem.

LEMMA 4. *Let $f \in L^1(\mathsf{R}^n)$ and $\alpha<n$. Suppose that $M_\alpha f(x)>1$ for all x, $x \in E$. Set*

$$E_\beta = \left\{x\ ;\ \limsup_{\delta\to 0} \frac{C_\alpha(E \cap B(x,\delta))}{\delta^\beta} > 0\right\}.$$

Then $M_\alpha f(x)\geqq 1$ C_β-a.e. on $E \cup E_\beta$.

PROOF. Let $x_0 \in E \cup E_\beta$. We may assume that $x_0 \notin E$. That is

$$\limsup_{\delta\to 0} \frac{C_\alpha(E \cap B(X_0,\delta))}{\delta^\beta} > 0\ .$$

Using lemma 3 we may also assume that

$$\lim_{\delta\to 0} \delta^{-\beta} \int_{B(x_0,\delta)} |f(y)|\, dy = 0 \qquad \text{for } \beta<n$$

and

$$\lim_{\delta\to 0} \delta^{-n} \int_{B(x_0,\delta)} |f(y)-f(x_0)|\, dy \quad \text{for } \beta=n\ .$$

Suppose now that the lemma is not true for x_0. Then there is a constant k, $k\geqq 1$, such that

$$M_\alpha f(x_0) \leqq \left(\frac{k}{k+1}\right)^\alpha .$$

For every x, $x \in E$, we choose a number $\delta(x)$, $\delta(x)>0$, so that

$$\delta(x)^{-\alpha} \int_{B(x,\delta(x))} |f(y)|\, dy > 1\ .$$

We get

$$M_\alpha f(x_0) \geqq (\delta(x)+|x-x_0|)^{-\alpha} \int_{B(x_0,\delta(x)+|x-x_0|)} |f(y)|\,dy$$
$$\geqq \left(\frac{\delta(x)}{\delta(x)+|x-x_0|}\right)^\alpha \delta(x)^{-\alpha} \int_{B(x,\delta(x))} |f(y)|\,dy$$
$$> \left(\frac{\delta(x)}{\delta(x)+|x-x_0|}\right)^\alpha .$$

Since

$$M_\alpha f(x_0) \leqq \left(\frac{k}{k+1}\right)^\alpha ,$$

it is easy to see that $\delta(x) \leqq k|x-x_0|$. Let χ_δ denote the characteristic function for the set $B(x_0, (k+1)\delta)$.

We now split the proof into two parts.

First let $\beta < n$. Set $F_\delta(x) = \chi_\delta |f(x)|$.

Now let $x \in E \cap B(x_0, \delta)$. If we use that $\delta(x) \leqq k|x-x_0|$, that is $\delta(x) \leqq k\delta$, we get

$$1 < \delta(x)^{-\alpha} \int_{B(x,\delta(x))} |f(y)|\,dy = \delta(x)^{-\alpha} \int_{B(x,\delta(x))} F_\delta(y)\,dy \leqq M_\alpha F_\delta(x) .$$

Lemma 2 gives

$$C_\alpha(E \cap B(x_0,\delta)) \leqq C_\alpha(\{x \; ; \; M_\alpha F_\delta(x) > 1\}) \leqq 3^\alpha A(C_\alpha) \|F_\delta\|_1 .$$

Finally we get

$$\frac{C_\alpha(E \cap B(x_0,\delta))}{\delta^\beta} \leqq 3^\alpha A(C_\alpha) \frac{1}{\delta^\beta} \int_{B(x_0,(k+1)\delta)} |f(y)|\,dy .$$

But this contradicts the facts that $x_0 \in E_\beta$ and

$$\lim_{\delta \to 0} \delta^{-\beta} \int_{B(x_0,\delta)} |f(y)|\,dy = 0 .$$

This proves the lemma for $\beta < n$.

If $\beta = n$ we must modify the proof. Set

$$g(x) = \begin{cases} \lim_{\delta\to 0} m(B(x,\delta))^{-1} \int_{B(x,\delta)} f(y)\,dy, & \text{if the limit exists.} \\ 0 & \text{elsewhere.} \end{cases}$$

We observe that $g(x)=f(x)$ a.e. (see Stein [8]), Set

$$G_\delta(x) = \chi_\delta(x)[f(x)-g(x_0)] .$$

$x \in E \cap B(x_0,\delta)$ gives

$$\begin{aligned} M_\alpha G_\delta(x) &\geqq \delta(x)^{-\alpha} \int_{B(x,\delta(x))} \chi_\delta(y)|f(y)-g(x_0)|dy \\ &= \delta(x)^{-\alpha} \int_{B(x,\delta(x))} |f(y)-g(x_0)|\,dy \\ &\geqq \delta(x)^{-\alpha} \int_{B(x,\delta(x))} |f(y)|\,dy - |g(x_0)|\delta(x)^{-\alpha} m\big(B(x,\delta(x))\big) \\ &> 1-|g(x_0)|\delta(x)^{-\alpha} m\big(B(x,\delta(x))\big) . \end{aligned}$$

Thus there is a δ_0, $\delta_0>0$, so that

$$M_\alpha 2G_\delta(x) > 1 \quad \text{for all } x \in E \cap B(x_0,\delta) \quad \text{if } \delta<\delta_0 .$$

The proof now proceeds exactly as for $\beta<n$.

We are going to define a set function $\tilde{H}_\alpha$, which we shall prove is equivalent to H_α. Let E be a set in R^n. The function $\tilde{H}_\alpha$ is defined by

$$\tilde{H}_\alpha(E) = \inf\{\|f\|_1 \; ; \; f \in L^1(\mathsf{R}^n) \text{ and } M_\alpha f(x)>1 \text{ on } E\} .$$

If $\{\|f\|_1; f \in L^1(\mathsf{R}^n) \text{ and } M_\alpha f(x)>1 \text{ on } E\}=\emptyset$, we set $\tilde{H}_\alpha(E)=\infty$. It is immediate that $\tilde{H}_\alpha$ is increasing.

LEMMA 5. *$\tilde{H}_\alpha$ is subadditive.*

PROOF. Let $E=\bigcup_{i=1}^\infty E_i$. We may assume that $\sum_{i=1}^\infty \tilde{H}_\alpha(E_i)<\infty$. Let $\varepsilon>0$. Choose $f_i \in L^1(\mathsf{R}^n)$ such that $M_\alpha f_i(x)>1$ on E_i and

$$\|f_i\|_1 \leqq \tilde{H}_\alpha(E_i)+\varepsilon 2^{-i}, \quad i=1,2,3,\ldots .$$

Set $f(x)=\sup|f_i(x)|$. We get

$$M_\alpha f(x) > 1 \text{ on } E_i, \quad i=1,2,3,\ldots$$

Thus

$$\begin{aligned} \tilde{H}_\alpha(E) &\leqq \|f\|_1 \leqq \int \sup|f_i(x)|\,dx \leqq \int \sum_{i=1}^\infty |f_i(x)|\,dx \\ &= \sum_{i=1}^\infty \int |f_i(x)|\,dx \leqq \sum_{i=1}^\infty \big(\tilde{H}_\alpha(E_i)+\varepsilon 2^{-i}\big) \leqq \sum_{i=1}^\infty \tilde{H}_\alpha(E_i)+\varepsilon , \end{aligned}$$

which gives the lemma.

We use the following notation:

$$\tilde{H}_\alpha(B(0,1)) = B_\alpha .$$

Since we are going to need to estimate B_α, the following lemma will be useful.

LEMMA 6. $0 < B_\alpha \leqq 1$.

PROOF. Let $\varepsilon > 0$. Denote by φ_ε a non-negative continuous function with support in $B(0,\varepsilon)$ so that $\int \varphi_\varepsilon(y)\,dy = 1$.

For $x \in B(0,1)$ we get

$$M_\alpha(1+\varepsilon)^{\alpha+1}\varphi_\varepsilon(x) \geqq \frac{1}{(1+\varepsilon)^\alpha}\int_{B(x,1+\varepsilon)} (1+\varepsilon)^{\alpha+1}\varphi_\varepsilon(y)\,dy = 1+\varepsilon > 1 .$$

Thus

$$\tilde{H}_\alpha(B(0,1)) \leqq \|(1+\varepsilon)^{\alpha+1}\varphi_\varepsilon\|_1 \leqq (1+\varepsilon)^{\alpha+1} ,$$

which gives $B_\alpha \leqq 1$.

Now let $f \in L^1(\mathrm{R}^n)$ so that $M_\alpha f(x) > 1$ on $B(0,1)$. We may assume that $\|f\|_1 \leqq 2$.

If $\delta^\alpha \geqq 2$ we get

$$\delta^{-\alpha}\int_{B(x,\delta)} |f(y)|\,dy \leqq \tfrac{1}{2}\|f\|_1 \leqq 1 .$$

Set $\delta_0 = 2^{1/\alpha}$. Let $z \in B(0,1)$. Then

$$M_\alpha f(z) = \sup_{0<\delta<\delta_0} \delta^{-\alpha}\int_{B(z,\delta)} |f(y)|\,dy .$$

Thus

$$M_\alpha f(z) \leqq \sup_{\delta>0} \delta^{-n}\int_{B(z,\delta)} \delta_0^{n-\alpha}|f(y)|\,dy = M_n\delta_0^{n-\alpha}f(z) .$$

This gives

$$B(0,1) \subset \{x \; ; \; M_n\delta_0^{n-\alpha}f(x) > 1\} .$$

Theorem 1.3 in Stein [8] finally gives

$$m(B(0,1)) \leqq m(\{x \; ; \; M_n\delta_0^{n-\alpha}f(x) > 1\}) \leqq A\delta_0^{n-\alpha}\|f\|_1 ,$$

where A is a constant. Thus

$$B_\alpha \geqq m(B(0,1))A^{-1}\delta_0^{\alpha-n} > 0 ,$$

which proves the lemma.

The following lemma shows that $\tilde{H}_\alpha$ is of "C_α-type".

LEMMA 7. $\tilde{H}(B(x,\delta))=B_\alpha\delta^\alpha$.

Since the proof is only a simple change of variables it is omitted.

LEMMA 8. *Let $\alpha<n$ and let E be a set in $\mathbf{R}^n$. Set*

$$E_\beta = \left\{x\,;\, \limsup_{\delta\to 0} \frac{\tilde{H}_\alpha(E\cap B(x,\delta))}{\delta^\beta} > 0\right\}.$$

Then there is a set $O_{x,\delta}$ such that $C_\beta(O_{x,\delta})=0$ and

$$\tilde{H}_\alpha(E\cap B(x,\delta)) = \tilde{H}_\alpha\big((E\cup(E_\beta\setminus O_{x,\delta}))\cap B(x,\delta)\big).$$

PROOF. Fix x and δ. Choose $f_j\in L^1(\mathbf{R}^n)$, $j=1,2,3,\ldots$, so that $M_\alpha f_j(x)>1$ on $E\cap B(x,\delta)$ and $\|f_j\|_1\leqq\tilde{H}_\alpha(E\cap B(x,\delta))+1/j$.

It is easy to see that

$$E_\beta\cap B(x,\delta)\subset(E\cap B(x,\delta))_\beta\,.$$

Lemma 4 gives that there is a set O_j, $C_\beta(O_j)=0$ and

$$M_\alpha f(x)\geqq 1 \quad\text{on } (E\cap B(x,\delta))\cup\big((E_\beta\cap B(x,\delta))\setminus O_j\big)\,.$$

That is,

$$M_\alpha f(x)\geqq 1 \quad\text{on } (E\cup(E_\beta\setminus O_j))\cap B(x,\delta)\,.$$

Set $O_{x,\delta}=\bigcup_{j=1}^\infty O_j$. We get $C_\beta(O_{x,\delta})=0$. Let $\varepsilon>0$. Then

$$M_\alpha(1+\varepsilon)f_j(x)>1 \quad\text{on } (E\cup(E_\beta\setminus O_{x,\delta}))\cap B(x,\delta)\,.$$

Thus

$$\tilde{H}_\alpha\big((E\cup(E_\beta\setminus O_{x,\delta}))\cap B(x,\delta)\big)\leqq(1+\varepsilon)\|f_j\|_1$$

$$\leqq(1+\varepsilon)\tilde{H}(E\cap B(x,\delta))+\frac{1+\varepsilon}{j}\,.$$

Since ε, $\varepsilon>0$, and j, $j=1,2,3,\ldots$, can be chosen arbitrarily, we get

$$\tilde{H}_\alpha(E\cap B(x,\delta))\leqq\tilde{H}_\alpha(E\cup(E_\beta\setminus O_{x,\delta})\cap B(x,\delta))\leqq\tilde{H}_\alpha(E\cap B(x,\delta))\,,$$

which proves the lemma.

We are now ready to compare H_α and $\tilde{H}_\alpha$.

LEMMA 9. $\tilde{H}_\alpha(E) \leqq B_\alpha H_\alpha(E)$.

PROOF. Let $\{B_j\}$ be a sequence of balls with radii r_j so that

$$E \subset \bigcup_{j=1}^{\infty} E_j .$$

We get

$$\tilde{H}_\alpha(E) \leqq \sum_{j=1}^{\infty} \tilde{H}_\alpha(B_j) = B_\alpha \sum_{j=1}^{\infty} r_j^\alpha .$$

Thus $\tilde{H}_\alpha(E) \leqq B_\alpha H_\alpha(E)$.

LEMMA 10. $H_\alpha(E) \leqq 3^\alpha \tilde{H}_\alpha(E)$.

PROOF. We may assume that $\tilde{H}_\alpha(E) < \infty$. Choose $f \in L^1(\mathsf{R}^n)$ so that

$$M_\alpha f(x) > 1 \text{ on } E .$$

Lemma 2 gives

$$H_\alpha(\{x \; ; \; M_\alpha f(x) > 1\}) \leqq 3^\alpha \|f\|_1 .$$

Thus $H_\alpha(E) \leqq 3^\alpha \|f\|_1$.

Let E be a subset of R^n and denote by $\delta(s)$ the radius of the ball S. We use the following notation:

$$\Delta_\alpha(x, E) = \limsup_{\delta(S) \to 0} \frac{H_\alpha(E \cap S)}{\delta(S)^\alpha} ,$$

where $x \in S$.

Notice that it is not needed that x is the centre of the ball S.

LEMMA 11. $H_\alpha(\{x; \Delta_\alpha(x, E) < 1\}) = 0$.

The lemma is proved by Kametani [6] for Hausdorff's measures. The proof we give is a small modification of Kametani's proof.

PROOF OF LEMMA 11. Set

$$E_m = \{x \in E \; ; \; \Delta_\alpha(x, E) < 1 - 1/m\}, \quad m = 2, 3, 4, \ldots .$$

Then

$$\{x; \Delta_\alpha(x, E) < 1\} \subset \bigcup_{m=2}^{\infty} E_m .$$

It is enough to show that

$$H_\alpha(E_m) = 0 \text{ for } m=2.,3,4,\ldots .$$

Let m be fixed. Set

$$E_{mn} = \left\{x \in E_m\ ;\ \frac{H_\alpha(E \cap S)}{\delta(S)^\alpha} < 1-\frac{1}{m} \text{ for all } S,\ x \in S \text{ and } \delta(S)<\frac{1}{n}\right\},$$

$n=1,2,3,\ldots$.

We get

$$E_m \subset \bigcup_{n=1}^{\infty} E_{mn} .$$

It is enough to show that

$$H_\alpha(E_{mn}) = 0 \quad \text{for } n=1,2,3,\ldots .$$

Suppose that there is an E_{mn} so that $H_\alpha(E_{mn})>0$. Choose balls B_j so that

$$E_{mn} \subset \bigcup_{j=1}^{\infty} B_j \quad \text{and} \quad \delta(B_j) < \frac{1}{2n} .$$

Then there exists a number k so that

$$H_\alpha(E_{mn} \cap B_k) > 0 .$$

Choose balls S_i so that $\delta(S_i)<1/n$, $E_{mn} \cap B_k \subset \bigcup_{i=1}^{\infty} S_i$, $S_i \cap E_{mn} \neq \varnothing$ and

$$\sum_{i=1}^{\infty} \delta(S_i)^\alpha < (1+1/m)H_\alpha(E_{mn} \cap B_k) .$$

We get

$$H_\alpha(E_{mn} \cap B_k) \leqq \sum_{i=1}^{\infty} H_\alpha(E_{mn} \cap S_i) \leqq \sum_{i=1}^{\infty} H_\alpha(E \cap S_i)$$

$$< (1-1/m) \sum_{i=1}^{\infty} \delta(S_i)^\alpha < (1-1/m^2)H_\alpha(E_{mn} \cap B_k) .$$

This is a contradiction. Thus $H_\alpha(E_{mn})=0$ for all m and n and the lemma is proved.

Lemma 12.

$$\limsup_{\delta\to 0} \frac{H_\alpha(E \cap B(x,\delta))}{\delta^\alpha} \geqq \frac{1}{2^\alpha}, \quad H_\alpha\text{-a.e. } on\ E .$$

Proof. Let $x \in E$. We may assume that $\Delta_\alpha(x,E)=1$.

Let S be a ball so that $x \in S$. We get

$$\frac{H_\alpha(E \cap B(x, 2\delta(S)))}{(2\delta(S))^\alpha} \geqq \frac{1}{2^\alpha} \frac{H_\alpha(E \cap S)}{\delta(S)^\alpha} .$$

Lemma 11 now gives the lemma.

REMARK 1. Let $n=2$ and $\alpha=1$ in Lemma 13. Then Besicovitch has shown in [3] that there is a set E, $E \subset \mathsf{R}^2$, such that there is equality in Lemma 12.

REMARK 2. Let $\alpha < n$. Then there exists a compact set F, $F \subset \mathsf{R}^n$, such that

$$\liminf_{\delta \to 0} \frac{H_\alpha(E \cap B(x,\delta))}{\delta^\alpha} = 0 \text{ for all } x \quad \text{and } H_\alpha(F) > 0 .$$

This can be deduced from Carleson [4].

LEMMA 13. *Let* $\alpha \leqq \beta$. *Then*

$$H_\beta(E)^{1/\beta} \leqq H_\alpha(E)^{1/\alpha} .$$

PROOF. See O'Farrell [7].

PROOF OF THE THEOREM. We may assume that $\alpha < \beta \leqq n$. Set

$$E_\beta = \left\{ x \, ; \, \limsup_{\delta \to 0} \frac{H_\alpha(E \cap B(x,\delta))}{\delta^\beta} > 0 \right\}.$$

Lemma 8 gives that there is a set $O_{x,\delta}$ such that $H_\beta(O_{x,\delta})=0$ and

$$\tilde{H}_\alpha(E \cap B(x,\delta)) = \tilde{H}_\alpha((E \cup (E_\beta \setminus O_{x,\delta})) \cap B(x,\delta)) .$$

If we use Lemma 9 and 10, we get

$$3^{-\alpha} H_\alpha((E \cup (E_\beta \setminus O_{x,\delta})) \cap B(x,\delta)) \leqq B_\alpha H_\alpha(E \cap B(x,\delta)) .$$

From lemma 13 we get

$$[H_\alpha(E \cap B(x,\delta))]^{1/\alpha} \geqq 3^{-1} B_\alpha^{-(1/\alpha)} H_\beta((E \cup (E_\beta \setminus O_{x,\delta})) \cap B(x,\delta))^{1/\beta} .$$

If we use that $H_\beta(O_{x,\delta})=0$, we find that

$$\left[\frac{H_\alpha(E \cap B(x,\delta))}{\delta^\alpha}\right]^{1/\alpha} \geqq 3^{-1} B_\alpha^{-(1/\alpha)} \left[\frac{H_\beta((E \cup E_\beta) \cap B(x,\delta))}{\delta^\beta}\right]^{1/\beta} .$$

Finally Lemma 12 gives

$$\limsup_{\delta\to 0} \frac{H_\alpha(E \cap B(x,\delta))}{\delta^\alpha} \geqq \frac{1}{6^\alpha B_\alpha} \quad H_\beta-\text{a.e. on } E \cup E_\beta ,$$

and this together with Lemma 6 gives the theorem.

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