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KOSZUL COMPLEXES WITH ISOMORPHIC HOMOLOGY

EUGENE GOVER and ALFIO RAGUSA*

Conditions under which Koszul complexes are isomorphic have been studied by Kirby [3]. Since many important attributes of a Koszul complex are homological, it is also natural to ask what the existence of an isomorphism on the homology level says about two complexes and about their generators. In the present article we consider this question for Koszul complexes over a fixed commutative noetherian ring. An isomorphism on homology in degree zero in this situation immediately forces the ideals defined by the two generating sets to be the same. We focus instead on the case in which there are isomorphisms on homology in all positive degrees. Assuming that these isomorphisms are induced by a chain map, we show (in Theorem 3.1) that the two Koszul complexes are either acyclic or else generated by the same number of elements. Over a local ring, if the isomorphisms are induced by a differential graded (DG) algebra map, we show (in Corollary 3.4) that the defining sequences of elements are either regular or else sets of generators for the same ideal. The hypothesis that the isomorphisms on homology come from an algebra map, or at least a chain map, is essential for obtaining these conclusions. An example (3.6) shows that one can have isomorphic positive homologies not induced by a chain map without implying either regularity or equality of the ideals.

The first section consists of notation and preliminary facts used in the rest of the paper. In the second section we collect some needed results about Koszul complexes and chain maps between them. The main results are stated and proved in the third section.

1. Notation and basic facts.

Throughout, we consider only commutative noetherian rings with identity. A Koszul complex over such a ring R defined by a set of elements

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$\boldsymbol{x}=x_1,\ldots,x_r$ will be denoted by $\boldsymbol{K}(x_1,\ldots,x_r;\, R)$ or, more briefly, $\boldsymbol{K}(\boldsymbol{x})$. Despite the fact that a Koszul complex depends on elements and not only on the ideal they generate, we will write $\boldsymbol{K}^I$ for $\boldsymbol{K}(\boldsymbol{x})$ when $(\boldsymbol{x})=I$ and it is clear which generators are being considered. In the same context, we will let Z^I denote the cycles and B^I the boundaries of this complex. In the complex $\boldsymbol{K}(x_1,\ldots,x_r;\, R)=\boldsymbol{K}^I$, $\boldsymbol{K}_0^I=R$ and $\boldsymbol{K}_1^I=R^r$. We will always fix a basis $e_1,\ldots,e_r$ for $\boldsymbol{K}_1^I$ and suppose that $d_1^I\colon \boldsymbol{K}_1^I \to \boldsymbol{K}_0^I$ is defined by $e_i \mapsto x_i$ for $i=1,\ldots,r$. Since $\boldsymbol{K}^I$ is the exterior algebra ΛR^r on the generators $\{e_i\}$, the generators for $\boldsymbol{K}_t^I$ in higher degrees will be all products $e_{i_1} \wedge \ldots \wedge e_{i_t} = e_{i_1 \ldots i_t}$ with $1 \leqq i_1 < \ldots < i_t \leqq r$. In the highest non-zero degree, $\Lambda^r R^r \cong R$ with the isomorphism given by $e_{1\ldots r} \mapsto 1$. Using this identification, we will consider elements in $\boldsymbol{K}_r^I$ to be elements of the ring.

Given a chain map $\varphi\colon \boldsymbol{K}^I \to \boldsymbol{K}^J$ between two Koszul complexes with $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_s)$, we will denote the basis of $\boldsymbol{K}_1^J$ by $e_1',\ldots,e_s'$. The requirement that $\varphi_0 d_1^I = d_1^J \varphi_1$ says that

$$\alpha x_j = \sum_{i=1}^{s} a_{ij} y_i \quad \text{for all } j=1,\ldots,r\,,$$

where multiplication by α defines φ_0 and (a_{ij}) is the matrix of φ_1 with respect to the chosen bases, i.e.

$$\varphi_1(e_j) = \sum_{i=1}^{s} a_{ij} e_i'\,.$$

In particular, $\alpha I \subseteqq J$.

Koszul complexes are, moreover, differential graded R-algebras. A chain map $\varphi\colon \boldsymbol{K}^I \to \boldsymbol{K}^J$ is a map of DG R-algebras precisely when $\varphi(x \wedge y)=\varphi(x) \wedge \varphi(y)$ and $\varphi_0=1_R$. When such an algebra map exists, the last condition forces $I \subseteqq J$. As an algebra, each Koszul complex is freely generated by elements of degree one and consequently, every DG R-algebra map φ can be obtained by starting with $\varphi_1\colon \boldsymbol{K}_1^I \to \boldsymbol{K}_1^J$ satisfying $d_1^I = d_1^J \varphi_1$ and lifting to $\Lambda\varphi_1=\varphi\colon \boldsymbol{K}^I \to \boldsymbol{K}^J$. The map φ is then an isomorphism if and only if φ_1 is an isomorphism, which in turn is equivalent to having $\det[\varphi_1]$ invertible where $[\varphi_1]$ is the matrix of φ_1 with respect to some pair of bases. Isomorphic Koszul complexes must have the same number of generators.

The homology of a Koszul complex is rigid in the sense that $H_p(\boldsymbol{K}^I)=0$ implies that $H_i(\boldsymbol{K}^I)=0$ for all $i \geqq p$ (cf. [1, 2.6]). It is also grade-sensitive: if $I=(x_1,\ldots,x_r)$ and grade $I=t$, then $t+q=r$, where q is the largest integer such that $H_q(\boldsymbol{K}^I)\neq 0$ (cf. [1, 1.7]). When $q=0$, i.e. when $H_i(\boldsymbol{K}^I)=0$ for all $i>0$, the complex is said to be acyclic. Whenever $x_1,\ldots,x_r$ is a regular sequence, $H_1(\boldsymbol{K}^I)=0$, and by rigidity, this is enough to imply the acyclicity of $\boldsymbol{K}^I$. If I is contained in the Jacobson radical of R, the converse also holds: acyclicity of $\boldsymbol{K}^I$ implies regularity of $x_1,\ldots,x_r$ (cf. [1, 2.8]).

2. Chain maps between Koszul complexes.

In this section we record some further properties of Koszul complexes and chain maps between them.

Start by considering two sets of generators $\boldsymbol{x}=x_1,\ldots,x_r$ and $\boldsymbol{y}=y_1,\ldots,y_s$ for the same ideal I. We can then write (not necessarily uniquely)

$$x_j = \sum_{i=1}^{s} a_{ij} y_i$$

for $j=1,\ldots,r$. The map $\varphi_1\colon R^r \to R^s$ defined by the matrix (a_{ij}) then makes the diagram

$$\begin{array}{ccc} R^r & \xrightarrow{\ x\ } & R \\ {\scriptstyle\varphi_1}\downarrow & & \| \\ R^s & \xrightarrow{\ y\ } & R \end{array}$$

commute. As observed in section 1, φ_1 lifts to a DG R-algebra map

$$\varphi = \Lambda\varphi_1\colon \boldsymbol{K}(\boldsymbol{x}) \to \boldsymbol{K}(\boldsymbol{y}) .$$

It is clear that φ is an isomorphism when $\boldsymbol{y}$ is a permutation $\sigma(\boldsymbol{x})$ of $\boldsymbol{x}$ and φ_1 is defined by the matrix expressing this permutation, that is $[\varphi_1] =(\delta_{i,\sigma(i)})$. It is also well-known (cf. [3]) that φ is an isomorphism when R is local and both $\boldsymbol{x}$ and $\boldsymbol{y}$ are minimal sets of generators for a proper ideal I. The proposition following the next lemma extends this last result to non-minimal sets of generators of the same length for any ideal in a local ring.

LEMMA 2.1. *Let $x_1,\ldots,x_r$ be elements of a ring R such that*

$$x_{t+j} = a_{1,t+j}x_1 + \ldots + a_{t,t+j}x_t$$

for some $t \leq r$ and for every $j=1,\ldots,r-t$. Then the map $\psi_1\colon R^r \to R^r$ defined by the matrix

$$M = \left(\begin{array}{c|c} I_t & -a_{i,t+j} \\ \hline 0 & I_{r-t} \end{array}\right)$$

induces a DG *R-algebra isomorphism*

$$\psi\colon \boldsymbol{K}(x_1,\ldots,x_t;\underbrace{0,\ldots,0}_{r-t};R) \xrightarrow{\ \cong\ } \boldsymbol{K}(x_1,\ldots,x_r;R)$$

where I_t and I_{r-t} are identity matrices of size t and $r-t$ respectively and $i=1,\ldots,t$, $j=1,\ldots,r-t$.

PROOF. A direct check shows that

$$\begin{array}{ccc} K_1(x_1,\ldots,x_t;0,\ldots,0) = R^r & \xrightarrow{(x_1,\ldots,x_t;\,0,\ldots,0)} & R \\ \psi_1 \downarrow & & \| \\ K_1(x_1,\ldots,x_r) = R^r & \xrightarrow{(x_1,\ldots,x_r)} & R \end{array}$$

commutes. Moreover, $\det M=1$ so the extension $\psi=\Lambda\psi_1$ is a DG R-algebra isomorphism.

PROPOSITION 2.2. *Let $\boldsymbol{x}=x_1,\ldots,x_r$ and $\boldsymbol{y}=y_1,\ldots,y_r$ be sequences that generate the same ideal I in a local ring A. Then the Koszul complexes $\boldsymbol{K}(\boldsymbol{x};A)$ and $\boldsymbol{K}(\boldsymbol{y};A)$ are isomorphic* DG *A-algebras.*

PROOF. Let $t=\nu(I)$ be the minimal number of generators of I. Using the observations made just before the lemma, we may assume that $x_1,\ldots,x_t$ and $y_1,\ldots,y_t$ minimally generate I.

If I is proper, it follows from [3] that there exists a DG A-algebra isomorphism

$$\varphi\colon \boldsymbol{K}(x_1,\ldots,x_t;A)\to \boldsymbol{K}(y_1,\ldots,y_t;A).$$

Otherwise $I=A$, $t=1$, x_1 and y_1 are invertible, and there is an obvious isomorphism

$$\varphi\colon \boldsymbol{K}(x_1;A)\to \boldsymbol{K}(y_1;A).$$

Thus for any ideal there is an isomorphism φ between Koszul complexes defined by minimal sets of generators. Let

$$\tilde{\varphi}\colon \boldsymbol{K}(x_1,\ldots,x_t;\underbrace{0,\ldots,0}_{r-t})\to \boldsymbol{K}(y_1,\ldots,y_t;\underbrace{0,\ldots,0}_{r-t})$$

be the ismorphism induced by the matrix

$$\left(\begin{array}{c|c} [\varphi_1] & 0 \\ \hline 0 & I_{r-t} \end{array}\right).$$

Using lemma 2.1 we can then construct the DG A-algebra isomorphism

$$\boldsymbol{K}(\boldsymbol{x})\xrightarrow[(\psi^{\boldsymbol{x}})^{-1}]{}\boldsymbol{K}(x_1,\ldots,x_t;0,\ldots,0)\xrightarrow[\tilde{\varphi}]{}\boldsymbol{K}(y_1,\ldots,y_t;0,\ldots,0)\xrightarrow[\psi^{\boldsymbol{y}}]{}\boldsymbol{K}(\boldsymbol{y}).$$

REMARK 2.3. If we drop the hypothesis that A be local, we can say slightly less: for every localization $A_{\mathfrak{m}}$ at a maximal ideal $\mathfrak{m}$, there exists a DG $A_{\mathfrak{m}}$-algebra isomorphism $\psi_{\mathfrak{m}}: \boldsymbol{K}(\boldsymbol{x}; A_{\mathfrak{m}}) \to \boldsymbol{K}(\boldsymbol{y}; A_{\mathfrak{m}})$.

When a chain map $\varphi: \boldsymbol{K}^I \to \boldsymbol{K}^J$ exists for (possibly) different ideals $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_s)$, we have already noted that $\alpha I \subseteq J$ where multiplication by α defines φ_0. If $r=s$, we also have

PROPOSITION 2.4. *Let $\varphi: \boldsymbol{K}^I \to \boldsymbol{K}^J$ be a chain map with $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_r)$. Then $\Delta J \subseteq I$ where multiplication by Δ defines φ_r.*

PROOF. Dualizing the commutativity of φ with the differentials of $\boldsymbol{K}^I$ and $\boldsymbol{K}^J$ in the highest degrees yields $(d_r^I)^* \varphi_{r-1}^* = \varphi_r^* (d_r^J)^*$. The conclusion then follows, since for Koszul complexes, Im $(d_r^I)^* = I$ and Im $(d_r^J)^* = J$.

REMARK 2.5. In the special case that $\varphi = \Lambda\varphi_1$ is an algebra map, one has $\alpha = \varphi_0 = 1$ and so $I \subseteq J$. In addition, $\Delta = \varphi_r = \det[\varphi_1]$. It then follows from 2.4 that $\det[\varphi_1] J \subseteq I$. (This inclusion can also be obtained from Cramer's rule.)

3. Isomorphic Koszul homologies.

What can be learned about Koszul complexes and their generators when their homologies are related? We now consider this problem for the case of two Koszul complexes over a ring R and a chain map between them that induces an isomorphism on homology in specified degrees. Observe first that if I and J are ideals of R and $\boldsymbol{K}^I$ and $\boldsymbol{K}^J$ are Koszul complexes defined by any two sets of generators for these ideals, then $H_0(\boldsymbol{K}^I) \cong H_0(\boldsymbol{K}^J)$ implies that $I=J$. This follows from the fact that R/I and R/J are the respective homologies in degree zero. It is therefore natural to ask what happens when there are isomorphisms between $H(\boldsymbol{K}^I)$ and $H(\boldsymbol{K}^J)$ in other degrees.

If there are isomorphisms on homology in all positive degrees, grade-sensitivity guarantees that $\operatorname{grade} J - \operatorname{grade} I = s - r$ where $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_s)$. More, in fact, can be said.

THEOREM 3.1. *Let $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_s)$ be two ideals in a ring R whose generators define Koszul complexes $\boldsymbol{K}^I$ and $\boldsymbol{K}^J$ respectively. Suppose that there exists a chain map $\varphi: \boldsymbol{K}^I \to \boldsymbol{K}^J$ such that $H(\varphi)$ is an isomorphism in all positive degrees. Then either $r=s$ or else both complexes are acyclic.*

PROOF. Consider the complexes $\boldsymbol{K}^I \otimes \boldsymbol{K}^I$ and $\boldsymbol{K}^I \otimes \boldsymbol{K}^J$ and "first" filtrations of these tensor products defined by

$$F_k(K^I \otimes K^I)_p = \sum_{\substack{i+j=p \\ i \leqq k}} K_i^I \otimes K_j^I$$

and

$$F_k(K^I \otimes K^J)_p = \sum_{\substack{i+j=p \\ i \leqq k}} K_i^I \otimes K_j^J .$$

Observe that $F_0(K^I \otimes K^I) \cong K^I$ and $F_0(K^I \otimes K^J) \cong K^J$. The natural inclusions

$$K^I \hookrightarrow F_1(K^I \otimes K^I) \quad \text{and} \quad K^J \hookrightarrow F_1(K^I \otimes K^J)$$

induce the following short exact sequences of complexes

$$(1) \qquad \begin{array}{ccccccccc} 0 & \to & K^I & \to & F_1(K^I \otimes K^I) & \to & C_1^I & \to & 0 \\ & & \downarrow \varphi & & \downarrow \Phi_1 & & \downarrow \psi_1 & & \\ 0 & \to & K^J & \to & F_1(K^I \otimes K^J) & \to & C_1^J & \to & 0 \end{array}$$

where Φ_1 is the restriction of $1_{K^I} \otimes \varphi$ to $F_1(K^I \otimes K^I)$ and ψ_1 is the induced chain map on the cokernels. The homology diagram induced by (1) is, in part,

$$(2) \qquad \begin{array}{ccccccccccc} \cdots \to & H_{p+1}(C_1^I) & \to & H_p(K^I) & \to & H_p(F_1(K^I \otimes K^I)) & \to & H_p(C_1^I) & \to & H_{p-1}(K^I) & \to \cdots \\ & \downarrow H(\psi_1) & & \downarrow H(\varphi) & & \downarrow H(\Phi_1) & & \downarrow H(\psi_1) & & \downarrow H(\varphi) & \\ \cdots \to & H_{p+1}(C_1^J) & \to & H_p(K^J) & \to & H_p(F_1(K^I \otimes K^J)) & \to & H_p(C_1^J) & \to & H_{p-1}(K^J) & \to \cdots \end{array}$$

It is easy to see that

$$(C_1^I)_p \cong K_1^I \otimes K_{p-1}^I \quad \text{and} \quad (C_1^J)_p \cong K_1^I \otimes K_{p-1}^J$$

from which it follows that $H_p(\psi_1) = 1_{K_1^I} \otimes H_{p-1}(\varphi)$, so $H_p(\psi_1)$ is an isomorphism for all $p \geqq 2$. Applying the five-lemma, $H_p(\Phi_1)$ is an isomorphism for all $p \geqq 2$. We now repeat this argument in general using induction on the filtration degree. The generalization of (1) is

$$(3) \qquad \begin{array}{ccccccccc} 0 & \to & F_{t-1}(K^I \otimes K^I) & \to & F_t(K^I \otimes K^I) & \to & C_t^I & \to & 0 \\ & & \downarrow \Phi_{t-1} & & \downarrow \Phi_t & & \downarrow \psi_t & & \\ 0 & \to & F_{t-1}(K^I \otimes K^J) & \to & F_t(K^I \otimes K^J) & \to & C_t^J & \to & 0 \end{array}$$

where Φ_{t-1} and Φ_t are the appropriate restrictions of $1_{K^I} \otimes \varphi$ and ψ_t is the induced chain map on the cokernels. Since

$$(C_t^I)_p \cong K_t^I \otimes K_{p-t}^I \quad \text{and} \quad (C_t^J)_p \cong K_t^I \otimes K_{p-t}^J ,$$

$H_p(\psi_t) = 1_{K_t^I} \otimes H_{p-t}(\varphi)$ and so $H_p(\psi_t)$ is an isomorphism for all $p \geqq t+1$. By induction, we may assume that $H_p(\Phi_{t-1})$ is an isomorphism for all $p \geqq t$. Again applying the five-lemma to the homology diagram coming from (3), $H_p(\Phi_t)$ is an isomorphism for all $p \geqq t+1$, so by induction this fact is true for all t. When $t=r$,

$$F_r(\boldsymbol{K}^I \otimes \boldsymbol{K}^I) \cong \boldsymbol{K}^I \otimes \boldsymbol{K}^I \quad \text{and} \quad F_r(\boldsymbol{K}^I \otimes \boldsymbol{K}^J) \cong \boldsymbol{K}^I \otimes \boldsymbol{K}^J.$$

Hence

$$\Phi_r = 1_{\boldsymbol{K}^I} \otimes \varphi$$

and so

$$H_p(1_{\boldsymbol{K}^I} \otimes \varphi)\colon H_p(\boldsymbol{K}^I \otimes \boldsymbol{K}^I) \to H_p(\boldsymbol{K}^I \otimes \boldsymbol{K}^J) \tag{4}$$

is an isomorphism for all $p \geqq r+1$.

Now start over from the beginning with a new pair of complexes $\boldsymbol{K}^I \otimes \boldsymbol{K}^J$ and $\boldsymbol{K}^J \otimes \boldsymbol{K}^J$. This time, using "second" filtrations defined by

$$F'_k(\boldsymbol{K}^I \otimes \boldsymbol{K}^J)_p = \sum_{\substack{i+j=p \\ j \leqq k}} \boldsymbol{K}_i^I \otimes \boldsymbol{K}_j^J$$

and

$$F'_k(\boldsymbol{K}^J \otimes \boldsymbol{K}^J)_p = \sum_{\substack{i+j=p \\ j \leqq k}} \boldsymbol{K}_i^J \otimes \boldsymbol{K}_j^J$$

and a similar argument, we can conclude that

$$H_p(\varphi \otimes 1_{\boldsymbol{K}^J})\colon H_p(\boldsymbol{K}^I \otimes \boldsymbol{K}^J) \to H_p(\boldsymbol{K}^J \otimes \boldsymbol{K}^J) \tag{5}$$

is an isomorphism for all $p \geqq s+1$. Combining (4) and (5), we have the following chain of isomorphisms for $q = \max\{r+1, s+1\}$:

$$\begin{aligned} \sum_{i+j=q} \Lambda^i R^r \otimes H_j(\boldsymbol{K}^I) &\cong H_q(\boldsymbol{K}(0,\ldots,0) \otimes \boldsymbol{K}^I) \cong H_q(\boldsymbol{K}^I \otimes \boldsymbol{K}^I) \\ &\cong H_q(\boldsymbol{K}^I \otimes \boldsymbol{K}^J) \cong H_q(\boldsymbol{K}^J \otimes \boldsymbol{K}^J) \\ &\cong H_q(\boldsymbol{K}(0,\underset{s}{\ldots},0) \otimes \boldsymbol{K}^J) \cong \sum_{i+j=q} \Lambda^i R^s \otimes H_j(\boldsymbol{K}^J)\,. \end{aligned} \tag{6}$$

Here, the first and last isomorphisms follow from the Künneth tensor formula, the second and fifth come from Lemma 2.1, and the middle two isomorphisms are $H_q(1_{\boldsymbol{K}^I} \otimes \varphi)$ and $H_q(\varphi \otimes 1_{\boldsymbol{K}^J})$ respectively.

If either $\boldsymbol{K}^I$ or $\boldsymbol{K}^J$ is acyclic, the hypothesis that $H(\varphi)$ is an isomorphism in positive dimensions guarantees that the other complex is also acyclic.

On the other hand if neither complex is asyclic, then by rigidity $H_1(\boldsymbol{K}^I) \cong H_1(K^J) \neq 0$. There then exists some maximal ideal $\mathfrak{m}$ such that after localizing at $\mathfrak{m}$, $H_1(\boldsymbol{K}^I)_{\mathfrak{m}} \cong H_1(\boldsymbol{K}^J)_{\mathfrak{m}} \neq 0$. We may therefore assume that the isomorphism (6) holds in a local ring with residue field k and with $H_1(\boldsymbol{K}^I) \cong H_1(\boldsymbol{K}^J) \neq 0$. After tensoring with k and counting dimensions, using the fact that $\Lambda^q R^r = \Lambda^q R^s = 0$, (6) leads to the equality

$$\sum_{\substack{i+j=q \\ j>0}} \binom{r}{i} \cdot v_j = \sum_{\substack{i+j=q \\ j>0}} \binom{s}{i} \cdot v_j \tag{7}$$

with $v_j = \dim_k H_j(\boldsymbol{K}^I) \otimes k = \dim_k H_j(\boldsymbol{K}^J) \otimes k \geqq 0$ for $j>0$, and at least $v_1 > 0$. It follows immediately from this that $r=s$.

The theorem that we have just proved shows what happens when two Koszul complexes with induced isomorphic positive homologies have different numbers of generators. The next theorem will be used to show what happens when they have the same number of generators.

THEOREM 3.2. *Let $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_r)$ be two ideals in a ring R whose generators define Koszul complexes $\boldsymbol{K}^I$ and $\boldsymbol{K}^J$. Suppose that there exists a chain map $\varphi\colon \boldsymbol{K}^I \to \boldsymbol{K}^J$ with the following two properties:*

i) *φ_r = multiplication by $\Delta \in \operatorname{rad} R$.*
ii) *$H_t(\varphi)$ is an isomorphism for all $t \geqq q$, for some fixed q.*

Then grade$I$ *and* grade$J$ *are both greater than $r-q$.*

PROOF. By the grade-sensitivity of Koszul complexes, we have to show that for all $t \geqq q$, $H_t(\boldsymbol{K}^I) \cong H_t(\boldsymbol{K}^J) = 0$. For $t=r$, this means showing that $0{:}I = 0{:}J = 0$. Take any $\alpha \in 0{:}I$. Since $H_r(\varphi)$ is multiplication by Δ, $\Delta\alpha \in 0{:}$J. Thus for every $y \in J$, $\Delta\alpha y = 0$, and since $H_r(\varphi)$ is injective, $\alpha y = 0$. Hence $\alpha \in 0{:}J$. This shows $0{:}I \subseteqq 0{:}J$. But $H_r(\varphi)$ is also surjective, so $\Delta(0{:}I) = 0{:}J$, from which we get $0{:}I \subseteqq \Delta(0{:}I)$. By Nakayama's lemma we conclude that $0{:}I = 0$ and consequently $0{:}J = 0$ as well.

Now consider $q \leqq t < r$. By decreasing induction we may assume

$$H_p(\boldsymbol{K}^I) \cong H_p(\boldsymbol{K}^J) = 0 \quad \text{for } p>t;$$

we must then prove $H_t(\boldsymbol{K}^I) \cong H_t(\boldsymbol{K}^J) = 0$. Take $z' \in Z_t^J$. Since $H_t(\varphi)$ is onto, there exists a cycle $z \in Z_t^I$ such that

$$\varphi_t(z) - z' = b^{(t)} = d^J w^{(t+1)} \in B_t^J$$

for some $w^{(t+1)} \in K^J_{t+1}$. Since $J \cdot H_t(K^J)=0$ and $H_t(\varphi)$ is injective, it follows that $J \cdot H_t(K^I)=0$. In particular, $y_i z \in B^I_t$ for $i=1,\ldots,r$. There are then elements $\mu_i \in K^I_{t+1}$ such that $d^I\mu_i = y_i z$ for $i=1,\ldots,r$. Set $\mu'_i = e'_i \wedge z' + y_i w^{(t+1)} \in K^J_{t+1}$.

We claim that $\varphi_{t+1}(\mu_i)-\mu'_i \in B^J_{t+1}$. This is shown by the following computation:

$$d^J[\varphi_{t+1}(\mu_i)-\mu'_i] = \varphi_t d^I(\mu_i)-d^J(\mu'_i) = \varphi_t(y_i z)-y_i z'-y_i b^{(t)} = 0$$

so $\varphi_{t+1}(\mu_i)-\mu'_i \in Z^J_{t+1}=B^J_{t+1}$ by the induction hypothesis. It follows for every $i=1,\ldots,r$ that there are elements $w_i^{(t+2)} \in K^J_{t+2}$ such that

$$\varphi_{t+1}(\mu_i)-\mu'_i = b_i^{(t+1)} = d^J w_i^{(t+2)} .$$

Moreover, for every $1 \leqq i_1 < i_2 \leqq r$,

$$y_{i_1}\mu_{i_2} - y_{i_2}\mu_{i_1} \in Z^I_{t+1} = B^I_{t+1} .$$

There are therefore elements $\mu_{i_1 i_2} \in K^I_{t+2}$ such that

$$d^I \mu_{i_1 i_2} = y_{i_1}\mu_{i_2} - y_{i_2}\mu_{i_1} .$$

As before, set

$$\mu'_{i_1 i_2} = e'_{i_1 i_2} \wedge z' + y_{i_1} w_{i_2}^{(t+2)} - y_{i_2} w_{i_1}^{(t+2)} \in K^J_{t+2} .$$

We next claim that $\varphi_{t+2}(\mu_{i_1 i_2})-\mu'_{i_1 i_2} \in B^J_{t+2}$. Again, this is established by computation:

$$\begin{aligned} d^J[\varphi_{t+2}(\mu_{i_1 i_2})-\mu'_{i_1 i_2}] &= \varphi_{t+1} d^I(\mu_{i_1 i_2}) - d^J(\mu'_{i_1 i_2}) \\ &= \varphi_{t+1}(y_{i_1}\mu_{i_2} - y_{i_2}\mu_{i_1}) - (y_{i_1}e'_{i_2} \wedge z' - y_{i_2}e'_{i_1} \wedge z' + y_{i_1} b_{i_2}^{(t+1)} - y_{i_2} b_{i_1}^{(t+1)}) \\ &= y_{i_1}(e'_{i_2} \wedge z' + y_{i_2} w^{(t+1)} + b_{i_2}^{(t+1)}) - y_{i_2}(e'_{i_1} \wedge z' + y_{i_1} w^{(t+1)} + b_{i_1}^{(t+1)}) - \\ &\quad - (y_{i_1}e'_{i_2} \wedge z' - y_{i_2}e'_{i_1} \wedge z' + y_{i_1} b_{i_2}^{(t+1)} - y_{i_2} b_{i_1}^{(t+1)}) = 0 \end{aligned}$$

so

$$\varphi_{t+2}(\mu_{i_1 i_2})-\mu'_{i_1 i_2} \in Z^J_{t+2} = B^J_{t+2} .$$

Thus for every $1 \leqq i_1 < i_2 \leqq r$ we have

$$\varphi_{t+2}(\mu_{i_1 i_2})-\mu'_{i_1 i_2} = b_{i_1 i_2}^{(t+2)} = d^J w_{i_1 i_2}^{(t+3)}$$

for some $w_{i_1 i_2}^{(t+3)} \in K^J_{t+3}$.

We now proceed in the same way for $t' = r-t$ steps until, for every $1 \leqq i_1 < \ldots < i_{t'} \leqq r$, we obtain elements $\mu_{i_1 \ldots i_{t'}} \in K^I_r$ with the property

$$\varphi_r(\mu_{i_1\ldots i_{t'}}) - \mu'_{i_1\ldots i_{t'}} \in Z_r^J = B_r^J = 0,$$

where

$$\mu'_{i_1\ldots i_{t'}} = e'_{i_1\ldots i_{t'}} \wedge z' + \sum_{k=1}^{t'} (-1)^{k+1} y_{i_k} w^{(r)}_{i_1\ldots \hat{i}_k\ldots i_{t'}}$$

and the elements $w^{(r)}_{i_1\ldots \hat{i}_k\ldots i_{t'}} \in K_r^J$ are constructed inductively in the same manner used for the earlier w's.

To go on, we need a sign convention and some combinatorial results. Given t' positive integers $i_1,\ldots,i_{t'}$ with $1 \leqq i_1 < \ldots < i_{t'} \leqq r$, let $\sigma_{i_1\ldots i_{t'}}$ stand for the sign of the permutation

$$(i_1,\ldots,i_{t'},(i_1,\ldots,i_{t'})\hat{\ })$$

where $(i_1,\ldots,i_{t'})\hat{\ } = 1,\ldots,\hat{i}_1,\ldots,\hat{i}_{t'},\ldots,r$. In this permutation each i_m, $m=1,\ldots,t'$, must move $(i_m-1)+(t'-m)$ steps to reach its final position in $1,2,\ldots,r$. Thus we need $i_1+\ldots+i_{t'}+\binom{t'}{2}-t'$ transpositions in order to convert this permutation into the identity. Therefore,

$$\sigma_{i_1\ldots i_{t'}} = (-1)^{i_1+\ldots+i_{t'}+\binom{t'+1}{2}}.$$

Now take any integer $j \in \{(i_1,\ldots,i_{t'})\hat{\ }\}$ and set

$$[j\,|\,i_1,\ldots,i_{t'}] = \text{number of } i\text{'s less than } j\,.$$

It is clear that

$$(8)\qquad [j|i_1,\ldots,i_{t'}]+[j|(i_1,\ldots,j,\ldots,i_{t'})\hat{\ }] = j-1\,.$$

With this notation, if $1 \leqq i_1 < \ldots < i_{t'+1} \leqq r$ are $t'+1$ integers, we claim that

$$(9)\qquad \sigma_{i_1\ldots\hat{i}_k\ldots i_{t'+1}} \cdot (-1)^{[i_k|(i_1\ldots i_{t'+1})\hat{\ }]}$$

does not depend on i_k but only on $i_1,\ldots,i_{t'+1}$ and k; so when k runs from 1 to $t'+1$, (9) is an alternating sign. This is shown, using (8), by the following computation:

$$\begin{aligned}\sigma_{i_1\ldots\hat{i}_k\ldots i_{t'+1}} &\cdot (-1)^{[i_k|(i_1\ldots i_{t'+1})\hat{\ }]}\\ &= (-1)^{i_1+\ldots+\hat{i}_k+\ldots+i_{t'+1}+\binom{t'+1}{2}+(i_k-1)-(k-1)}\\ &= (-1)^{i_1+\ldots+i_{t'+1}+\binom{t'+1}{2}+k}.\end{aligned}$$

Returning to the proof, using the identification of K_r^I with R stated in section 1, we may think of the element $\mu_{i_1\ldots i_{t'}}$ as an element of R. We then claim that

$$\sum_{1 \leqq i_1 < \ldots < i_{t'} \leqq r} \sigma_{i_1 \ldots i_{t'}} \mu_{i_1 \ldots i_{t'}} e'_{(i_1 \ldots i_{t'})\hat{}} \in Z_t^J .$$

To establish this, it is enough to show that for every $1 \leqq i_1 < \ldots < i_{t'+1} \leqq r$,

$$\sum_{k=1}^{t'+1} \sigma_{i_1 \ldots \hat{i}_k \ldots i_{t'+1}} \mu_{i_1 \ldots \hat{i}_k \ldots i_{t'+1}} (-1)^{[i_k | (i_1 \ldots i_{t'+1})\hat{}\,]} y_{i_k} = 0 ,$$

which by the previous result can be expressed as

$$\sum_{k=1}^{t'+1} (-1)^{k+1} y_{i_k} \mu_{i_1 \ldots \hat{i}_k \ldots i_{t'+1}} = 0 .$$

But this follows, as in the other dimensions, because

$$\sum_{k=1}^{t'+1} (-1)^{k+1} \mu_{i_1 \ldots \hat{i}_k \ldots i_{t'+1}} y_{i_k} \in Z_r^I = B_r^I = 0 .$$

From

$$\mu'_{i_1 \ldots i_{t'}} = \varphi_r(\mu_{i_1 \ldots i_{t'}}) = \Delta \mu_{i_1 \ldots i_{t'}}$$

we now obtain

$$\sum_{i_1 \ldots i_{t'}} \sigma_{i_1 \ldots i_{t'}} \mu'_{i_1 \ldots i_{t'}} e'_{(i_1 \ldots i_{t'})\hat{}} = \Delta \sum_{i_1 \ldots i_{t'}} \sigma_{i_1 \ldots i_{t'}} \mu_{i_1 \ldots i_{t'}} e'_{(i_1 \ldots i_{t'})\hat{}} \in \Delta Z_t^J ,$$

which by substitution implies

$$z' + \sum_{k=1}^{t'} \sum_{i_1 \ldots i_{t'}} \sigma_{i_1 \ldots i_{t'}} (-1)^{k+1} y_{i_k} w^{(r)}_{i_1 \ldots \hat{i}_k \ldots i_{t'}} e'_{(i_1 \ldots i_{t'})\hat{}} \in \Delta Z_t^J .$$

The second term in this sum is the boundary of the element

$$\sum_{i_1 \ldots i_{t'-1}} \sigma_{i_1 \ldots i_{t'-1}} w^{(r)}_{i_1 \ldots i_{t'-1}} e'_{(i_1 \ldots i_{t'-1})\hat{}} .$$

We therefore have shown that $z' \in \Delta Z_t^J + B_t^J$ and hence $Z_t^J \subseteqq \Delta Z_t^J + B_t^J$. By Nakayama's lemma it follows that $Z_t^J = B_t^J$, and from this $H_t(\mathbf{K}^J) \cong H_t(\mathbf{K}^I) = 0$.

We now exhibit some of the consequences of Theorem 3.2.

THEOREM 3.3. *Let $I = (x_1, \ldots, x_r) \subseteqq (y_1, \ldots, y_s) = J$ be two ideals of a ring R such that $(I:J) \subseteqq \operatorname{rad}(R)$. If there is a chain map $\varphi: \mathbf{K}^I \to \mathbf{K}^J$ making $H_t(\varphi)$ an isomorphism for $t > 0$, then $x_1, \ldots, x_r$ is a regular sequence and $\mathbf{K}^J$ is acyclic.*

PROOF. If $r \neq s$, the conclusion follows from Theorem 3.1 and the fact that $I \subseteq \operatorname{rad}(R)$. Otherwise, by 2.4, we have $\Delta J \subseteq I$ where multiplication by Δ defines φ_r. The condition on $I:J$ then forces $\Delta \in \operatorname{rad}(R)$, which means that the hypotheses of Theorem 3.2 are satisfied with $t=1$. The conclusion now follows from 3.2 and having $I \subseteq \operatorname{rad}(R)$.

COROLLARY 3.4. *Let $I=(x_1,\ldots,x_r)$ and $J=(y_1,\ldots,y_s)$ be ideals in a local ring $(A,\mathfrak{m})$. If there is an algebra map $\varphi\colon \boldsymbol{K}^I \to \boldsymbol{K}^J$ inducing isomorphisms $H_t(\varphi)$ for $t>0$, then either $r=s$ and $I=J$, or else $x_1,\ldots,x_r$ and $y_1,\ldots,y_s$ are both regular sequences.*

PROOF. By Remark 2.5, $I \subseteq J$ under this hypothesis. If the sequences are not both regular, then by 3.1 $r=s$, and applying 3.3 in the local case it follows that $I=J$.

REMARK 3.5. i) Combining Theorem 3.1 and Corollary 3.4, one can say: if I and J are two different ideals that cannot be generated by regular sequences, then there is no algebra map between $\boldsymbol{K}^I$ and $\boldsymbol{K}^J$ (for any two sets of generators) that induces isomorphisms for all positive homologies. When $I \subseteq J$, the same is also true for chain maps.

ii) In 3.2 and 3.3 the hypothesis that the isomorphism $H_+(\boldsymbol{K}^I) \cong H_+(\boldsymbol{K}^J)$ is induced by a chain map is essential, as can be seen by the following example.

EXAMPLE 3.6. Consider $A=k[[X,Y,Z]]/(XY)=k[[x,y,z]]$, where k is any field and x,y,z are the images of X,Y,Z under the natural map. Set $I=(xz)$ and $J=(x)$. We then have that $H_1(\boldsymbol{K}^I)=0:xz=(y)$ and $H_1(\boldsymbol{K}^J)=0:x=(y)$ so trivially, $H_1(\boldsymbol{K}^I) \cong H_1(\boldsymbol{K}^J)$. But this isomorphism (the identity) is certainly not induced by a chain map from $\boldsymbol{K}^I$ to $\boldsymbol{K}^J$. Moreover, an easy computation shows that no chain map can induce an isomorphism on H_1.

If $I=(\boldsymbol{x})=(x_1,\ldots,x_r) \subsetneq J=(\boldsymbol{y})=(y_1,\ldots,y_s)$ are two ideals in a local ring $(A,\mathfrak{m})$, then Theorem 3.3 shows, in particular, that if $H_+(\boldsymbol{K}^I)$ and $H_+(\boldsymbol{K}^J)$ are isomorphic by means of an isomorphism coming from a chain map, then $\boldsymbol{x}$ is a minimal set of generators for I. This conclusion of minimality can be obtained more generally when $H_1(\boldsymbol{K}^I) \cong H_1(\boldsymbol{K}^J)$ by means of any isomorphism.

PROPOSITION 3.7. *With the preceding notation, if $H_t(\boldsymbol{K}^I) \cong H_t(\boldsymbol{K}^J)$, then $\nu(I) \geqq r+1-t$.*

PROOF. Suppose $\nu=\nu(I)<r+1-t$, that is $t \leqq r-\nu$. We may assume that $\boldsymbol{x}_\nu = x_1,\ldots,x_\nu$ minimally generate I. By Lemma 2.1 we have

$$
\begin{aligned}
H_t(K^I) &\cong H_t(K(x_\nu)\otimes K(\underbrace{0,\ldots,0}_{r-\nu}))\\
&\cong H_t(K(x_\nu)) \oplus [H_{t-1}(K(x_\nu))\otimes A^{r-\nu}] \oplus \ldots\\
&\quad \ldots \oplus [H_0(K(x_\nu))\otimes \Lambda^t A^{r-\nu}]\,.
\end{aligned}
$$

Since $H_0(K(x_\nu))=A/I$ and $t \leqq r-\nu$, we have that $H_t(K^I)$ contains A/I as a direct summand. On the other hand, $J\cdot H_t(K^J)=0$. Hence, because $H_t(K^I)\cong H_t(K^J)$, $J\cdot H_t(K^I)=0$. Therefore, in particular, $J\cdot A/I=0$, which means $J\subseteqq I$ contradicting $I\subsetneqq J$.

In particular, for $t=1$ we get

COROLLARY 3.8. *If $H_1(K^I)\cong H_1(K^J)$, then x is a minimal set of generators for I.*

When $r=s$, 3.8 says that both sets of generators are minimal.

REMARK 3.9. Given a ring homomorphism $f_0: R\to R'$, taking an ideal $I\subset R$ to an ideal $I'\subset R'$, that can be lifted to a chain map $f: K^I(R) \to K^{I'}(R')$ considered over R via f_0, one can ask what the consequences are when $H(f)$ is an isomorphism. Examples of such an isomorphism can be given, for instance when (R',I') is the I-adic completion of (R,I) and $f_0: R\to R'$ is the natural map. It might be interesting to find conditions on the pairs (R,I) and (R',I') that will imply the isomorphism $H(f)$ on the homology level. Note that in the case of local rings $(A,\mathfrak{m},k)$ and $(A',\mathfrak{m}',k')$ and Koszul complexes $K^A=K^{\mathfrak{m}}(A)$ and $K^{A'}=K^{\mathfrak{m}'}(A')$ defined by minimal sets of generators for the two maximal ideals, a DG A-algebra map $\psi: K^A\to K^{A'}$ induces the k-algebra map $H(\psi)$. When $H(\psi)$ is an isomorphism, ψ preserves matric Massey-product structure (cf. [4, Theorem 1.5]). If, in addition, the rings A and A' have the same embedding dimension, it follows from a result of Avramov [2] that they also have the same Poincaré series and Betti numbers.

REFERENCES

1. M. Auslander and D. Buchsbaum, *Codimension and multiplicity*, Ann. of Math. 68 (1958), 625–657.
2. L. Avramov, *On the Hopf algebra of a local ring*, Math. USSR-Izv. 8 (1974), 259–284.
3. D. Kirby, *Isomorphic Koszul complexes*, Mathematika 20 (1973), 53–57.
4. J. P. May, *Matric Massey products*, J. Algebra 12 (1969), 533–568.

DEPARTMENT OF MATHEMATICS
NORTHEASTERN UNIVERSITY
BOSTON, MA. 02115
U.S.A.

DIPARTMENTO DI MATEMATICA
UNIVERSITÀ DI CATANIA
95125 CATANIA
ITALY