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DIFFERENTIATION UNDER THE INTEGRAL SIGN AND HOLOMORPHY

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Abstract.

In this paper we prove the following general exponential law for holomorphic functions in infinite dimensions:

$$H_e(U \times V, G) \cong H_e(U, H_e(V, G)).$$

This generalizes exponential laws for holomorphic functions in [7], [1], [6].

1. Introduction.

The integral $\int_a^b f(t)dt$ of a continuous function $f:[a,b] \to E$ into a convergence vector space (abbreviated cvs) E was in [5] defined to be the element $l \mapsto \int_a^b l \circ f(t)dt$ of the second dual $LL_{co}E$, where co denotes the compact-open topology. This definition, combined with continuous convergence (see [2]) on function spaces, led for an L_c-embedded E (see [14], [3]) promptly to useful results concerning preservation of limits and continuity of $x \mapsto \int_a^b f(t,x)dt$ (cf. Lemma 3.2). The main reasons for this simplicity are: 1) the reflexivity $LL_{co}C_c(X) = C(X)$ of the algebra of continuous functions on a convergence space X, when endowed with continuous convergence; 2) the cartesian closedness of the category of convergence spaces.

In the present paper a notion of continuous differentiability (called (D2)) is shown to combine well with the integral and leads in a natural way to, for instance, a theorem on differentiation under the integral sign. The class of L_c-embedded cvs is large enough to contain all Hausdorff locally convex spaces. There are, however, important classes of cvs which ar not L_c-embedded. For instance, bornological vector spaces (see [11]), endowed with bornological convergence, are not L_c-embedded in general. We therefore develop a parallel theory for functions with values in L_e-embedded spaces (see [3]). The class of these spaces contains not only all Hausdorff locally convex spaces but all polar bornological vector spaces as well. A structure, which we call "local

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uniform convergence" and denote with the subscript e is now used on function spaces.

The theory developped in section 3 provides a basis for infinite-dimensional analysis in general. But in this paper it is only applied to holomorphic functions and spaces of holomorphic functions. It is shown that a function $f: U \to F$ defined on a τ-open set and with values in an L_c- or L_c-embedded space is holomorphic (i.e. Gâteaux holomorphic and continuous) iff it is complex differentiable in the sense (D2) when considered as a function into a sequential completion of F. Power series expansions for holomorphic functions on τ-open subsets of equable cvs with values in L_e-embedded spaces are derived. These expansions have stronger convergence properties than the ones in [5]. A new general exponential law

$$H_e(U \times V, G) \cong H_e(U, H_e(V, G))$$

for spaces of holomorphic functions is derived. Its connection with results of Colombeau [6] for bornological spaces will be studied in a forthcoming paper.

As the referee has pointed out to the author, A. Kriegl and L. D. Nel discuss another form of holomorphy in [13]. They also prove an exponential law.

A *convergence space* X [8] is a set, on which with each point $x \in X$ is associated a set of filters, which are said to converge to x, and are such that the following conditions hold:

1) The trivial ultrafilter associated with x, always convergences to x;

2) If $\mathscr{F} \geqq \mathscr{G}$ and $\mathscr{G}$ converges to x, then $\mathscr{F}$ converges to x;

3) If $\mathscr{F}$ and $\mathscr{G}$ converge to x, then $\mathscr{F} \cap \mathscr{G}$ converges to x.

A *convergence vector space* (cvs) (see [8], [9]) is a convergence space with a vector structure, such that the vector operations are continuous (a map is continuous if it preserves convergence). All vector spaces in this paper have the scalar field $\boldsymbol{K}$ (= R or C). A cvs E is said to be *equable* (see [9]) if each filter which converges to 0 in E contains a filter $\mathscr{G}$, such that $\boldsymbol{V}\mathscr{G} = \mathscr{G}$ and $\mathscr{G}$ converges to 0 in E. Here $\boldsymbol{V}$ denotes the 0-neighbourhood filter of $\boldsymbol{K}$. Clearly there exists on E a coarsest equable vector convergence structure finer than the original structure on E. The vector space E endowed with this equable structure is denoted by E^e. For a convergence space X and a cvs E $C_c(X, E)$ denotes the vector space of all continuous $f: X \to E$ endowed with continuous convergence (see [2]). A net $(f_\iota)_\iota$ converges to zero in $C_c(X, E)$ iff for each $x \in X$ and each net $(x_\kappa)_\kappa$, which converges to x, the net $(f_\iota(x_\kappa))_{\iota\kappa}$ converges to zero in E. The convergence structure of

$$C_e(X, E) = (C_c(X, E))^e$$

is in this paper called "local uniform convergence" (in fact, it *is* local uniform convergence if E is a normed space). A net $(f_\iota)_\iota$ converges to zero in $C_e(X, E)$ (where E is a locally convex cvs; see below) iff for each $x \in U$ and each $(x_\kappa)_\kappa$, which converges to x, there is a filter $\mathscr{G}$, converging to zero in F, such that for each $G \in \mathscr{G}$ exists a κ_0, such that $q_G(f_\iota(x_\kappa))$ converges to zero uniformly on $\{x_\kappa : \kappa \geqq \kappa_0\}$ (q_G denotes the gauge of G). A cvs E is said to be *L_c-embedded* (or *L_e-embedded*) if the mapping $j_E : E \to L_c L_c E$ (or $j_E : E \to L_e L_e E$), $j_E(x)l = l(x)$, into the second dual is an embedding (see [14], [3]). We say that E is *circled* (or *locally convex* if each filter which converges to 0 in E contains a filter base $\mathscr{B}$ of circled (or circled convex) sets, such that $\mathscr{B}$ converges to zero in E.

With each point x in a locally convex (topological) vector space E is associated the set of filters, which converge to x with respect to the topology. Thus E can be considered as a cvs. A bornological vector space E with bornology $\mathscr{B}$ can be identified with a cvs in the following way: Each filter on E, which contains a filter of the form $\boldsymbol{V}B$ for some $B \in \mathscr{B}$, is said to converge to zero (bornological convergence). Convergence to other points is obtained by translation. All Hausdorff locally convex topological vector spaces are L_c- and L_e-embedded and all polar bornological vector spaces (i.e. such with a bornology base of sets B with $B^{\circ\circ} = B$) are L_e-embedded. However, most cvs belong to neither of these classes. As an example, consider the algebra $L_c(\mathscr{D}, \mathscr{D})$ $(= L_e(\mathscr{D}, \mathscr{D}) \cong L_e(\mathscr{D} \otimes L_e\mathscr{D}))$ of continuous endomorphisms on the locally convex space $\mathscr{D}$ of C^∞-functions on R^n with compact support. This space is L_c-embedded and L_e-embedded, but is neither topological nor bornological, since $L_e\mathscr{D}$ is a bornological vector space, which is not a normed space.

2. Differentiable functions.

Let E be a separated cvs and let I be an interval. We say that a function $f : I \to E$ is *differentiable* if the limit of $h^{-1}(f(t+h) - f(t))$ as $h \to 0$, called the derivative $f'(t)$ of f at t, exists in E (we form the one-sided limit if t is an endpoint of I) and *continuously* differentiable if $f' : I \to E$ is continuous.

For any cvs E we denote by E_τ the vector space E, endowed with the finest locally convex topology, which is coarser than the convergence structure of E.

LEMMA 2.1. *Let E be a real* cvs. *If a function $f \in C([a, b], E)$ is differentiable on $]a, b[$, then*

$$f(b)-f(a)\in(b-a)\mathrm{cl}_\tau\Gamma f']a,b[,$$

where the closed convex hull $\mathrm{cl}_\tau\Gamma f']a,b[$ *is formed in* E_τ (cf. *also* [8]).

PROOF. For each $l\in LE$ there is a number $\xi\in]a,b[$, such that

$$l(f(b)-f(a)) = (b-a)(l\circ f')(\xi).$$

Consequently

$$l((f(b)-f(a))/(b-a)) \geqq c,$$

if $l(f']a,b[)\geqq c$. Thus

$$f(b)-f(a)\in(b-a)\mathrm{cl}_\sigma\Gamma f']a,b[,$$

where the closed hull is formed in the weak topology $\sigma(E,LE)$. But this hull coincides with the hull formed in E_τ.

LEMMA 2.2. *Let* E *be an* L_c-*or* L_e-*embedded* cvs *and let*

$$R = [a_1,b_1]\times\cdots\times[a_n,b_n].$$

If a function $f: R\to E$ *and its partial derivatives* $f'_k: R\to E$ *are continuous, then there is a bounded subset* B *of* E *with* $B^{\circ\circ}=B$ *and* $f(\beta)-f(\alpha)\in\|\beta-\alpha\|\cdot B$ *for all* $\alpha,\beta\in R$.

PROOF. Since E is L_c- or L_e-embedded, the sets $f'_k(R)$, $k=1,\dots,n$, are compact and hence bounded (see [4]). The bornology of E^b is polar (see [3]). Consequently there is a bounded set $B=B^{\circ\circ}$, such that $2nf'_k(R)\subseteqq B$ for $k=1,\dots,n$. In the complex case, for instance, there is then for each $\alpha,\beta\in R$ and $l\in LE$ a point $\xi\in R$ with

$$\mathrm{Re}(l\circ f(\beta)-l\circ f(\alpha)) = \sum\mathrm{Re}(l\circ f'_k)(\xi)(\beta_k-\alpha_k)\in 2^{-1}\|\beta-\alpha\|\cdot\mathrm{Re}\,l(B).$$

A corresponding relation holds for the imaginary part. Thus

$$l(f(\beta)-f(\alpha))\in l(B)\|\beta-\alpha\|$$

for all $l\in LE$, which yields:

$$f(\beta)-f(\alpha)\in\|\beta-\alpha\|\cdot B^{\circ\circ} = \|\beta-\alpha\|B.$$

Let E and F be cvs over $\boldsymbol{K}$ (= R or C) and let U be an open subset of E. We say that a function $f: U\to F$ is *differentiable* (or *has a derivative*) in the sense (Dk), $k=0,1,2$, if there is, for each $x\in U$, a continuous linear $f'(x): E\to F$ (the *derivative*) and a remainder $r: V\to F$, where V is an open set with $0\in V$ and $x+V\subseteqq U$, such that $f(x+h)=f(x)+f'(x)h+r(h)$ and the condition (Rk), $k = 0, 1, 2$, holds (cf. [9], [10], [12]:

(R0) $r(h) \to 0$ as $h \to 0$, $\dfrac{r(sh)}{s} \to 0$ as $s \to 0$ for each $h \in V$;

(R1) $r(h) \to 0$ as $h \to 0$, $\dfrac{r(sh)}{s} \to 0$ as $(s, h) \to (0, 0)$.

(R2) The set V can be chosen to be circled and the function ε, defined by

$$\varepsilon(s, h) = \begin{cases} \dfrac{r(sh)}{s} & \text{for } s \neq 0 \\ 0 & \text{for } s = 0, \end{cases}$$

is continuous on $S \times V$, where $S = \{\lambda \in \boldsymbol{K} : |\lambda| < 1\}$.

REMARKS.

a) Let $U \subseteqq \mathsf{R}^n$ be open and let F be an L_c-embedded cvs. Using (1) (below) and Lemma 3.2 it is easy to show that a function $f : U \to F$ is differentiable in the sense (D2), if the partial derivatives f'_k, $k = 1, \ldots, n$, exist and are continuous. For an L_e-embedded F, Proposition 3.7 can be used for the derivation of a corresponding result.

b) Holomorphic functions are differentiable in the sense (D2) under general conditions by Proposition 4.1.

c) Clearly (R2) implies (R0) and (R1).

d) Observe, that the condition (R2) requires the set U to be *c-open* in the sense that for each $x \in U$ exists a circled open set V with $x + V \subseteqq U$.

PROPOSITION 2.3. *Let E and F be* cvs *over $\boldsymbol{K}$, E equable and F circled. If a function f: $U \to F$ on an open subset U of E is differentiable in the sense* (D1), *then $f : U \to F^e$ is continuous.*

PROOF. Let $x \in U$ and let $\mathscr{F} = \boldsymbol{V}\mathscr{F}$ be a filter, which converges to 0 in E. We shall prove that $f(x+\mathscr{F})$ converges to $f(x)$ in F^e. Now

$$f(x+\mathscr{F}) - f(x) \geqq f'(x)(\mathscr{F}) + r(\mathscr{F}).$$

Let $\mathscr{G}$ be a filter, which converges to 0 in F, has a filter base consisting of circled sets and is coarser than $\varepsilon(\boldsymbol{V}, \mathscr{F})$, where the function ε is defined as in (R2). For each δ with $0 < \delta < 1$ and each circled $F \in \mathscr{F}$ (with $F \subseteqq V$)

$$r(\delta F) = \delta \cdot \varepsilon(D_\delta, F), \quad \text{where} \quad D_\delta = \{\lambda \in \boldsymbol{K} : |\lambda| < \delta\}.$$

Thus $r(\mathscr{F}) \geqq V\varepsilon(V, \mathscr{F}) \geqq V\mathscr{G}$, and hence

$$f(x+\mathscr{F})-f(x) \geqq Vf'(x)(\mathscr{F})+V\mathscr{G},$$

i.e. $f(x+\mathscr{F})$ converges to $f(x)$ in F^e.

We say that a cvs E is *τ-regular*, if for each filter $\mathscr{F}$, which converges to 0 in E, the filter base $\{\mathrm{cl}_\tau F : F \in \mathscr{F}\}$ of closed hulls $\mathrm{cl}_\tau F$ in E_τ, converges to 0 in E.

PROPOSITION 2.4. *Let I be an interval, X a convergence space and F a τ-regular, locally convex* cvs. *A function $f: I \times X \to F$ has a continuous partial derivative $f'_1: I \times X \to F$ iff $\hat{f}: I \to C_c(X, F)$, defined by $\hat{f}(t)(x) = f(t, x)$, has a continuous derivative, and then $\hat{f}' = (f'_1)\tilde{}$.*

PROOF. Suppose that f'_1 is continuous. Let $\mathscr{G}$ be a filter, which converges to a point $x \in X$ and let $t \in I$. By Lemma 2.1

$$f(t+h, x)-f(t, x) \in h\, \mathrm{cl}_\tau \Gamma f'_1(I_\delta, G)$$

for $|h| < \delta$, $t+h \in I$ and $x \in G$, where $G \in \mathscr{G}$ and $I_\delta =]t-\delta, t+\delta[$. The filter base

$$\{\mathrm{cl}_\tau \Gamma f'_1(I_\delta, G) : \delta > 0, G \in \mathscr{G}\}$$

converges to $f'_1(t, x)$, since f'_1 is continuous and F is locally convex and τ-regular. Thus the quotient $h^{-1}(\tilde{f}(t+h)-\tilde{f}(t))$ converges to $(f'_1)\tilde{}(t)$ in $C_c(X, F)$ for each $t \in I$, i.e. $\tilde{f}'$ is the continuous function $(f'_1)\tilde{}$. Conversely, if $\tilde{f}'$ exists and is continuous, then

$$h^{-1}(f(t+h, x)-f(t, x)) = h^{-1}(\hat{f}(t+h)-\hat{f}(t))(x)$$

converges to $\tilde{f}'(t)(x)$ as $h \to 0$ for each $t \in I$ and $x \in X$, that is f'_1 exists and is continuous.

Proposition 2.3 can be applied to a continuously differentiable function $f: R \to E$, which is defined on a closed rectangle R in $\mathbb{R}^n$, if a continuously differentiable extension of f to an open neighbourhood of R exists. Proposition 2.4 can be used as a tool for the construction of such an extension:

COROLLARY 2.4.1. *Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and let E be a τ-regular, locally convex* cvs. *Each $f \in C(R, E)$ with continuous partial derivatives $f'_k: R \to E$, $k = 1, \ldots, n$, has a continuous extension $\tilde{f}: R_\varepsilon \to E$ with continuous partial derivatives, where*

$$R_\varepsilon =]a_1-\varepsilon, b_1+\varepsilon[\times \cdots \times]a_n-\varepsilon, b_n+\varepsilon[$$

and $\varepsilon > 0$.

PROOF. With f is associated a function

$$\tilde{f}:[a_1,b_1]\to C_c([a_2,b_2]\times\cdots\times[a_n,b_n],E),$$

which is continuously differentiable by Proposition 2.4. A continuously differentiable extension of $\tilde{f}$ to $]a_1-\varepsilon,b_1+\varepsilon[$ is constructed by applying a (possibly empty) straight line segment at each end of the "curve" $\tilde{f}$ in the directions indicated by the derivatives. With this extension is associated a continuously differentiable function

$$[a_2,b_2]\to C_c(]a_1-\varepsilon,b_1+\varepsilon[\times[a_3,b_3]\times\cdots\times[a_n,b_n],E),$$

which is extended in the same way, and so on. The desired extension of f is obtained by induction.

3. Integration of differentiable functions.

Let $R=[a_1,b_1]\times\cdots\times[a_n,b_n]$ and let $f\in C(R,E)$, where E is a cvs. We define the integral

$$\int\cdots\int_R f(t_1,\ldots,t_n)dt_1\ldots dt_n$$

of f over R to be the linear form

$$\left(l\mapsto\int\cdots\int_R l\circ f(t_1,\ldots,t_n)dt_1\ldots dt_n\right)\in LL_{co}E,$$

where co denotes the compact-open topology (see [4]). For brevity we write t, dt and $\int_R$ for $(t_1,\ldots,t_n)$, $dt_1\ldots dt_n$ and $\int\cdots\int_R$. For a cvs E let E_c^ω and $a_c^\omega E$ (respectively E_e^ω and $a_e^\omega E$) denote the sequentially complete hull and the sequential adherence of $j_E(E)$ in L_cL_cE (respectively L_eL_eE). We recall and slightly generalize the lemmas [5; Lemma 5.2, 4.2, and 4.3]:

LEMMA 3.1. *Let* $R=R_1\times R_2$, *where*

$$R_1=[a_1,b_1]\times\cdots\times[a_k,b_k]$$

and

$$R_2=[a_{k+1},b_{k+1}]\times\cdots\times[a_n,b_n],\quad 1\leqq k\leqq n,$$

and let F *be a* cvs. *For any* $f\in C(R,F)$ *the equalities*

$$\int_R f(t)dt=\int_{R_1}ds_1\int_{R_2}f(s_1,s_2(dt_2=\int_{R_2}ds_2\int_{R_1}f(s_1,s_2)ds_1$$

hold, where $t=(s_1,s_2)$, $s_1=(t_1,\ldots,t_k)$, $s_2=(t_{k+1},\ldots,t_n)$, *and* $\int_R f(t)dt\in a_c^\omega F$.

PROOF. The lemma is an immediate consequence of elementary analysis and the fact, that $LF = L(F_c^\omega)$ separates points in F_c^ω. According to [4] the value of an integral is an element of $a_c^\omega F$.

LEMMA 3.2. *Let* $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, *let* X *be a convergence space and let* F *be an* L_c*-embedded* cvs:

(i) *For any* $f \in C(R \times X, F)$ *the integral* $\int_R \bar{f}(t)dt$ *of the function* $\bar{f}: R \to C_c(X, F)$, *defined by* $\bar{f}(t)(x) = f(t, x)$, *can be (canonically) identified with the function*

$$\left(x \mapsto \int_R f(t, x)dt\right) \in C(X, a_c^\omega F).$$

(ii) *If a net* $(f_\iota)_{\iota \in I}$ *converges to* f *in* $C_c(R \times X, F)$, *then*

$$\lim_\iota \int_R f_\iota(t, x)dt = \int_R f(t, x)dt,$$

where the limit is formed in $C_c(X, a_c^\omega F)$.

PROOF. (i) If $F = \boldsymbol{K}$ then

$$\int_R \bar{f}(t)dt \in LL_{co}C_c(X) \cong C(X)$$

(cf. [2]). Thus the integral can be identified with a continuous function, the actual form of which one obtains by letting the integral operate on

$$(i_X(x): g \mapsto g(x)) \in LC_c(X).$$

Now, let F be an arbitrary L_c-embedded cvs. Since the mapping $\alpha: C_c(X, F) \to C_c(X \times L_cF), \alpha(g)(x, l) = l \circ g(x)$, is an embedding, the restriction of $L_cL_c(\alpha)$ to $a_c^\omega C_c(X, F)$ is an injective mapping into

$$L_cL_cC_c(X \times L_cF) \cong C_c(X \times L_cF).$$

Hence the integral of $\bar{f}$ can be identified with the integral of

$$\bar{\bar{f}}: R \to C_c(X \times L_cF), \quad \bar{\bar{f}}(t)(x, l) = l \circ f(x).$$

This in turn, can be identified with the continuous function

$$(x, l) \mapsto \int_R l \circ f(t, x)dt = \left(\int_R f(t, x)dt\right)(l)$$

and hence also with $(x \mapsto \int_R f(t,x)dt) \in C(X, L_c L_c F)$. But since $\int_R f(t,x)dt \in a_c^\omega F$ the assertion follows.

(ii) is proved in the same way as [5, Lemma 4.3] using (i).

When $f(t,x)$ is assumed to be continuously differentiable in the variable x, more can be said about the function $x \mapsto \int_R f(t,x)dt$:

THEOREM 3.3. *Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, let U be a c-open subset of a* cvs *E and let F be an L_c-embedded* cvs. *If a function $f \in C(R \times U, F)$ has a continuous partial derivative $f'_x: R \times U \to F$ in the sense* (D2) *with respect to the U-variable, then the function*

$$I(f): U \to a_c^\omega F, \quad I(f)(x) = \int_R f(t,x)dt,$$

is differentiable in the sense (R2) *and the derivative is given by*

$$I(f)'(x)h = \int_R f'_x(t,x)h dt.$$

PROOF. For $x \in U$ let V be a circled open set, such that $x + V \subseteq U$,

$$f(t, x+h) = f(t,x) + f'_x(t,x)h + r(t,h)$$

and the function $\varepsilon(s,t,h)$ (with the value $s^{-1}r(t,sh)$ for $s \neq 0$ and the value 0 for $s = 0$) is continuous for $|s| < 1$, $t \in R$ and $h \in V$. Since $h \mapsto \int_R f'_x(t,x)h dt$ is linear and continuous (Lemma 3.2 (i)) it suffices to note that $r_1: h \mapsto \int_R r(t,h)dt$ satisfies the remainder conditon (R2): The function

$$\varepsilon_1 : (s,h) \mapsto \int_R \varepsilon(s,t,h)dt$$

corresponding to r_1 is continuous for $|s| < 1$ and $h \in V$ by Lemma 3.2 (i).

THEOREM 3.4. *Let U be a c-open subset of a* cvs *E and F an L_c-embedded* cvs. *Suppose that a net $(f_\iota)_\iota$ and a function f in $C(U,F)$ are such that*

(i) *the net $(f_\iota(x))_\iota$ converges to $f(x)$ in F for each $x \in U$, and*
(ii) *each function f_ι is continuously differentiable in the sense* (D2) *and the net $(f'_\iota)_\iota$ converges to a function g in $C_c(U, L_c(E,F))$.*

Then f is (*continuously*) *differentiable in the sense* (D2) *and $f' = g$.*

REMARK. The condition (i) can be weakened if U is, for instance a translation of a circled open set: Using (1) and (ii) one realizes, that one only needs to assume that $(f_\iota(x_0))_\iota$ converges to $f(x_0)$ at some point $x_0 \in U$.

PROOF. For $x \in U$, let V be a circled open set with $x + V \subseteqq U$. If a function $G \in C(U, F)$ is continuously differentiable, then $t \mapsto l(G'(x+th)h)$ is the continuous derivative of $t \mapsto l \circ G(x+th)$ for $h \in V$, $t \in [0, 1]$. Thus

$$G(x+h) = G(x) + \int_0^1 G'(x+th)h\,dt \quad \text{for } h \in V. \tag{1}$$

Applied to the function f_ι, (1) yields:

$$f_\iota(x+h) = f_\iota(x) + f'_\iota(x)h + r_\iota(h),$$

where

$$r_\iota(h) = \int_0^1 (f'_\iota(x+th)h - f'_\iota(x)h)\,dt.$$

The net $((t, h) \mapsto f'_\iota(x+th)h - f'_\iota(x)h)_\iota$ converges to $(t, h) \mapsto g(x+th)h - g(x)h$ in $C_c([0, 1] \times V, F)$. Hence the net (r_ι) converges to

$$r : h \mapsto \int_0^1 (g(x+th)h - g(x)h)\,dt$$

in $C_c(V, a_c^\omega F)$ by Lemma 3.1. Now

$$f(x+h) = f(x) + g(x)h + r(h),$$

so we only have to prove that r satisfies the remainder condition (R2). Let ε be the function defined by r as in (R2). Then

$$\varepsilon(s, h) = \int_0^1 (g(x+sth)h - g(x)h)\,dt$$

and is continuous by Lemma 3.1. Thus f is differentiable in the sense (D2) and $f' = g$.

For local uniform convergence there seems to be no direct counterpart to Lemma 3.2, which turned out to be very useful in [5]. But if we restrict our attention to functions, which are sufficiently differentiable, then an analogous theorem (Theorem 3.7) can be proved. But first we need a few preparatory results.

We use the following notations: For a circled convex subset B of a cvs

E, E_B denotes the normed space generated by B, and $\hat{E}_B$ is its completion. Further E^b denotes the vector space E endowed with bornological convergence with respect to the bornology of bounded sets in E.

PROPOSITION 3.5. *Let E be an L_c- or L_e-embedded* cvs, *let*

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

and assume that a function $f \in C(R, E)$ has continuous partial derivatives $f'_k: R \to E$. Then

(i) *the function $f: R \to E^b$ is uniformly continuous, and*
(ii) *the integral $\int_R f(t)dt$ is an element of the inductive limit $\lim_{\rightarrow} \hat{E}_B$, where B runs through the set of all circled convex bounded subsets of E, and consequently an element of $a_e^{\omega} E$.*

PROOF. By Lemma 2.2 there is a bounded set $B = B^{\circ\circ}$, such that

$$\{f(\beta)-f(\alpha): \|\beta-\alpha\| < \delta\} \subseteqq \delta B$$

for each $\delta > 0$. Thus $f: R \to E^b$ is uniformly continuous. But f is even uniformly continuous as a function into E_B, where B is as above. Hence the integral is in $\hat{E}_B$ and, consequently, in $\lim_{\rightarrow} \hat{E}_B$. Since $j_E: E^b \to L_e L_e E$ is continuous, it follows that the integral is an element of $a_e^{\omega} E$.

The straightforward proof for the following lemma is left to the reader.

LEMMA 3.6. *For any convergence space X and any* cvs *F the mapping*

$$C_e(X, F^e) \to C_e(X, F),$$

$f \mapsto i \circ f$, where i: $F^e \to F$ is the identity map, is an embedding.

Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, let U be a c-open subset of a cvs and let F be a cvs. For brevity we shall denote by D the set of functions $f \in C(R \times U, F)$, which have continuous partial derivatives $f'_k: R \times U \to F$, $k = 1, \ldots, n$, with respect to the R-variables, and continuous partial derivatives

$$f'_x: R \times U \times E \to F \quad \text{and} \quad f''_{xx}: R \times U \times E \times E \to F$$

in the sense (D2) with respect to the U-variable.

THEOREM 3.7. *Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, let U be a c-open subset of an equable* cvs *E and let F be an L_e-embedded* cvs:

(i) *For any $f \in D$ the integral $\int_R \bar{f}(t)dt$ of the function $\bar{f} = R \to C_e(U, F)$,* *defined by* $\bar{f}(t)x = f(t, x)$, *can be canonically identified with the function*

$$I(f) = \left(x \mapsto \int_R f(t, x)dt\right) \in C(U, a_e^\omega F),$$

which is continuously differentiable in the sense (D2) *with the derivative*

$$I(f)' : U \to L_e(E, a_e^\omega F)$$

given by

$$I(f)'(x)h = \int_R f'_x(t, x)h dt.$$

(ii) *If a net* $(f_\iota)_\iota$ *of functions in* D *converges in* $C_e(R \times U, F)$ *to a function* $f \in D$, *then*

$$\lim_\iota \int_R f_\iota(t, x)dt = \int_R f(t, x)dt,$$

where the limit is formed in $C_e(U, a_e^\omega F)$.

PROOF. (i) The function $\bar{f}: R \to C_e(U, F)$ is continuous by Corollary 2.4.1 and Proposition 2.3. First we shall show that the integral of $\bar{f}$ can be identified with the integral of $\bar{f}: R \to C_e(U \times L_e F)$, defined by $\bar{f}(t)(x, l) = l \circ f(t, x)$: Since F is L_e-embedded, we have (by Lemma 3.6) a canonical embedding

$$\alpha = (C_e(U, F) \to C_e(U, C_e L_e F) \to C_e(U, C_c L_e F) \cong C_e(U \times L_e F)),$$

and therefore the restriction of $LL_e\alpha$ to $a_e^\omega C_e(U, F)$, which contains the integral of $\bar{f}$, is injective. The integral of $\bar{f}$ is an element of

$$LL_{co}C_e(U \times L_e F) \cong C(U \times L_e F)$$

(recall that $LL_c C_e(X) \cong C(X)$ for any convergence space X; cf. [1]). Thus it may be identified with

$$\left((x, l) \mapsto \int_R l \circ f(t, x)dt\right) \in C(U \times L_e F),$$

and hence also with

$$I(f) = \left(x \mapsto \int_R f(t, x)dt\right) \in C(U, L_c L_e F).$$

For each $x \in U$ let V be a circled open set with $x+V \subseteqq U$. By (1)

$$f(t, x+h) = f(t, x)+f'_x(t, x)h+r(t, h)$$

for $t \in R$ and $h \in V$, where the remainder

$$r(t, h) = \int_0^1 (f'_x(t, x+uh)h-f'_x(t, x)h)du.$$

Integrating over R and using Lemma 3.1 one obtains:

$$I(f)(x+h) = I(f)(x)+\int_R f'_x(t, x)hdt+r_1(h)$$

where

$$r_1(x, h) = \int_T (f'_x(t, x+uh)h-f'_x(t, x)h)dtdu \quad \text{and} \quad T = R\times[0, 1].$$

The function ε_1, corresponding to r_1, now has the form

$$\varepsilon_1(s, h) = \int_T (f'_x(t, x+ush)h-f'_x(t, x)h)dtdu,$$

and is continuous, considered as a function $S\times V \to L_cL_eF$ with

$$S = \{\lambda \in \boldsymbol{K} : |\lambda| < 1\},$$

according to Lemma 3.2 (i). The function $I(f): U \to L_cL_eF$ is hence continuously differentiable in the sense (D2) and factors therefore continuously through $L_eL_eF = (L_cL_eF)^e$ by Proposition 2.3. According to the same proposition, the derivative

$$I(f)'(x): E \to L_cL_eF, h \mapsto \int_R f'_x(t, x)dt$$

and the functions $I(f)': U \to L_c(E, L_eL_eF)$ and $\varepsilon_1 : S\times V \to L_cL_eF$ factor continuously through $L_eL_eF = (L_cL_eF)^e$, $L_e(E, L_eL_eF)$ and L_eL_eF respectively, since they are differentiable in the sense (D2) by Theorem 3.3. Thus $I(f)$: $U \to L_eL_eF$ is continuously differentiable in the sense (D2). The values of the functions $I(f)$ and $I(f)'$ are elements of the subspace $a_e^\omega \underset{\sim}{F}$ of L_eL_eF by Proposition 3.5.

(ii) The linear mapping $I: C_c(R\times U, F) \to C_c(U, L_cL_cF)$, $f \mapsto I(f)$, is continuous by Lemma 3.2 (ii). Consequently $I: C_e(R\times U, F) \to C_e(U, L_cL_cF)$ is continuous, too. The embedding $a_e^{\omega}F \to L_eL_cF$ (recall that L_eL_cF is a closed subspace of L_eL_eF; cf. [3]) defines an embedding $C_e(U, a_e^{\omega}F) \to C_e(U, L_eL_cF)$ and by Lemma 3.6, $C_e(U, L_eL_cF)$ is a subspace of $C_e(U, L_cL_cF)$. Moreover, according to (i), $I(f) \in C(U, a_e^{\omega}F)$ for each $f \in D$. Thus $I: D \to C_e(U, a_e^{\omega}F)$ is continuous with respect to the convergence structure, which $C_e(R \times U, F)$ induces on D.

We say that an open subset of a cvs is *τ-open* if it is a union of translations of circled convex open sets.

THEOREM 3.8. *Let U be a τ-open subset of an equable* cvs *E and let F be an L_c-embedded* cvs. *Suppose that a net $(f_\iota)_\iota$ in $C(U, F)$ has the properties:*

(i) *the net $(f_\iota(x))_\iota$ converges in F to a point $f(x)$ for each $x \in U$;*
(ii) *the derivatives $f'_\iota: U \to L_c(E, F)$ and $f''_\iota: U \to L_c(E \otimes E, F)$ exist and are continuous for each ι, and the nets $(f'_\iota)_\iota$ and $(f''_\iota)_\iota$ converge in $C_c(U, L_c(E, F))$ and $C_c(U, L_c(E \otimes E, F))$ to functions g and m respectively.*

Then the function $f: U \to F^e$ is differentiable in the sense (D2) *and $f' = g$.*

REMARK. The condition (i) can often be weakened (cf. Remark after Theorem 3.4).

PROOF. Let $x \in U$ be arbitrary and consider the equation

$$f_\iota(x+h) = f_\iota(x) + f'_\iota(x)h + r_\iota(h),$$

where

$$r_\iota(h) = \int_0^1 (f'_\iota(x+uh)h - f'_\iota(x)h)du \in F,$$

which is valid for each h in a circled convex open set V with $x + V \subseteqq U$, according to (1). The function ε_ι, which is defined as in (R2) by the remainder r_ι, is now

$$\varepsilon_\iota(s, h) = \int_0^1 \gamma_\iota(u, s, h)du,$$

where $\gamma_\iota(u, s, h) = f'_\iota(x+suh)h - f'_\iota(x)h$. Let

$$r: h \mapsto \int_0^1 (g(x+uh)h - g(x)h)du$$

and

$$\varepsilon\colon (s,h) \mapsto \int_0^1 (g(x+suh)h - g(x)h)du$$

denote the limits of the nets $(r_\iota)_\iota$ and $(\varepsilon_\iota)_\iota$ in $C_c(V,F)$ and $C_c(S \times V, F)$ respectively (Lemma 3.2 (ii)). Then $f(x+h) = f(x)+g(x)h+r(h)$, and ε is the function corresponding to the remainder r. We shall prove that

$$\varepsilon\colon S \times V \to a_c^{\omega}F, \quad \text{where } S = \{\lambda \in \mathbf{K} : |\lambda| < 1\},$$

is differentiable in the sense (D2). Since f_ι'' exists and is continuous, one easily finds that the partial derivative $D_{(s,h)}\gamma_\iota$ of γ_ι with respect to (s,h) (in the sense (D2)) exists and is continuous, and that

$$D_{(s,h)}\gamma_\iota(u,s,h)(\Delta s, \Delta h) = uf_\iota''(x+ush)(h \otimes h)\Delta s + \\ + usf_\iota''(x+ush)(\Delta h \otimes h) + f_\iota'(x+us)\Delta h - f_\iota'(x)\Delta h.$$

By Theorem 3.3

$$\varepsilon_\iota'(s,h)(\Delta s, \Delta h) = \int_0^1 D_{(s,h)}\gamma_\iota(u,s,h)(\Delta s, \Delta h)du.$$

Since the nets $(f_\iota')_\iota$ and $(f_\iota'')_\iota$ converge, the net $(\varepsilon_\iota')_\iota$ converges to a function η in $C_c(S \times V,\ L_c(\mathbf{K} \times E, a_c^{\omega}F))$, according to Lemma 3.2 (ii). Theorem 3.4 now yields that $\varepsilon\colon S \times V \to a_c^{\omega}F$ is differentiable in the sense (D2) and that $\varepsilon' = \eta$. Hence $\varepsilon\colon S \times V \to F$ factors continuously through F^e. Since the linear $g(x)\colon E \to F$ also factors continuously through F^e, the function $f\colon U \to F^e$ is differentiable in the sense (D2) and $f' = g$.

4. Applications to holomorphic functions.

All vector spaces in this section are complex. In [5] Lemma 3.2 was used for the study of holomorphic functions with values in L_c-embedded spaces and spaces of such holomorphic functions, endowed with continuous convergence. In order to get similar results with the same method but with continuous convergence replaced by local uniform convergence, we have to prove that holomorphic functions are sufficiently differentiable, so that Theorem 3.7 can be applied.

First, we recall a few defintions (cf. [5], [7]). A function $f\colon U \to F$ into a cvs F is *Gâteaux holomorphic* (or *G-holomorphic*) if the function $\lambda \mapsto l \circ f(x+\lambda h)$ is holomorphic in a neighbourhood of zero for each $x \in U$, $h \in E$ and

$l \in LF$. It is *holomorphic*, if it is G-holomorphic and continuous, and it is said to be *finitely continuous*, if it is continuous on $U \cap L$ with respect to the euclidean topology for each finite-dimensional subspace L of E. Thus f is finitely continuous iff it is continuous with respect to the finest vector convergence structure on E.

PROPOSITION 4.1. *Let U be a τ-open subset of a* cvs E (*respectively equable* cvs E) *and let F be an L_c-embedded* (*respectively L_e-embedded*) cvs. *The following statements are equivalent for a function $f: U \to F$:*

(i) *f is holomorphic;*
(ii) *$f: U \to F_c^\omega$ (respectively $f: U \to F_e^\omega$) is complex differentiable in the sense* (*D*0);
(iii) *$f: U \to F_c^\omega$ (respectively $f: U \to F_c^\omega$) is complex differentiable in the sense* (D2).

PROOF. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. Thus, it remains to show that (i) $\Rightarrow$ (iii). For this, let $f: U \to F$ be holomorphic and assume first that F is L_c-embedded. Since

$$(\lambda-1)^{-1} = \lambda^{-1}+\lambda^{-2}+\lambda^{-2}(\lambda-1)^{-1} \quad \text{for } \lambda \neq 0, 1,$$

we have for $x \in U$ and a suitable circled, convex, open V, such that $x+2V \subseteqq U$:

$$f(x+h) = \frac{1}{2\pi i} \int_{|\lambda| = \varrho > 1} \frac{f(x+\lambda h)}{\lambda-1}\, d\lambda = f(x)+\hat{d}f(x)h+r(h),$$

where

$$\hat{d}f(x)h = \frac{1}{2\pi i} \int_{|\lambda| = \varrho} \frac{f(x+\lambda h)}{\lambda^2}\, d\lambda, \quad r(h) = \frac{1}{2\pi i} \int_{|\lambda| = \varrho > 1} \frac{f(x+\lambda h)}{\lambda^2(\lambda-1)}\, d\lambda$$

and where $h \mapsto \hat{d}f(x)h$ can be shown to be the restriction to V of a continuous linear mapping $E \to a_c^\omega F$ (cf. [5], [7], [4]). For the function ε corresponding to the remainder r we obtain the expression

$$\varepsilon(s, h) = \frac{s}{2\pi i} \int_{|\mu| = 2} \frac{f(x+\mu h)}{\mu^2(\mu-s)}\, d\mu \quad (|s| < 1),$$

which shows that ε is continuous (Lemma 3.2). Thus (iii) is valid. The corresponding expression for $l \circ \varepsilon(s, h)$ where $l \in LF$, immediately shows that ε is G-holomorphic as well, and hence holomorphic. Now, assume that E is equable, F L_e-embedded and $f: U \to F$ holomorphic. Then f is holomorphic as a function into the L_c-embedded reflection F_c of F (i.e. the canonical image of F in L_cL_cF) and factors therefore, since it is differentiable in the sense (D2), continuously through $(F_c)^e = F$ (Proposition 2.3). By the same argument the functions ε and $\hat{d}f(x)(\cdot)$, being holomorphic functions into $a_c^\omega(F_c)$, factor continuously through $(a_c^\omega(F_c))^e$. Since ε and $\hat{d}f(x)(\cdot)$ take their values in $a_e^\omega F$, which is a subspace of $(a_c^\omega(F_c))^e, f: U \to a_e^\omega F$ is differentiable in the sense (D2). Thus (iii) holds in both cases.

The derivative $\hat{d}f: U \times E \to F_c^\omega$, $(x, h) \mapsto \hat{d}f(x)h$, of a holomorphic function into an L_c-embedded cvs F, is continuous by Lemma 3.2 and hence holomorphic, since it is G-holomorphic. According to Proposition 4.1, $\hat{d}f: U \times E \to F_c^\omega$ is differentiable in the sense (D2). But this is obviously the same as to say that $\hat{d}f: U \to L_c(E, F_c^\omega)$ is differentiable in the sense (D2) because of linearity in the second variable. By induction one shows that $f: U \to F_c^\omega$ is infinitely differentiable in the sense (D2). If E is equable and F is L_e-embedded, a corresponding result is obtained using the factorization technique provided by Proposition 2.3:

COROLLARY 4.1.1. *Let U be a τ-open subset of a* cvs E (*respectively equable* cvs E) *and let F be a sequentially complete L_c-embedded* (*respectively L_e-embedded*) cvs. *A holomorphic function $f: U \to F$ is infinitely differentiable,* i.e. *it has continuous derivatives $f^{(k)}: U \times E \times \cdots \times E \to F$ in the sense* (D2) *of all orders $k = 0, 1, 2, \ldots$.*

We shall now see how a theory of holomorphic functions can be developped using local uniform convergence instead of continuous convergence on function spaces (cf. [5]). Consider first a finitely continuous, G-holomorphic function $f: U \to F$ on a τ-open subset U of an equable cvs E with values in an L_e-embedded cvs F. The function $f: U \to F_c$, being holomorphic with respect to the finest vector convergence structure Λ_0 on E, has for each $x \in U$ an expansion (cf. [5])

$$(2) \qquad f(x+h) = \sum_{m=0}^{\infty} \frac{\hat{d}^m f(x)}{m!} h,$$

$$\hat{d}^m f(x)h = \frac{m!}{2\pi i} \int_{|\lambda| = \varrho} \frac{f(x+\lambda h)}{\lambda^{m+1}} \, d\lambda \quad (m = 0, 1, \ldots)$$

for $h \in V$, where V is a circled convex open set with $x + V \subseteqq U$. Let U_0 and E_0 be the sets U and E endowed with Λ_0. According to [5],

$$\hat{d}^m f: U_0 \times E_0 \to (F_c)_c^\omega$$

is holomorphic (and an m-homogeneous polynomial in h) and is therefore differentiable in the sense (D2) by Proposition 4.1. By Propositions 2.3 and 3.5, $\hat{d}^m f : U_0 \times E_0 \to (F_c)_c^\omega$ factors continuously through $a_e^\omega F$, which is a subset of $(F_c)_c^\omega$. Exactly as in [5] one proves that the sum (2) converges in $L_e L_{co} F$ and hence also in $a_e^\omega F$ for a fixed $h \in V$.

Let $H(U, F)$ denote the set of holomorphic functions $U \to F$. In [5] we proved that $H_e(U, F)$ is complete for a complete L_c-embedded cvs F. A similar theorem holds for an L_e-embedded F:

PROPOSITION 4.2. *If U is a τ-open subset of an equable* cvs *E and F is a (sequentially) complete L_e-embedded* cvs, *then $H_e(U, F)$ is (sequentially) complete.*

PROOF. The canonical mapping $H_e(U, F) \to H_e(U, L_c L_e F)$ is an embedding (Lemma 3.6) and $H_e(U, L_c L_e F)$ is complete (see [5]). Therefore we only have to show that $H(U, F)$ is (sequentially) closed in $H_e(U, L_c L_e F)$. Let $(f_\iota)_\iota$ be a net (or sequence) in $H(U, F)$, which converges to f in $H_e(U, L_c L_e F)$. Since $f : U \to L_c L_e F$ is holomorphic, the same factorization technique as in the proof for Proposition 4.1 yields, that $f : U \to L_e L_e F$ is differentiable in the sense (D2). Since $f(x) = \lim f_\iota(x)$ for each $x \in E$ and since F is (sequentially) complete, $f : U \to F$ is differentiable in the sense (D2) and hence holomorphic by Proposition 4.1. Thus $H_e(U, F)$ is (sequentially) complete.

The proofs for the next two theorems are omitted. The first part of Theorem 4.3 is a direct consequence of the corresponding assertions for continuous convergence (cf. [5]), Corollary 4.1.1 and the Propositions 2.3 and 3.5. The proofs for Theorem 4.3 (ii) and Theorem 4.4 (i), (ii) are almost identical with the corresponding proofs in [5], but this time Theorem 3.7 (together with Corollary 4.1.1) is the working tool instead of Lemma 3.2. Finally, for establishing the third part of Theorem 4.4 one uses the embedding

$$C_e(U, C_e(V, G)) \to C_e(U \times V, G)$$

(cf. Lemma 3.6) instead of the isomorphism $C_c(U, C_c(V, G)) \cong C_c(U \times V, G)$.

THEOREM 4.3. *Let U be a τ-open subset of an equable* cvs *E and let F be an L_e-embedded* cvs:

(i) *The function $\hat{d}^m : H_e(U, F) \times U \times E \to a_e^\omega F$, $(f, x, h) \mapsto \hat{d}^m f(x)h$, is continuous for each $m \geqq 0$.*
(ii) *There is a τ-open subset X of $H_e(U, F) \times U \times E$, containing $H(U, F) \times U \times \{0\}$, such that the formula*

$$S = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m,$$

where $S(f, x, h) = f(x+h)$, *is valid in the* cvs $C_e(X, a_e^{\omega} F)$.

THEOREM 4.4. *Let* U *and* V *be* τ*-open subsets of equable* cvs E *and* F *respectively and let* G *be an* L_e*-embedded* cvs:

(i) *If* $f: U \times V \to G$ *is holomorphic, then the function* $\hat{f}: U \to H_e(V, a_e^{\omega} G)$, *defined by* $\hat{f}(x)y = f(x, y)$, *is holomorphic.*

(ii) *If* $g: U \to H_e(V, G)$ *is holomorphic, then the function* $\check{g}: U \times V \to G$, *defined by* $\check{g}(x, y) = g(x)y$, *is holomorphic.*

(iii) *If* G *is sequentially complete, then* $H_e(U \times V, G) \cong H_e(U, H_e(V, G))$.

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