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THE F. AND M. RIESZ THEOREM REVISITED

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1. Introduction.

The celebrated F. and M. Riesz theorem states: if μ is a complex Borel measure on the unit circle $\boldsymbol{T}$ such that

$$\hat{\mu}(n) = \int_{\boldsymbol{T}} e^{-in\theta} d\mu(\theta) = 0 \quad \text{for } n = -1, -2, \ldots,$$

then μ is absolutely continuous with respect to Lebesque mesaure on $\boldsymbol{T}$. Some forty years later, Helson and Lowdenslager [4] generalized the F. and M. Riesz theorem to compact Abelian groups with ordered duals. deLeeuw and Glicksberg [1], Doss [2], [3], and Yamaguchi [11] shortly afterwards obtained a number of related results. In this note we present simple and perspicuous proofs of these theorems by using the Helson-Lowdenslager theorem and some other well-known facts. In particular, we will prove Yamaguchi's theorem without using the theory of disintegration.

We are very grateful to Professor S. Saeki for showing us the proof of Theorem C for $G = \mathbf{R}^n$. His idea was also a guide for our proof for the case in which G contains a compact open subgroup.

2. Preliminaries and four theorems.

Let G be an Abelian group. We say that G is an *ordered group* if G contains a subsemigroup P such that $P \cup (-P) = G$ and $P \bigcap (-P) = \{0\}$. (We will refer to P as an *order in* G.) It is well known that G is an ordered group if and only if G is torsion-free (see [5]).

Let G be a locally compact Abelian group and let $\hat{G}$ be its dual group. (The term "locally compact Abelian group" means "locally compact Abelian group satisfying Hausdorff's separation axiom".) A fixed but arbitrary Haar measure on G will be denoted by m_G. The symbol $M(G)$ will denote the Banach algebra of all bounded regular complex Borel measures on G under con-

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volution multiplication and the total variation norm. For an element x in G, δ_x denotes the Dirac measure at x. For μ in $M(G)$, let μ_a and μ_s be respectively the absolutely continuous and singular parts of μ with respect to m_G. We denote the Fourier-Stieltjes transform of a measure μ by $\hat{\mu}$ and convolution of measures μ and ν by $\mu * \nu$. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish on $\hat{G}\backslash E$.

All notation and terminology not explained in the sequel is as in [5].

We now state the four Theorems mentioned in section 1.

THEOREM A (Helson–Lowdenslager, cf. [9, Theorem 8.2.3]). *Let G be a compact Abelian group with ordered dual $\hat{G}$ and let P be an order in $\hat{G}$. If μ is a measure in $M_P(G)$, then μ_a and μ_s belong to $M_P(G)$ and moreover* $\hat{\mu}(0) = 0$.

THEOREM B (deLeeuw–Glicksberg [1, Proposition 5.1]). *Let G be a compact Abelian group and let ψ be a nontrivial homomorphism of $\hat{G}$ into the additive group $\mathbf{R}$ of real numbers. If μ is a measure in $M(G)$ such that $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \hat{G}$ with $\psi(\gamma) \leqq 0$, then $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \hat{G}$ with $\psi(\gamma) \leqq 0$.*

THEOREM C (Doss [3, Lemma 1]). *Let G be a locally compact Abelian group with ordered dual $\hat{G}$ and let P be an order in $\hat{G}$. If μ is a measure in $M_P(G)$, then μ_a and μ_s also belong to $M_P(G)$ and moreover* $\hat{\mu}_s(0) = 0$.

THEOREM D (Yamaguchi [11]). *Let G be a locally compact Abelian group and let P be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If μ is a measure in $M_P(G)$, then μ_a and μ_s also belong to $M_P(G)$.*

REMARK 2.1. In his paper [11], Yamaguchi also showed the following. Let G, $\hat{G}$, and P be as in Theorem D. If μ is a measure in $M_{P^c}(G)$, then μ_a and μ_s belong to $M_{P^c}(G)$. To prove Theorems C and D, it suffices to prove them with $M_P(G)$ replaced by $M_{P^c}(G)$: Yamaguchi proved this in [11, pp. 244–245]. We will prove Theorems C and D in this form.

The cited proof of Theorem A is unexceptionable, and Theorem A will be used in our work. We will generalize Theorem B. Doss's proof of Theorem C is flawed, since he tacitly assumes that P is Haar measurable (which as shown in [5] need not be the case). It seems worthwhile to present a short proof of Theorem C. Yamaguchi's proof of Theorem D is in part impenetrable, and again our simple proof seems preferable.

3. Generalized Theorem B.

In this section we prove a generalization of Theorem B (see Theorem 3.6). We first prove the theorem for a compact metrizable Abelian group by using a result on measurable selections. We next prove it for all compact Abelian groups by using a lemma due to Pigno and Saeki ([8, Lemma 4]).

We first prove a simple corollary of Theorem A.

LEMMA 3.1. *Let G be a compact Abelian group with torsion-free dual $\hat{G}$ and let P be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If μ is a measure in $M_{P^c}(G)$, then μ_a and μ_s also belong to $M_{P^c}(G)$.*

PROOF. We may suppose that $P \cap (-P) \neq \{0\}$: otherwise the lemma is Theorem A. Since $P \cap (-P)$ is a torsion-free Abelian group, there is a subsemigroup Q of $P \cap (-P)$ such that $Q \cup (-Q) = P \cap (-P)$, and $Q \cap (-Q) = \{0\}$ (see [5, Remark (2.6)]). We write

$$P_1 = (P \backslash Q) \cup \{0\}$$

and

$$P_2 = (P \backslash (-Q)) \cup \{0\}.$$

A short argument, which we omit, shows that P_1 and P_2 are subsemigroups of $\hat{G}$, that $P_1 \cup (-P_1) = P_2 \cup (-P_2) = \hat{G}$, that $P_1 \cap (-P_1) = P_2 \cap (-P_2) = \{0\}$, and that $P_1 \cup P_2 = P$. Suppose that μ is a measure in $M_{P^c}(G)$. In particular we have $\hat{\mu}(\gamma) = 0$ for all $\gamma \in P_1$. Theorem A shows that $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P_1$; similarly $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P_2$. Since $P_1 \cup P_2 = P$, we have $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P$.

The following Lemma 3.2 is due to Ryll–Nardzewski [7], [10].

LEMMA 3.2. *Let X be a metric space and let Y be a separable and complete metric space. Let $\mathscr{F}$ be the family of all nonvoid closed subsets of Y. Let Σ be a mapping from X to $\mathscr{F}$ with the following property: $\{x \in X : \Sigma(x) \subset K\}$ is closed in X for each closed subset K of Y. Then there exists a mapping σ from X into Y such that:*

(i) $\sigma(x) \in \Sigma(x)$ *for each* $x \in X$;

and

(ii) $\sigma^{-1}(U)$ *is a Borel subset of X for each open subset U of Y.*

LEMMA 3.3. *Let G be a compact metrizable Abelian group, let H be a closed subgroup of G, and let π be the natural homomorphism of G onto G/H. Then there exists a mapping σ from G/H into G with the following properties:*

(i) $\pi \circ \sigma(\dot{x}) = \dot{x}$ *for each* $\dot{x} \in G/H$;

(ii) $\sigma^{-1}(U)$ *is a Borel subset of G/H for each open subset U of G.*

PROOF. We use Lemma 3.2 with $X = G/H$, $Y = G$, and $\Sigma(\dot{x}) = x+H$, where $\pi(x) = \dot{x}$. We need only to verify that the set

$$A_k = \{\dot{x} \in G/H\,;\ \Sigma(\dot{x}) = x+H \subset K\}$$

is closed in G/H for each closed subset K of G. This is simple. Indeed, let $\{\dot{x}_n\}$ be a sequence in A_k such that $\{\dot{x}_n\}$ converges to an element $\dot{x}$ in G/H. Choose elements x_n and x in G such that $\pi(x_n) = \dot{x}_n$ for $n = 1, 2, \ldots$ and $\pi(x) = \dot{x}$. Since G is compact and metric, a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges to an element x_0 of G. Then $\{\dot{x}_{n_j}\}$ converges to $\dot{x}_0 = \pi(x_0)$, and so $\dot{x}_0 = \dot{x}$. For $h \in H$, we have $x_0 + h \in K$ because $\{x_{n_j} + h\}$ converges to $x_0 + h$ and $x_{n_j} + h \in K$. That is, $x_0 + H \subset K$; and so $x + H \subset K$ because $\dot{x}_0 = \dot{x}$.

LEMMA 3.4. *Let G be a compact metrizable Abelian group and let P be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If μ is a measure in $M_{P^c}(G)$, then μ_a and μ_s are also in $M_{P^c}(G)$.*

PROOF. Since G is a compact metrizable Abelian group, G may be regarded as a closed subgroup of the countably infinite dimensional torus G_0 (see [9, Theorem 2.2.6]).

By Lemma 3.3, there exists a mapping σ from G_0/G to G_0 with the following properties:

(i) $\pi \circ \sigma(\dot{x}) = \dot{x}$ for each $\dot{x} \in G_0/G$;
(ii) $\sigma^{-1}(U)$ is a Borel subset of G_0/G for each open subset U of G_0,

where π denotes the natural homomorphism from G_0 onto G_0/G.

It suffices to show that μ_s belongs to $M_{P^c}(G)$ if μ does. Assume the contrary: there is a measure μ in $M_{P^c}(G)$ such that $\hat{\mu}_s(\gamma_0) \neq 0$ for some $\gamma_0 \in P$. By considering $\bar{\gamma}_0\mu$, we may suppose that $\hat{\mu}_s(0) \neq 0$. We will also consider μ as a measure in $M(G_0)$.

Now we define a function on G_0/G for each bounded Borel function on G_0 and each $\nu \in M(G_0)$ as follows:

$$(\dagger) \qquad \dot{x} \to \nu * \delta_{\sigma(\dot{x})}(f) \quad \text{for } \dot{x} \in G_0/G.$$

It is obvious that the mapping $(\dagger)$ is a bounded Borel function on G_0/G for each bounded Borel function f on G_0 and each $\nu \in M(G_0)$. Thus we can define measures λ, λ_1, and λ_2 in $M(G_0)$ as follows:

$$(\ddagger)\qquad \begin{cases} \lambda(f) = \displaystyle\int_{G_0/G} \mu * \delta_{\sigma(\dot{x})}(f)\,dm_{G_0/G}(\dot{x}); \\ \lambda_1(f) = \displaystyle\int_{G_0/G} \mu_a * \delta_{\sigma(\dot{x})}(f)\,dm_{G_0/G}(\dot{x}); \\ \lambda_2(f) = \displaystyle\int_{G_0/G} \mu_s * \delta_{\sigma(\dot{x})}(f)\,dm_{G_0/G}(\dot{x}) \end{cases}$$

for $f \in C(G_0)$.

Note that the equalities ($\ddagger$) hold for each bounded Borel function g on G_0. This can be easily verified by approximating a bounded Borel function on G_0 by continuous functions on G_0.

We will show that λ_1 and λ_2 are respectively the absolutely continuous and singular parts of λ with respect to m_{G_0}. Once we have proved this fact, the Lemma can be established as follows. Define

$$\tilde{P} = \{\gamma \in \hat{G}_0 : \gamma|G \in P\},$$

where $\gamma|G$ denotes the restriction of γ to G. It is obvious that $\tilde{P}$ is a subsemigroup of $\hat{G}_0$ and that $\tilde{P} \cup (-\tilde{P}) = \hat{G}_0$. If γ is an element of $\tilde{P}$, then $\gamma|G \in P$ and therefore we have

$$\begin{aligned} \mu * \delta_{\sigma(\dot{x})}(\bar{\gamma}) &= \hat{\mu}(\gamma)(-\sigma(\dot{x}), \gamma) \\ &= \hat{\mu}(\gamma|G)(-\sigma(\dot{x}), \gamma) \\ &= 0 \end{aligned}$$

for each $\dot{x} \in G_0/G$. We infer that

$$\begin{aligned} \hat{\lambda}(\gamma) &= \lambda(\bar{\gamma}) \\ &= \int_{G_0/G} \mu * \delta_{\sigma(\dot{x})}(\bar{\gamma})\,dm_{G_0/G}(\dot{x}) \\ &= 0 \end{aligned}$$

for each $\gamma \in \tilde{P}$. On the other hand, we have

$$\begin{aligned}\hat{\lambda}_s(0) &= \hat{\lambda}_2(0)\\ &= \int_{G_0/G} \mu_s * \delta_{\sigma(\dot{x})}(1) dm_{G_0/G}(\dot{x})\\ &= \hat{\mu}_s(0) \neq 0.\end{aligned}$$

The group $\hat{G}_0$ is the weak direct sum of countably many copies of the integers and so is torsion-free. This contradicts Lemma 3.1.

To complete the present proof, we need to prove that λ_1 and λ_2 are respectively the absolutely continuous and singular parts of λ with respect to m_{G_0}. Since $\lambda = \lambda_1 + \lambda_2$, it is sufficient to show the following:

(I) λ_1 is absolutely continuous with respect to m_{G_0};
(II) λ_2 is singular with respect to m_{G_0}.

To prove (I), let E be a Borel subset of G_0 such that $m_{G_0}(E) = 0$. Since

$$\begin{aligned}0 &= m_{G_0}(E)\\ &= \int_{G_0/G} \int_G 1_E(x+y) dm_G(y) dm_{G_0/G}(\dot{x}) \quad (\dot{x} = \pi(x)),\end{aligned}$$

there exists a Borel subset A of G_0/G such that $m_{G_0/G}(A) = 0$ and

$$\int_G 1_E(x+y) dm_G(y) = 0$$

for all x in G_0 such that $\pi(x) \in A^c \subset G_0/G$. For $\dot{x} \in A^c$, we have

$$\mu_a * \delta_{\sigma(\dot{x})}(1_E) = \int_G 1_E(\sigma(\dot{x}) + y) \frac{d\mu_a}{dm_G}(y) dm_G(y).$$

Since $\sigma(\dot{x}) \in \pi^{-1}(\dot{x})$, it follows that $\mu_a * \delta_{\sigma(\dot{x})}(1_E) = 0$. Accordingly we have

$$\lambda_1(E) = \int_{G_0/G} \mu_a * \delta_{\sigma(\dot{x})}(1_E) dm_{G_0/G}(\dot{x})$$

$$= \int_A \mu_a * \delta_{\sigma(\dot{x})}(1_E) dm_{G_0/G}(\dot{x}) +$$

$$+ \int_{A^c} \mu_a * \delta_{\sigma(\dot{x})}(1_E) dm_{G_0/G}(\dot{x})$$

$$= 0.$$

This proves (I).

We now prove (II). Employing the canonical decomposition of μ_s as a linear combination of nonnegative (singular!) measures, we may suppose that μ_s is nonnegative and that $\|\mu_s\| = 1$.

Since G_0 is compact and metrizable, $C(G_0)$ contains a countable dense subset $\{f_n\}$. Let ε be a positive real number. For $n = 1, 2, \ldots$, Luzin's theorem shows that there exists a compact subset E_n of G_0/G such that $m_{G_0/G}(E_n^c) < \varepsilon/2^n$ and $\dot{x} \to \mu_s * \delta_{\sigma(\dot{x})}(f_n)$ is continuous on E_n. We write $E = \bigcap_{n=1}^{\infty} E_n$. Then E is compact and $m_G(E^c) < \varepsilon$ and $\dot{x} \to \mu_s * \delta_{\sigma(\dot{x})}(f_n)$ is continuous on E for $n = 1, 2, \ldots$. Since $\{f_n\}$ is dense in $C(G_0)$, $\dot{x} \to \mu_s * \delta_{\sigma(\dot{x})}(h)$ is continuous on E for each $h \in C(G_0)$. The measure $\mu_s * \delta_{\sigma(\dot{x})}$ is singular with respect to m_G for each $\dot{x} \in G_0/G$. A short argument, which we omit, shows that for each $\dot{x} \in E$, there is an $f \in C(G_0)$ with $0 \leqq f \leqq 1$ such that:

$$(\dagger\dagger) \qquad \begin{cases} 1 = \|\mu_s * \delta_{\sigma(\dot{x})}\| < \mu_s * \delta_{\sigma(\dot{x})}(f) + \varepsilon; \\ \delta_x * m_G(f) < \varepsilon \quad \text{for } x \in \pi^{-1}(\{\dot{x}\}), \end{cases}$$

since $G + \sigma(\dot{x}) = \pi^{-1}(\{\dot{x}\})$.

Since $\dot{x} \to \mu_s * \delta_{\sigma(\dot{x})}(f)$ is continuous on E and $x \to \delta_x * m_G(f)$ is continuous on G, the inequalities ($\dagger\dagger$) hold on some neighborhood $U_{\dot{x}}$ of $\dot{x}$ in E. Since E is compact, there exist $\dot{x}_1, \dot{x}_2, \ldots$, and $\dot{x}_k$ in E such that $\bigcup_{j=1}^{k} U_{\dot{x}_j} = E$. We denote the f's that correspond to $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_k$ by $f_1, f_2, \ldots, f_k$, respectively. Now we define a function g on G_0 as follows:

$$g = \begin{cases} f_1 & \text{on } \pi^{-1}(U_{\dot{x}_1}); \\ f_2 & \text{on } \pi^{-1}(U_{\dot{x}_2} \setminus U_{\dot{x}_1}); \\ \vdots & \vdots \\ f_k & \text{on } \pi^{-1}(U_{\dot{x}_k} \setminus \bigcup_{j=1}^{k-1} U_{\dot{x}_j}); \\ 0 & \text{on } \pi^{-1}(E^c). \end{cases}$$

Then g is a Borel function on G_0 such that $0 \leqq g \leqq 1$, $1-\varepsilon < \mu_s * \delta_{\sigma(\dot{x})}(g)$, and $\delta_x * m_G(g) < \varepsilon$ for each $\dot{x} \in E$ and $x \in \pi^{-1}(\{\dot{x}\})$. Hence we have

$$\lambda_2(g) = \int_{G_0/G} \mu_s * \delta_{\sigma(\dot{x})}(g) dm_{G_0/G}(\dot{x}) > 1-2\varepsilon$$

and

$$\begin{aligned} m_{G_0}(g) &= \int_{G_0} g\, dm_{G_0} \\ &= \int_{G_0/G} \int_G g(x+y) dm_G(y) dm_{G_0/G}(\dot{x}) \\ &= \int_E \int_G g(x+y) dm_G(y) dm_{G_0/G}(\dot{x}) \\ &< \varepsilon m_{G_0/G}(E) \leqq \varepsilon. \end{aligned}$$

Since this holds for each $\varepsilon > 0$, λ_2 is singular with respect to m_{G_0}.

We quote the following lemma from Pigno and Saeki [8, Lemma 4].

LEMMA 3.5. *Let G be a nonmetrizable locally compact Abelian group, and let D be a σ-compact subset of G with $m_G(D) = 0$. Then, given a σ-compact subset Δ of $\hat{G}$, we can find a σ-compact, non-compact, open subgroup Γ of $\hat{G}$ which contains Δ and satisfies $m_G(D+\Gamma^{\perp}) = 0$.*

THEOREM 3.6. *Let G be a compact Abelian group and let P be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If μ is a measure in $M_{P^c}(G)$, then μ_a and μ_s also belong to $M_{P^c}(G)$.*

PROOF. It suffices to show that $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P$. Since μ_s is a singular measure, we can choose a σ-compact subset E of G such that $m_G(E) = 0$ and $|\mu_s|(E^c) = 0$. Let γ_0 be any element of P. By Lemma 3.5, there is a countable subgroup Γ of $\hat{G}$ containing γ_0 such that

$$(1) \qquad m_G(E+\Gamma^{\perp}) = 0.$$

Let π be the natural homomorphism from G onto $G/\Gamma^{\perp}$. By (1), we have

$$(2) \qquad (\pi(\mu))_s = \pi(\mu_s),$$

where $\pi(\mu)$ denotes the image of μ under $\pi : \pi(\mu)(A) = \mu(\pi^{-1}(A))$ for Borel subsets A of $G/\Gamma^{\perp}$. Write $P' = P \cap \Gamma$. Then P' is a subsemigroup of Γ such that $P' \cup (-P') = \Gamma$, and $(\pi(\mu))\hat{}(\gamma') = 0$ for all $\gamma' \in P'$. (Recall that the dual group of $G/\Gamma^{\perp}$ is Γ and therefore $\hat{\mu}(\gamma) = (\pi(\mu))\hat{}(\gamma)$ for all $\gamma \in \Gamma$.) Since $\Gamma = (G/\Gamma^{\perp})\hat{}$ is countable $G/\Gamma^{\perp}$ is metrizable and hence (2) and Lemma 3.4 imply that $(\pi(\mu_s))\hat{}(\gamma') = (\pi(\mu)_s)\hat{}(\gamma') = 0$ for all $\gamma' \in P'$. It follows that $\hat{\mu}_s(\gamma_0) = (\pi(\mu_s))\hat{}(\gamma_0) = 0$. Since γ_0 is an arbitrary element of P, we have $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P$.

REMARK 3.7. Let ψ be as in Theorem B. If we put

$$P = \{\gamma \in \hat{G} ; \psi(\gamma) \leqq 0\}$$

and apply Theorem 3.6, we obtain Theorem B.

Theorem 3.6 is strictly stronger than Theorem B. To see this, consider the compact group T^3 and its dual group $\mathbf{Z}^3$. Let P be

$$\{(x, y, z) \in \mathbf{Z}^3 ; z > 0\} \cup \{(x, y, 0) \in \mathbf{Z}^3 ; x \geqq 0\}.$$

Plainly P is a subsemigroup of $\mathbf{Z}^3$ such that $P \cup (-P) = \mathbf{Z}^3$. If ψ is a nonzero homomorphism of $\mathbf{Z}^3$ into $\mathbf{R}$ nonnegative on P, we have $\psi((x, y, z)) = \alpha z$ with $\alpha > 0$. Thus ψ vanishes for all $(x, y, 0)$ and

$$P \subsetneqq \psi^{-1}(\{x \in \mathbf{R} | x \geqq 0\}).$$

Thus Theorem B cannot prove Theorem 3.6.

4. Proof of Theorem C.

As we noted in Remark 2.1, we will prove Theorem C with $M_P(G)$ replaced by $M_{P^c}(G)$. We make use of the structure theorem for locally compact Abelian groups (see [6, Theorem (24.30)]) and examine two cases.

We may suppose that $\hat{G}$ is nondiscrete: otherwise the Theorem reduces to Theorem A. It suffices to show that if μ is a measure in $M_{P^c}(G)$, then μ_s belongs to $M_{P^c}(G)$. By the structure theorem, $\hat{G}$ has the form $\mathbf{R}^n \oplus X$, where n is a nonnegative integer and X is a locally compact Abelian group containing a compact open subgroup Λ. We examine two cases.

CASE I: $n = 0$. Since $X = \hat{G}$ is nondiscrete, Λ is infinite, and $\hat{G}/\Lambda$ is discrete. If we put $H = \Lambda^{\perp}$, then H is a compact open subgroup of G, and G/H is discrete. The dual group of G/H is of course $H^{\perp} = \Lambda$. Let μ be a measure in $M_{P^c}(G)$. Since μ has σ-compact support, there exists a sequence $\{x_n\}$ of elements in G such that $\mu = \sum_{n=1}^{\infty} \mu_{x_n+H}$ and $x_i + H \neq x_j + H$ if $i \neq j$, where μ_{x_n+H} denotes the restriction of μ to $x_n + H$. Observe that

$$(1) \qquad \|\mu\| = \sum_{n=1}^{\infty} \|\mu_{x_n+H}\|.$$

Put

$$(2) \qquad \lambda_n = \mu_{x_n+H} * \delta_{-x_n} \quad \text{for } n = 1, 2, \ldots$$

so that $\lambda_n \in M(H)$ and $\mu = \sum_{n=1}^{\infty} \lambda_n * \delta_{x_n}$. We obtain

$$(3) \qquad \hat{\mu}(\gamma) = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma)(-x_n, \gamma) \quad \text{for } \gamma \in \hat{G}.$$

Since H is open in G, it is obvious that $\mu_s = \sum_{n=1}^{\infty} (\lambda_n)_s * \delta_{x_n}$, and so

$$(4) \qquad \hat{\mu}_s(\gamma) = \sum_{n=1}^{\infty} ((\lambda_n)_s)\hat{\ }(\gamma)(-x_n, \gamma) \quad \text{for } \gamma \in \hat{G}.$$

We will now show that if $\gamma \in P$, then $\hat{\lambda}_n(\gamma + \Lambda) = 0$ for $n = 1, 2, \ldots$. (Recall that the dual group of H is $\hat{G}/\Lambda$.) As a measure in $M(H)$, λ_n has a Fourier-Stieltjes transform constant on cosets of $H^{\perp} = \Lambda$. Thus we may write $\hat{\lambda}_n(\gamma + \Lambda)$ for $\gamma \in \hat{G}$. For a fixed γ in P, define

$$\nu = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma + \Lambda)(-x_n, \gamma)\delta_{x_n+H}.$$

(This series converges in the total variation norm on $M(G/H) = l^1(G/H)$ because of (1) and (2).) By (3), we have for every $\gamma' \in \Lambda$

$$\hat{\nu}(\gamma') = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma+\Lambda)(-x_n,\gamma)(-x_n+H,\gamma')$$

$$= \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma+\gamma')(-x_n,\gamma+\gamma')$$

$$= \hat{\mu}(\gamma+\gamma').$$

Since Λ is a compact infinite torsion-free Abelian group, $P \cap \Lambda$ is dense in Λ (see [5, Theorem (3.2)]). Thus for $\gamma' \in \Lambda$, there exists a net $\{\gamma_\alpha\}$ in $P \cap \Lambda$ such that $\{\gamma_\alpha\}$ converges to γ'. Since all $\gamma+\gamma_\alpha$ are in P, we have $\hat{\nu}(\gamma_\alpha) = \hat{\mu}(\gamma+\gamma_\alpha) = 0$ for all α. Since $\hat{\nu}$ is continuous, we have

$$\hat{\nu}(\gamma') = \lim_\alpha \hat{\nu}(\gamma_\alpha) = 0.$$

Since this holds for each $\gamma' \in \Lambda$, ν must be the zero measure, which is to say that $\hat{\lambda}_n(\gamma+\Lambda) = 0$ for $n = 1, 2, \ldots$.

Now let π be the natural homomorphism from $\hat{G}$ onto $\hat{G}/\Lambda$ and put $\tilde{P} = \pi(P)$. Then $\tilde{P}$ is a subsemigroup of $\hat{G}/\Lambda$ and $\hat{G}/\Lambda = \tilde{P} \cup (-\tilde{P})$. We have just shown that $\hat{\lambda}_n(\gamma+\Lambda) = 0$ for each $\gamma+\Lambda \in \tilde{P}$ and $n = 1, 2, \ldots$. Theorem 3.6 implies that $(\lambda_n)_s\hat{}(\gamma+\Lambda) = 0$ for each $\gamma+\Lambda \in \tilde{P}$ and $n = 1, 2, \ldots$. From (4) we conclude that

$$\hat{\mu}_s(\gamma) = \sum_{n=1}^{\infty} (\lambda_n)_s\hat{}(\gamma+\Lambda)(-x_n,\gamma)$$

$$= 0$$

for each $\gamma \in P$.

CASE II: $n > 0$. We write elements of $\mathbf{R}^n \oplus X$ as (a,γ) where $a \in \mathbf{R}^n$ and $\gamma \in X$. Define

$$H = (\mathbf{Z}^n \oplus X)^\perp (= \mathbf{Z}^n \oplus \{0\})$$

and put $P' = P \bigcap (\mathbf{Z}^n \oplus X)$. Let π be the natural homomorphism from $G = \mathbf{R}^n \oplus \hat{X}$ onto $\mathbf{R}^n \oplus \hat{X}/\mathbf{Z}^n \oplus \{\dot{0}\}$. Let μ be a measure in $M_{P^c}(G)$. Fix an element (a_0,γ_0) in P and define $\sigma = (-a_0, -\gamma_0)\mu$. Let $\pi(\sigma)$ denote the image of σ under $\pi : \pi(\sigma)$ is an element of $M(\mathbf{R}^n \oplus \hat{X}/\mathbf{Z}^n \oplus \{0\})$. We have

$$(\pi(\sigma))\hat{}((m,\gamma)) = \hat{\mu}((a_0,\gamma_0)+(m,\gamma))$$

$$= 0$$

for all $(m, \gamma) \in P'$ because $(a_0, \gamma_0) + (m, \gamma) \in P$. Note that the dual group of $\mathbf{R}^n \oplus \hat{X}/\mathbf{Z}^n \oplus \{0\}$ is $(\mathbf{Z}^n \oplus \{0\})^\perp = \mathbf{Z}^n \oplus X$. Since $\mathbf{Z}^n \oplus X$ is a group dealt with in Case I, we have

$$(5) \qquad (\pi(\sigma))_s^{\wedge}((m, \gamma)) = 0$$

for all $(m, \gamma) \in P'$. The group $\mathbf{Z}^n \oplus \{0\}$ is countable and so if E is a Borel subset of G with $m_G(E) = 0$, we have $m_G(E + (\mathbf{Z}^n \oplus \{0\})) = 0$. This implies that $\pi(\sigma_s)$ is singular. Since $\pi(L^1(G)) = L^1(G/H)$ if π is the natural homomorphism of G onto G/H, it follows that $\pi(\sigma_s) = (\pi(\sigma))_s$.

Combine this with (5) to obtain

$$\begin{aligned} \hat{\mu}_s((a_0, \gamma_0)) &= \hat{\sigma}_s((0, 0)) \\ &= (\pi(\sigma_s))^{\wedge}((0, 0)) \\ &= ((\pi(\sigma))_s)^{\wedge}((0, 0)) \\ &= 0. \end{aligned}$$

Since (a_0, γ_0) is an arbitrary element of P, we have $\hat{\mu}_s((a, \gamma)) = 0$ for all $(a, \gamma) \in P$.

REMARK 4.1. We may use Theorem C and the argument in the proof of Lemma 3.1 to obtain the following special case of Theorem D.

Let G be a locally compact Abelian group with torsion-free dual group $\hat{G}$ and let P be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If μ is a measure in $M_{P^c}(G)$, then μ_a and μ_s are also in $M_{P^c}(G)$.

5. Proof of Theorem D.

To prove Theorem D, we will make use of two fundamental facts about locally compact Abelian groups.

By [6, Theorem (A.15) and Theorem (25.32)(a)], we can find a divisible locally compact Abelian group D such that D contains G as an open subgroup. We define

$$\tilde{P} = \{\gamma \in \hat{D}; \gamma | G \in P\}.$$

It is obvious that $\tilde{P}$ is a subsemigroup of $\hat{D}$ and $\hat{D} = \tilde{P} \cup (-\tilde{P})$. Let μ be a measure in $M_{P^c}(G)$. It suffices to prove that $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P$. We will regard μ as being a measure in $M(D)$. Since G is open in D, μ_a and μ_s are respectively the absolutely continuous and singular parts of μ with repect to m_D. Our first aim is to prove that $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \tilde{P}$ when μ is regarded as a measure in $M(D)$. If $\gamma \in \tilde{P}$, then $\gamma|G$ is in P and therefore

$$\hat{\mu}(\gamma) = \int_D (-x, \gamma)d\mu(x)$$
$$= \int_G (-x, \gamma|G)d\mu(x)$$
$$= 0.$$

Since $\hat{D}$ is torsion-free (see [6, Theorem (24.23)]), Remark 4.1 gives us $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \tilde{P}$.

Next we take an element γ in P. There is an element γ_0 of $\tilde{P}$ such that $\gamma_0|G = \gamma$. We find that

$$\hat{\mu}_s(\gamma) = \int_G (-x, \gamma)d\mu_s(x)$$
$$= \int_G (-x, \gamma_0|G)d\mu_s(x)$$
$$= \int_D (-x, \gamma_0)d\mu_s(x) = 0.$$

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