MATH. SCAND. 60 (1987), 109–128

ANALYTICALLY NORMED SPACES *

V. DOBRIĆ

Summary.

The problem of transferring results concerning almost sure convergence from a family of seminormed spaces generating the topology of a locally convex space to the locally convex space requires, in general, determining when an uncountable union of null-sets is a null-set. A solution of this problem is obtained in a large class of locally convex spaces viz. analytically normed spaces with respect to a certain σ-algebra on the space. The main result is that a sequence of random vectors taking values in an analytically normed space converges almost sure if and only if it converges almost sure with respect to every seminorm in a generating family. Then it is proved that such a class of analytically normed spaces is stable under "nice" countable inductive and projective limits, that this class contains a long series of standard spaces e.g. $C(T)$ in its compact-open topology with T Polish, $C(T)$ in its strict topology and with T Polish, the Schwartz test function space as well as the Schwartz distribution space.

1. Introduction.

Limit theorems for stochastic processes with values in locally convex spaces, particularly in the test function space, have recently been studied in connection with the infinite system of particles. The topology of a locally convex space is generaded by a family of seminorms, so knowing some limit theorems for seminormed spaces, one could try to transfer them to the locally convex spaces. Particularly we are interested in limit theorems concerning almost sure convergence, for example the law of large numbers. But to transfer almost sure convergence results for seminormed spaces to locally convex spaces requires in general determining when the uncountable union of null-sets is a null-set. We offer a solution of this problem by introducing the concept of analytically normed spaces with respect to a certain σ-algebra on the space. Basically this means that an ordered structure on the generating family of seminorms is sufficiently smooth. This class of analytically normed spaces includes the test function spaces as well as many other locally convex spaces.

---

* The present paper is a part of my doctorial thesis and was carried out under supervision of Professor J. Hoffmann-Jørgensen during my stay at The Mathematics Institute at Aarhus University.

Received August 6, 1986.

We shall also show that under very weak conditions, the inductive and projective limits of analytically normed spaces are again analytically normed spaces.

Our main reference is the forthcoming book of J. Hoffmann-Jørgensen [4], in particular Chapter 2, where the author studies the smoothly ordered spaces with the purpose of solving problems connected with infinite dimensional stochastic processes.

2. Basic definitions.

We let $(\mathsf{N}^{\mathsf{N}}, \mathscr{B}(\mathsf{N}^{\mathsf{N}}), \leqq)$ be the ordered, measurable space of all sequences of integers, where $\mathscr{B}(\mathsf{N}^{\mathsf{N}})$ is the Borel σ-algebra with respect to the product topology on N^{N} and $\leqq$ is the product ordering on N^{N}, i.e.

$$(\sigma_i') \leqq (\sigma_i'') \Leftrightarrow \sigma_i' \leqq \sigma_i'' \quad \forall i \in \mathsf{N}.$$

Let us recall that a subset Γ_0 of an ordered space $(\Gamma, \leqq)$ is cofinal in $(\Gamma, \leqq)$ if for every $\alpha \in \Gamma$ there exists $\beta \in \Gamma_0$ so that $\sigma \leqq \beta$.

We shall follow the notation of [2] and [6] concerning Souslin schemes, $\mathscr{F}$-Souslin sets, and the Souslin operations. If $\mathscr{F}$ is a family of sets in Ω, then $S(\mathscr{F})$ denotes the family of all $\mathscr{F}$-Souslin sets, and $CS(\mathscr{F})$ the family of all $C\mathscr{F}$-Souslin sets that is

$$CS(\mathscr{F}) = \{A \subseteqq \Omega \mid \Omega \setminus A \in S(\mathscr{F})\}.$$

It is well-known that $S(S(\mathscr{F})) = S(\mathscr{F})$ and that $S(\mathscr{F})$ is stable on countable unions and countable intersections.

DEFINITION 2.1. Let $(\Gamma, \mathscr{G})$ be a measurable space with an ordering $\leqq$. Then $(\Gamma, \mathscr{G}, \leqq)$ is called a smoothly ordered measurable space if there exists a measurable, increasing, cofinal map φ from $(\mathsf{N}^{\mathsf{N}}, \mathscr{B}(\mathsf{N}^{\mathsf{N}}, \leqq)$ into $(\Gamma, \mathscr{G}, \leqq)$. We say that $(\Gamma, \mathscr{G}, \leqq)$ is a σ-smoothly ordered measurable space if there exists a sequence $\{\Gamma_n\}$ of subsets of Γ so that $\cup \Gamma_n$ is cofinal in $(\Gamma, \leqq)$ and for all $n \in \mathsf{N}$ the spaces $(\Gamma_n, \mathscr{G}_n, \leqq_n)$ are smoothly ordered, where $\mathscr{G}_n = \mathscr{G}_{|\Gamma_n}$ and $\leqq_n$ is the restriction of $\leqq$ on Γ_n.

DEFINITION 2.2. Let $(\Omega, \mathscr{F})$ be a measurable space and let $\Lambda \subseteqq \mathsf{R}^{\Omega}$ be a set of real functions on Ω. Then we say that Λ is smoothly (respectively σ-smoothly) filtering upwards on $(\Omega, \mathscr{F})$ if there exists a smoothly (respectively σ-smoothly) ordered space $(\Gamma, \mathscr{G}, \leqq)$ and an increasing cofinal map φ from $(\Gamma, \mathscr{G})$ into Λ with its pointwise ordering so that

$$\{(\alpha, \omega) \in \Gamma \times \Omega \mid \varphi(\alpha, \omega) > a\} \in S(\mathscr{G} \otimes \mathscr{F}) \quad \forall a \in R,$$

where $\varphi(\alpha, \omega) = \varphi(\alpha)(\omega)$.

Note that since φ is increasing,

$$\varphi(\alpha, \omega) \leqq \varphi(\beta, \omega) \quad \forall \alpha \leqq \beta, \forall \omega \in \Omega,$$

and that cofinality of φ means

$$\forall f \in \Lambda \ \exists \alpha \in \Gamma; \quad \varphi(\alpha, \omega) \geqq f(\omega) \quad \forall \omega \in \Omega.$$

DEFINITION 2.3. If $(-\Lambda) = \{f \in \mathrm{R}^{\Omega} \mid (-f) \in \Lambda\}$ is smoothly (respectively σ-smoothly) filtering upwards on $(\Omega, \mathscr{F})$, then we say that Λ is smoothly (respectively σ-smoothly) filtering downwards on $(\Omega, \mathscr{F})$.

DEFINITION 2.4. If $\mathscr{K}$ is a family of subsets on Ω, then we say that $\mathscr{K}$ is smoothly (σ-smoothly) filtering upwards on Ω if $\Lambda = \{1_K \mid K \in \mathscr{K}\}$ is so.

Throughout all of this paper F denotes a locally convex space, Π the family of all continuous seminorms on F and $\mathscr{F}$ a σ-algebra on F.

The following definition is due to J. Hoffmann-Jørgensen.

DEFINITION 2.5. We say that F is an $\mathscr{F}$-analytically normed space if:

(2.5.1) Π is σ-smoothly filtering upwards on $(F, \mathscr{F})$,

or equivalently, since Π is upwards directed,

(2.5.2) Π is smoothly filtering upwards on $(F, \mathscr{F})$

(see Lemma 2.5 in [4]); or equivalently, if there exists a generating family Π_0 of seminorms for the topology on F satisfying

(2.5.3) Π_0 is σ-smoothly filtering upwards on $(F, \mathscr{F})$

(see Lemma 2.5 in [4]); or equivalently, if there exists an upwards directed subset Π_1 of Π generating the topology on F and satisfying

(2.5.4) Π_1 is smoothly filtering upwards on $(F, \mathscr{F})$

(see Lemma 2.5) in [4]); or equivalently:

(2.5.5) there exists a smoothly ordered measurable space $(\Gamma, \mathscr{G}, \leqq)$ and a map $\varphi: \Gamma \to \Pi$ defined by $\varphi: \alpha \to q_\alpha$ and satisfying:

(i) $a_\alpha \leqq q_\beta \quad \forall \alpha \leqq \beta$,
(ii) $\{q_\alpha \in \Pi \mid \alpha \in \Gamma\}$ is cofinal in Π,
(iii) $\{(\alpha, x) \in \Gamma \times F \mid q_\alpha(x) > a\} \in S(\mathscr{G} \otimes \mathscr{F}) \quad \forall a > 0$,

or equivalently:

(2.5.6) there exists a map $\sigma \to q_\sigma$ from N^N into Π so that:

(i) $q_0 \leqq q_\tau \quad \forall \sigma \leqq \tau$,
(ii) $\{q_\sigma \in \Pi \mid \sigma \in \mathsf{N}^\mathsf{N}\}$ generates topology on F,
(iii) $\{(\sigma, x) \in \mathsf{N}^\mathsf{N} \times F \mid q_0(x) > a\} \in S(\mathscr{B}(\mathsf{N}^\mathsf{N}) \otimes \mathscr{F}) \quad \forall a > 0$.

Notice that in (2.5.5 (iii)) or (2.5.6 (iii)) we may replace $>$ by $\geqq$ since the family of Souslin sets is stable on countable unions and countable intersections.

If $(\Gamma, \leqq)$ is countably cofinal, then evidently $(\Gamma, 2^\Gamma, \leqq)$ is a σ-smoothly ordered measurable space. If $(\Gamma, \leqq)$ is upwards directed and coundably cofinal, then $(\Gamma, 2^\Gamma, \leqq)$ is a smoothly ordered measurable space.

The following definition is from [2]:

DEFINITION 2.6. Let f be a map from a measurable space $(E, \mathscr{E})$ into a topological space F. We say that:

(2.6.1) f is $\mathscr{E}$-simple if there exists a finite partition $\{E_1, \ldots, E_n\} \subseteqq \mathscr{E}$ of E such that f is constant on each E_k, $1 \leqq k \leqq n$.

(2.6.2) f is $\mathscr{E}$-elementary if there exists a countable partition $\{E_k\}_{k=1}^\infty \subseteqq \mathscr{E}$ of E so that f is constant on each E_k, $k \geqq 1$.

DEFINITION 2.7. Let $\mathscr{F}$ be a σ-algebra on F. Then F admits a $\mathscr{F}$-elementary ($\mathscr{F}$-simple) resolution of the identity on F if and only if there exists $\mathscr{F}$-elementary ($\mathscr{F}$-simple) functions, $\{s_k\}_{k=1}^\infty$ from F into F, such that

$$\lim_{k \to \infty} s_k(x) = x \quad \forall x \in F.$$

3. The analytically normed spaces.

Our first result is a description of a neighbourhood base at 0 for the topology on an $\mathscr{F}$-analytically normed space.

PROPOSITION 3.1. *If F is an $\mathscr{F}$-analytically normed space, then*

$$\mathscr{U} = \{V \in CS(\mathscr{F}) \mid V \text{ is closed, symmetric, convex, } \; 0 \in \operatorname{int} V\}$$

is a neighbourhood base at 0 *for the topology on F.*

PROOF. If F is an $\mathscr{F}$-analytically normed space, than (2.5.6) holds, and since

$$S(\mathscr{B}(\mathsf{N}^{\mathsf{N}} \otimes \mathscr{F}) = S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \times \mathscr{F}),$$

see [6], we have that every N^{N} section of a set in $S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{F})$ belongs to $S(\mathscr{F})$ (and every F section of a set in $S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{F})$ belongs to $S(\mathscr{B}(\mathsf{N}^{\mathsf{N}})))$. Particularly, if $a > 0$, if

$$A = \{(\sigma, x) \in \mathsf{N}^{\mathsf{N}} \times F \mid q_\sigma(x) > a\},$$

and if $\sigma \in \mathsf{N}^{\mathsf{N}}$ is fixed, then by (2.5.6 (iii)) we have

$$A(\sigma) = \{x \in F \mid (\sigma, x) \in A\} = \{x \in F \mid q_\sigma(x) > a\} \in S(\mathscr{F}),$$

and therefore

$$\{x \in F \mid q_\sigma(x) \leqq a\} = V_a(\sigma) = A^c(\sigma) \in CS(\mathscr{F}). \tag{3.1.1}$$

For every $\sigma \in \mathsf{N}^{\mathsf{N}}$ the set $V_a(\sigma)$ is closed, symmetric, convex, and $0 \in \operatorname{int} V$. So if

$$\mathscr{U} = \{V_a(\sigma) \mid \sigma \in \mathsf{N}^{\mathsf{N}}, \quad a > 0\},$$

then by (2.5.6(ii)), $\mathscr{U}$ is a neighbourhood base at 0 for the topology on F, and by (3.1.1) the Proposition is proved.

In the case when a locally convex space is metrizable, we have a converse of Proposition 3.1 which we show below.

PROPOSITION 3.2. *Let F be a metrizable locally convex space and $\mathscr{F}$ an σ-algebra on F so that $A \in \mathscr{F}$ implies $aA \in \mathscr{F}$ for all $a > 0$. If*

(3.2.1) $\mathscr{W} = \{V \in CS(\mathscr{F}) \mid V$ *is closed, symmetric, convex* $0 \in \operatorname{int} V\}$ *is a neighbourhood base at* 0 *for the topology on F, then F is $\mathscr{F}$-analytically normed.*

PROOF. Since F is metrizable, there exists a countable decreasing subfamily $\mathscr{U}$ of $\mathscr{W}$, which is also a neighbourhood base at 0 for the topology on F. Let $\{p_n \mid n \in \mathsf{N}\} \subseteqq \Pi$ be the increasing sequence of seminorms associated with $\mathscr{U}$, and let $\varphi: \mathsf{N} \to \Pi$ be a map defined by

$$\varphi(n) = p_n.$$

Then φ is increasing and cofinal in Π. We shall prove that

$$A_a = \{(n, x) \in \mathsf{N} \times F \mid \varphi(n)(x) > a\} \in S(\mathscr{B}(\mathsf{N}) \otimes \mathscr{F}) \quad \forall a > 0.$$

By assumption (3.2.1) we have that

$$\{x \in F \mid p_n(x) > 1\} \in S(\mathscr{F}) \quad \forall n \in \mathsf{N},$$

and so

$$\{x \in F \mid p_n(x) > a\} = a\{x \in F \mid p_n(x) > 1\} \in S(\mathscr{F}) \quad \forall n \in \mathsf{N}, \quad \forall a > 0,$$

since $aA \in \mathscr{F}$ for all $A \in \mathscr{F}$ and all $a > 0$. If we rewrite A_a as follows:

$$A_a = \bigcup_{n=1}^{\infty} \{n\} \times \{x \in F \mid p_n(x) > a\},$$

then, since

$$\{n\} \times \{x \in F \mid p_n(x) > a\} \in S(\mathscr{B}(\mathsf{N}) \otimes \mathscr{F}) \quad \forall n \in \mathsf{N}, \quad \forall a > 0,$$

we have that

$$A_a \in S(\mathscr{B}(\mathsf{N}) \otimes \mathscr{F}) \quad \forall a > 0.$$

Since the space $(\mathsf{N}, \mathscr{B}(\mathsf{N}), \leqq)$ is smoothly ordered (take $\psi: \mathsf{N}^{\mathsf{N}} \to \mathsf{N}$ to be $\psi(\sigma) = \sigma_1$), we have proved that F is $\mathscr{F}$-analytically normed.

The following result is the main result of this paper. It shows that in an $\mathscr{F}$-analytically normed space, almost sure convergence is equivalent to the almost sure convergence in (F, q) for all q in a generating family of seminorms for the topology on F.

THEOREM 3.3. *Let $(S, \mathscr{S}, \mu)$ be a probability space, and let F be $\mathscr{F}$-analytically normed. If $f_n: (S, \mathscr{S}) \to (F, \mathscr{F})$ are measurable random variables such that*

(3.3.1) $$\mu^*\{s \in S \mid q(f_n(s)) \to 0\} = 1 \quad \forall q \in \Pi,$$

then

(3.3.2) $$f_n(s) \to 0 \quad \textit{in} \quad F \quad \mu\text{-a.s.}$$

PROOF. If F is $\mathscr{F}$-analytically normed, there exists an increasing and cofinal map $\varphi: \mathsf{N}^{\mathsf{N}} \to \Pi$ such that

$$\{(\sigma, x) \in \mathsf{N}^{\mathsf{N}} \times F \mid \varphi(\sigma)(x) > a\} \in S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{F}) \quad \forall a > 0. \tag{3.3.3}$$

Let

$$T_q = \{s \in S \mid q(f_n(s)) \not\to 0\} \quad \forall q \in \Pi,$$

and let $\mathscr{T} = \{T_q \mid q \in \Pi\}$. If we prove that $\mathscr{T}$ is smoothly filtering upwards on $(S, \mathscr{S})$, then, because $\mu_*(T_q) = 0$ for every $q \in \Pi$, Corollary 2.3 in [4] states that

$$T = \bigcup_{q \in \Pi} T_q$$

is a μ-measurable set and $\mu(T) = 0$ which then implies (3.3.2). Let

$$\psi: \mathsf{N}^{\mathsf{N}} \to \{1_{T_q} \mid q \in \Pi\}$$

be defined by

$$\psi(\sigma) = 1_{T_{\varphi(\sigma)}}.$$

It is very easy to check that ψ is increasing and cofinal. It remains to prove that

$$S_a = \{(\sigma, s) \in \mathsf{N}^{\mathsf{N}} \times S \mid \psi(\sigma)(s) > a\} \in S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{S}) \quad \forall a > 0. \tag{3.3.4}$$

If $a \geqq 1$, then $S_a = \phi \in S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{S})$. If $0 < a < 1$, then

$$\begin{aligned} S_a &= \{(\sigma, s) \in \mathsf{N}^{\mathsf{N}} \times S \mid s \in T_{\varphi(\sigma)}\} = \{(\sigma, s) \in \mathsf{N}^{\mathsf{N}} \times S \mid \varphi(\sigma)(f_n(s)) \not\to 0\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{(\sigma, s) \in \mathsf{N}^{\mathsf{N}} \times S \mid \varphi(\sigma)(f_k(s)) \geqq 1/n\}. \end{aligned} \tag{3.3.5}$$

So if we prove that

$$\{(\sigma, s) \in \mathsf{N}^{\mathsf{N}} \times S \mid \varphi(\sigma)(f_k(s)) \geqq 1/n\} \in S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \times \mathscr{S}), \tag{3.3.6}$$

we have finished. Let us define $h_n: \mathsf{N}^{\mathsf{N}} \times S \to \mathsf{N}^{\mathsf{N}} \times F$ by

$$h_n(\sigma, s) = (\sigma, f_n(s)) \quad \forall n \in \mathsf{N}.$$

Then for every $n \in \mathsf{N}$, h_n is a $(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{S}, \mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{F})$-measurable random variable and

$$\{(\sigma, s) \in \mathrm{N}^{\mathrm{N}} \times S \mid \varphi(\sigma)(f_k(s)) \geqq 1/n\} = \tag{3.3.7}$$
$$= h_k^{-1}\{(\sigma, x) \in \mathrm{N}^{\mathrm{N}} \times F \mid \varphi(\sigma)(x) \geqq 1/n\}.$$

Now by (3.3.3), (3.3.7) and measurability of h_n, $n \in \mathrm{N}$, we have (3.3.6).

COROLLARY 3.4. *If F is $\mathscr{F}$-analytically normed and if $f: (S, S, \mu) \to (F, \mathscr{F})$ is a random variable such that*

$$(\mu^\infty)_*\{(s_i) \in S^\infty \mid q(S_n((s_i))/n) \to 0\} = 1 \quad \forall q \in \Pi, \tag{3.4.1}$$

where μ^∞ is the countable product of μ with itself and

$$S_n((s_i)) = \sum_{i=1}^{n} f(s_i) \quad \forall n \in \mathrm{N},$$

then

$$(\mu^\infty)_*\{(s_i) \in S^\infty \mid q(S_n((s_i))/n) \to 0 \quad \forall q \in \Pi\} = 1.$$

PROOF. Since f is $(\mathscr{S}, \mathscr{F})$-measurable, the functions

$$S_n : S^\infty \to F$$

are $(\mathscr{S}^\infty, \mathscr{F})$-measurable for every $n \in \mathrm{N}$, and since (3.4.1) holds, the statement of the Corollary is a direct consequence of Theorem 3.3.

4. Examples of $\mathscr{F}$-analytically normed spaces.

The following result is a basic result for this section.

THEOREM 4.1. *Let F be a linear space with a σ-algebra $\mathscr{F}$. Let $(T, \mathscr{T})$ be a Blackwell space (see [4]) and let $\mathscr{K}$ be a family of subsets of T which is σ-smoothly filtering upwards on $(T, \mathscr{T})$. Let $t \to t^*$ be a map from T into the algebraic dual F^* of F such that*

$$\{(x, t) \in F \times T \,\big|\, |t^*(x)| > a\} \in S(\mathscr{F} \otimes \mathscr{T}) \quad \forall a > 0. \tag{4.1.1}$$

If the topology on F is induced by the family of seminorms:

$$q_K(x) = \sup_{t \in K} |t^*(x)| \quad \forall K \in \mathscr{K}, \quad \forall x \in F, \tag{4.1.2}$$

then F is $\mathscr{F}$-analytically normed.

PROOF. Since $\mathscr{K}$ is σ-smoothly filtering upwards on $(T, \mathscr{T})$ there exists a σ-smoothly ordered measurable space $(G, \mathscr{G}, \leqq)$ and a map $\varphi: G \to \mathscr{K}$, which is increasing and cofinal with respect to $\subseteqq$ on $\mathscr{K}$, and such that

(4.1.3) $$A = \{(g, t) \in G \times T \mid t \in \varphi(g)\} \in S(\mathscr{G} \otimes \mathscr{T}).$$

Now let

$$\psi(g) = q_{\varphi(g)} \quad \text{for } g \in G.$$

Then ψ is increasing from G into $\Pi = \{q_K | K \in \mathscr{K}\}$, and since φ is cofinal, we have that $\{\psi(g) | g \in G\}$ generates the topology on F. Hence by (2.5.5) it suffices to show that

(4.1.4) $$B(a) = \{(g, x) \in G \times F \mid q_{\varphi(g)}(x) > a\} \in S(\mathscr{G} \otimes \mathscr{F})$$

for all $a > 0$. Let $p(g, x, t) = (g, x)$ be the projection of $G \times F \times T$ onto $G \times F$; then

$$\begin{aligned} B(a) &= \{(g, x) \in G \times F \mid \exists t \in \varphi(g): |t^*(x)| > a\} \\ &= p\{(g, x, t) \in G \times F \times T \mid t \in \varphi(g), |t^*(x)| > a\}. \end{aligned}$$

Since

$$\begin{aligned} &\{(\varphi, x, t) \in G \times F \times T \mid t \in \varphi(g), |t^*(x)| > a\} \\ &\quad = A \times T \cap \{(g, x, t) \in G \times F \times T \,\big|\, |t^*(x)| > a\} \in S(\mathscr{G} \otimes \mathscr{F}) \end{aligned}$$

by (4.1.1) and (4.1.3), and since $(T, \mathscr{T})$ is Blackwell, we have that (4.1.4) follows from the projection theorem for Blackwell spaces (see Theorem 1.5 in [4]).

Let us recall that the weak*-Baire σ-algebra on the topological dual F' of F is the smallest σ-algebra on F' making the evolution maps $x \to x(t)$ (from F' into R) measurable for every $t \in F$.

Let us also recall that the Mackey topology on the topological dual F' is the linear topology on F' having family $\mathscr{K}^\circ = \{K^\circ | K \in \mathscr{K}\}$ as a subbase at 0, where

$$\mathscr{K} = \{K \subseteqq F \mid K \quad \text{is symmetric, convex, weakly compact}\}$$

and K° is the polar of K, i.e.

$$K^\circ = \{x' \in F' \,\big|\, |x'(x)| \leqq 1 \quad \forall x \in K\}.$$

PROPOSITION 4.2. *Let F be a locally convex space with dual space F'. Let $\mathscr{F}'$ be the weak*-Baire σ-algebra on F', and let κ be the compact-open topology on F'* (i.e. *the topology of uniform convergence on all compact subsets of F*), *and β the strong topology on F'* (i.e. *the topology of uniform convergence on all weakly bounded subsets of F*), *then we have*:

(4.2.1) *(F', β) is $\mathscr{F}'$-analytically normed if F is separable and metrizable and $(F, \mathscr{B}(F))$ is Blackwell.*

(4.2.2) *(F', κ) is $\mathscr{F}'$-analytically normed if $F = \bigcup_{n=1}^{\infty} S_n$, where S_n is Polish for all $n \geqq 1$, and every compact set in F is contained in S_n for some $n \geqq 1$.*

PROOF. In both cases we have that $(F, \mathscr{B}(F))$ is Blackwell. We shall apply Theorem 4.1 with $(T, \mathscr{T}) = (F, \mathscr{B}(F))$ and $t \to t^*$ to be the canonical injection of F into the algebraic dual F'^* of F'. The map from $F \times F'$ into $\mathbf{R}$ defined by

$$(x, x') \to |x'(x)|$$

is continuous in the first variable and $\mathscr{F}'$-measurable in the second. Since F is separable and metrizable in the first case and F is standard in the second case, it follows from [2, Theorem IV.2.1] that in both cases we have that F admits a $\mathscr{B}(F)$-simple resolution of the identity. Hence by [2, Theorem IV.2.6] we have

$$\{(x, x') \in F \times F' \mid |x'(x)| > a\} \in \mathscr{B}(F) \otimes \mathscr{F}' \quad \forall a > 0, \tag{4.2.3}$$

i.e. condition (4.1.1) holds.

Now we shall prove (4.2.1). Let $\mathscr{K}$ be the set of all bounded subsets of F. To show that $\mathscr{K}$ is smoothly filtering upwards on $(F, \mathscr{B}(F))$, let q_n, $n \geqq 1$ be an increasing countable family of seminorms inducing the topology on F, and let

$$\varphi(\sigma) = \{x \in F \mid q_n(x) \leqq \sigma(n) \quad \forall n \geqq 1\} \quad \forall \sigma \in \mathbf{N}^{\mathbf{N}}.$$

Clearly φ is an increasing map from $(\mathbf{N}^{\mathbf{N}}, \leqq)$ into $(\mathscr{K}, \subseteqq)$, and since $\sup\{q_n(x) \mid x \in K\} < \infty$ for all $n \geqq 1$ and all $K \in \mathscr{K}$, we see that φ is cofinal. Moreover, we have

$$\begin{aligned}\{(\sigma, x) \in \mathbf{N}^{\mathbf{N}} \times F \mid x \in \varphi(\sigma)\} &= \{(\sigma, x) \in \mathbf{N}^{\mathbf{N}} \times F \mid q_n(x) \leqq \sigma(n) \quad \forall n \geqq 1\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{\sigma \in \mathbf{N}^{\mathbf{N}} \mid \sigma(n) = k\} \times \{x \in F \mid q_n(x) \leqq k\} \in \mathscr{B}(\mathbf{N}^{\mathbf{N}}) \otimes \mathscr{B}(F).\end{aligned}$$

Thus $\mathscr{K}$ is smoothly filtering upwards on $(F, \mathscr{B}(F))$ and so (4.2.1) follows from Theorem 4.1.

To prove (4.2.2) we let $\mathscr{K}$ be the set of all compact subsets of F. Then $\mathscr{K}$ is σ-smoothly filtering upwards on $(F, \mathscr{B}(F))$ by [4, Corollary 2.8] and so (4.2.2) follows from Theorem 4.1.

REMARKS. (1). Suppose that F is a separable Fréchet space; then the strong topology on F' equals the Mackey topology on F' by [5, Theorem 23.5 (3)], and so F' with the Mackey topology is $\mathscr{F}$-analytically normed.

(2). Suppose that F is quasi-complete (e.g. a Fréchet space). Then the Mackey topology coincides on F' with the compact-open topology on F' when F has its weak topology. Hence if F is quasi-complete and F is a countable union of weakly Polish sets exhausting the weak compact sets, then F' with the Mackey topology is $\mathscr{F}'$-analytically normed.

(3). Let F be a separable Fréchet space such that every weakly compact set is strongly compact (e.g. $F = l^1$); then the Mackey topology on F' coincides with the compact-open topology on F', and so F' with its Mackey topology is $\mathscr{F}$-analytically normed.

If T is a topological space, we let $\mathscr{C}(T)$ be the set of all continuous real valued functions, and we let

$$P_t(x) = x(t) \quad \forall\, t \in T, \quad \forall\, x \in \mathscr{C}(T).$$

PROPOSITION 4.3. *Let T be a Polish space, let $F = \mathscr{C}(T)$ and let $\mathscr{F} = \sigma\{P_t \mid t \in T\}$. Then F is $\mathscr{F}$-analytically normed if $\mathscr{C}(T)$ is topologized by the compact open topology.*

PROOF. If $\mathscr{T} = \mathscr{B}(T)$, then $(T, \mathscr{T})$ is a Blackwell space, see [2], and $\mathscr{K} = \{K \subseteqq T \mid K \text{ is compact}\}$ is smoothly filtering upwards on $(T, \mathscr{T})$, see [4]. Since the compact-open topology on $\mathscr{C}(T)$ is generated by the family of seminorms $\{P_K \mid K \in \mathscr{K}\}$ defined by

$$P_K(x) = \sup_{t \in K} |x(t)| \quad \forall\, K \in \mathscr{K},$$

it is by Theorem 4.1 enough to prove that

$$\{(x,t) \in \mathscr{C}(T) \times T \;\big|\; |x(t)| > a\} \in \mathscr{F} \otimes \mathscr{T} \quad \forall\, a > 0. \tag{4.3.1}$$

By definition of $\mathscr{F}$, the functions

$$x \to |x(t)| = |P_t(x)| \quad \forall\, t \in T,$$

are $\mathscr{F}$-measurable. The functions

$$t \to |x(t)| \quad x \in \mathscr{C}(T)$$

are continuous, and since every Polish space T admits $\mathscr{B}(T)$-elementary resolution of the identity, Theorem IV.2.6 in [2] states that

$$(x, t) \to |x(t)|$$

is a jointly measurable function. Hence (4.3.1) holds.

If T is a topological space, we let

$$C(T) = \{f : \mathrm{T} \to R \mid f \text{ is bounded and continuous}\}.$$

Let us recall that the strict topology β on $C(T)$ is the topology generated by the family of seminorms

$$q(f) = \sup_{n \geqq 1} \{a_n \|f\|_{K_n}\} \quad \forall f \in C(T),$$

where $K_1, K_2, \ldots$ are compact subsets of T, and $(a_n)_{n \in \mathrm{N}}$ is a sequence of positive reals with $\lim_{n \to \infty} a_n = 0$, see [3].

PROPOSITION 4.4. *Let T be a Polish space, let $F = C(T)$ and let $\mathscr{F} = \sigma\{P_t \mid t \in T\}$. Then F is $\mathscr{F}$-analytically normed if $C(T)$ has the strict topology.*

PROOF. Let $C(\mathrm{Q}^+ | 0)$ be the set of all sequences of positive rationals that tend to 0, topologized by the discrete product topology, i.e. if $\{(s_{j_\lambda}) \mid \lambda \in \Lambda\}$ is a net in $C(\mathrm{Q}^+ | 0)$, then

$$\lim_{\lambda} (s_{j_\lambda}) = (s_j) \Leftrightarrow \forall j \geqq 1 \; \exists \lambda_0 \in \Lambda : s_{j_\lambda} = s_j \quad \forall \lambda \geqq \lambda_0.$$

If $\vdash$ is a subsequence ordering on $C(\mathrm{Q}^+ | 0)$, i.e.

$$(s'_j) \vdash (s''_j) \Leftrightarrow (s'_j) \text{ is a subsequence of } (s''_j),$$

then $(C(\mathrm{Q}^+ | 0), \mathscr{B}(C(\mathrm{Q}^+ | 0)), \vdash)$ is a smoothly ordered measurable space, see [4, Theorem 2.10]. Let $\mathscr{K}(T)$ be the set of all compact subsets of T, let $v(T)$ be the Vietori's topology on $\mathscr{K}(T)$, i.e. the topology generated by the sets

$$\{K \in \mathscr{K}(T) \mid K \subseteqq G\} \quad \text{and} \quad \{K \in \mathscr{K}(T) \mid K \cap G \neq \emptyset\},$$

for G open in T, and let $\mathscr{E}(T)$ be the Borel σ-algebra of $(\mathscr{K}(T), v(T))$ called the Effros-Borel structure. The space $(\mathscr{K}(T), \mathscr{E}(T), \subseteqq)$ is a smoothly ordered measurable space by Theorem 2.7 in [4] since T is a Polish space, and so is the product space

$$(\mathscr{K}(T)^{\mathbf{N}}, \bigotimes_{i=1}^{\infty} \mathscr{E}(T), \subseteqq).$$

Let φ be the map

$$\varphi: C(\mathbf{Q}^+ | 0) \times \mathscr{K}(T)^{\mathbf{N}} \to \Pi$$

defined by

$$\varphi((r_i), (K_i)) = \sup_{i \in \mathbf{N}} (r_i \|f\|_{K_i}) = q(f).$$

For abbreviation we let

$$\Gamma = C(\mathbf{Q}^+ | 0) \times \mathscr{K}(T)^{\mathbf{N}}.$$

If we prove that φ satisfies (2.5.5), then F is $\mathscr{F}$-analytically normed since Γ is a smoothly ordered space. That φ is increasing is trivial. If $q \in \Pi$, i.e.

$$q(f) = \sup_{i \in \mathbf{N}} (a_i \|f\|_{K_i}),$$

and if we take $(r_i) \in C(\mathbf{Q}^+ | 0)$ so that $r_i \geqq a_i$, $\forall i \in \mathbf{N}$, we have the cofinality of φ. It remains to prove

$$A_a = \{(r_i), (K_i), f) \in \Gamma \times F \mid \sup_{i \in \mathbf{N}} (r_i \|f\|_{K_i}) > a\} \in S(\mathscr{B}(\Gamma) \otimes \mathscr{F}) \quad \forall a > 0.$$

Since

$$A_a = \bigcup_{i=1}^{\infty} \bigcup_{q \in \mathbf{Q}} \{((r_j), (K_j), f) \in \Gamma \times F \mid r_i = q, \quad \|f\|_{K_i} > a/q\},$$

it is enough to show that

(4.4.1) $$B_c = \{(K, f) \in \mathscr{K} \times C(T) \mid \|f\|_K > c\} \in \mathscr{E}(\mathscr{K}) \otimes \mathscr{F}$$

for all $c \geqq 0$. Let d be a metric defining the topology on T and let

$$b(t, r) = \{u \in T \mid d(u, t) < r\}.$$

Let D be a countable dense subset of T, then we have:

$$B_c = \{(K, f) \in \mathscr{K} \times C(T) \mid \exists t \in K : |f(t)| > c\}$$

$$= \bigcup_{n=1}^{\infty} \{(K, f) \in \mathscr{K} \times C(T) \mid \exists t \in K : |f(t)| > c + 2^{-n}\}$$

$$= \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \{(K, f) \in \mathscr{K} \times C(T) \mid \exists t \in D : b(t, 2^{-k}) \cap K \neq \emptyset, |f(t)| > c + 2^{-n}\}$$

$$= \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{t \in D}$$

$$\{K \in \mathscr{K} \mid K \cap b(t, 2^{-k}) \neq \emptyset\} \times \{f \in C(T) \bigm| |f(t)| > c + 2^{-n}\}$$

$$\in \mathscr{E}(\mathscr{K}) \otimes \mathscr{F}.$$

Thus (4.4.1) holds and so the Theorem is proved.

Let us point out that if T is a completely regular space, then the dual of $(C(T), \mathscr{B})$ equals $M(T)$, where $M(T)$ is the set of all real valued, finite Radon measures on $(T, \mathscr{B}(T))$, see [3].

5. Inductive and projective limit of analytically normed spaces.

We shall prove first that under very weak conditions an inductive limit of analytically normed spaces is again such a space.

THEOREM 5.1. *Let $F = \lim_{\rightarrow}(E_n, T_n)$ be an inductive limit of $\{(E_n, T_n) \mid n \in \mathbf{N}\}$, where E_n is a locally convex space and $T_n : E_n \to F$ is linear for all $n \in \mathbf{N}$, and suppose that*

$$F = \operatorname{span}\left(\bigcup_{n=1}^{\infty} T_n(E_n)\right). \tag{5.1.1}$$

Let Π_n be the set of all continuous seminorms on E_n and let $\psi_n : \mathbf{N}^{\mathbf{N}} \to \Pi_n$ be increasing cofinal maps satisfying

$$\psi_n(\cdot, x) \text{ is upper semicontinuous} \quad \forall n \in \mathbf{N}, \quad \forall x \in E_n, \tag{5.1.2}$$

where $\psi_n(\sigma, x) = \psi_n(\sigma)(x)$ for all $n \in \mathbf{N}$, $\sigma \in \mathbf{N}^{\mathbf{N}}$, and $x \in E_n$. If $\mathscr{F}$ is a σ-algebra on F such that F admits an $\mathscr{F}$-elementary resolution of the identity, then F is $\mathscr{F}$-analytically normed.

PROOF. Let $\tau \in \mathsf{N}^{\mathsf{N}}$ and $\gamma = (\gamma_i) \in (\mathsf{N}^{\mathsf{N}})^{\mathsf{N}}$. Then we put

$$\psi(\tau, \gamma, x) = \inf\left\{ \sum_{i=1}^{n} \tau(i)\psi_i(\gamma_i, x_i) \mid n \in \mathsf{N}, (x_1, \ldots, x_n) \in E^n(x) \right\}$$

for all $x \in F$ where

$$E^n(x) = \left\{ (x_1, \ldots, x_n) \in \prod_{j=1}^{n} E_j \mid x = \sum_{j=1}^{n} T_j(x_j) \right\}.$$

Then by (5.1.1) and the definition of the inductive limit topology, it follows that

$$\{\psi(\tau, \gamma, \cdot) \mid (\tau, \gamma) \in \mathsf{N}^{\mathsf{N}} \times (\mathsf{N}^{\mathsf{N}})^{\mathsf{N}}\}$$

is a family of continuous seminorms on F, which generates the topology on F (recall that ψ_i is cofinal). Thus ψ is a cofinal map from $M = \mathsf{N}^{\mathsf{N}} \times (\mathsf{N}^{\mathsf{N}})^{\mathsf{N}}$ into the set of all continuous seminorms on F. Clearly ψ is increasing, so consider the set

$$\text{(5.1.3)} \qquad A(a) = \{(\tau, \gamma, x) \in M \times F \mid \psi(\tau, \sigma, x) \geqq a\} \quad \text{for } a \geqq 0.$$

If we can show that $A(a) \in S(\mathscr{B}(M) \otimes \mathscr{F})$, we have finished since $(M, \mathscr{B}(M), \leqq)$ is a smoothly ordered space. Now, since $\psi(\tau, \sigma, \cdot)$ is continuous, then by the assumption that F admits an $\mathscr{F}$-elementary resolution of the identity and [2, Theorem IV.2.6] it is sufficient to show that

$$A(a, x) = \{(\tau, \sigma) \in M \mid \psi(\tau, \sigma, x) \geqq a\} \in \mathscr{B}(M), \quad \forall\, x \in F, \quad \forall\, a \geqq 0.$$

Observe that

$$A(a, x) = \bigcup_{n=1}^{\infty} \bigcap_{(a_1, \ldots, x_n) \in E^n(x)} \left\{ (\tau, \sigma) \in M \mid \sum_{i=1}^{n} \tau(i)\psi_i, x_i) \geqq a \right\},$$

and since $(\tau, \sigma) \to \tau(i)$ is non-negative continuous on M and since $(\tau, \sigma) \to \psi_i(\sigma_i, x_i)$ is non-negative upper semicontinuous by (5.1.2), it follows that

$$(\tau, \sigma) \to \sum_{i=1}^{n} \tau(i)\psi_i(\sigma_i, x_i)$$

is upper semicontinuous. Hence $A(a, x)$ is closed in M and so (5.1.3) holds. Thus the Theorem is proved.

One of the conditions in Theorem 5.1 is that F should admit an $\mathscr{F}$-elementary resolution of the identity. The following proposition shows the case where it is so.

PROPOSITION 5.2. *Let F be the inductive limit of a family of locally convex spaces $\{(E_i, T_i) | i \in \mathsf{N}\}$, where E_i is a locally convex space and $T_i: E_i \to F$ is an injective linear map for all $i \in \mathsf{N}$. Suppose that*

$$F = \bigcup_{i=1}^{\infty} T_i(E_i) \tag{5.2.1}$$

and that E_i admits $\mathscr{F}_i$-resolution of the identity for every $i \in \mathsf{N}$, where $\mathscr{F}_i$ is an σ-algebra on E_i for all $i \in \mathsf{N}$. If

$$\mathscr{F} = \sigma\{T_i(A_i) \mid i \in \mathsf{N}, A_i \in \mathscr{F}_i\},$$

then F admits $\mathscr{F}$-elementary resolution of the identity.

PROOF. Let $\{s_{in}\}_{n=1}$ be $\mathscr{F}_i$-elementary resolution of the identity on E_i for all $i \in \mathsf{N}$. Since (5.2.1) holds, every $x \in F$ belongs to at least one of $T_i(E_i)$, $i \in \mathsf{N}$. Let

$$k(x) = \min\{n \in \mathsf{N} \mid x \in T_n(E_n)\} \tag{5.2.2}$$

and let $f_n: F \to F$ be defined by

$$f_n(x) = T_{k(x)}(s_{k(x),n}(T_{k(x)}^{-1}(x))) \quad \forall\, n \in \mathsf{N}.$$

Further let $\{F_{inj} | j \in \mathsf{N}\} \subseteqq \mathscr{F}_i$ be a disjoint partition of E_i and $\{a_{inj} | j \in \mathsf{N}\} \subseteqq E_i$, such that

$$s_{in}(y) = a_{inj} \quad \forall\, y \in F_{inj}.$$

If we denote $b_{inj} = T_i(a_{inj})$, and

$$H_{inj} = T_i(F_{inj}) \setminus \bigcup_{m=1}^{i-1} T_m(E_m),$$

then we have

$$f_n(x) = b_{inj} \quad \forall\, x \in H_{inj}$$

and $\{H_{inj} | (i, j) \in \mathsf{N}^2\}$ is a disjoint partition of F contained in $\mathscr{F}$. Moreover, if $x \in F$ and $y = T_{k(x)}^{-1}(x)$, then

$$\lim_{n\to\infty} f_n(x) = T_{k(x)}\left(\lim_{n\to\infty} s_{k(x)n}(y)\right) = T_{k(x)}(y) = x,$$

because $T_{k(x)}$ is continuous by definition of the inductive topology on F.

EXAMPLE 5.3. Let $\Omega \subseteqq \mathrm{R}^n$ be an open set, $K \subseteqq \Omega$ a compact set and let the space

$$\mathscr{D}_k = \{f \in C^\infty(\Omega) \mid \text{support}(f) \subseteqq K\}$$

have the topology induced by the family of seminorms

$$P_{K,n}(f) = \max_{|\alpha| \leqq n} \sup_{x \in K} |D^\alpha f(x)|,$$

where $\Omega \in \mathrm{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and

$$D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n},$$

and $C^\infty(\Omega)$ is the space of all infinitely differentiable functions (see [7]). Let $\{K_i\}_{i=1}^\infty$ be a sequence of compact subsets of Ω such that

$$K_i \subseteqq \operatorname{int} K_{i+1} \quad i \in \mathrm{N} \quad \text{and} \quad \Omega = \bigcup_{i=1}^\infty K_i,$$

and let

$$\mathscr{D}(\Omega) = \{f \in C^\infty(\Omega) \mid \text{support } (f) \text{ is a compact subset of } \Omega\}.$$

Then

$$\mathscr{D}(\Omega) = \bigcup_{i=1}^\infty \mathscr{D}_{K_i}$$

and let $\mathscr{D}(\Omega)$ have the inductive topology with respect to the family $\{\mathscr{D}_{K_i} | i \in \mathrm{N}\}$ and the canonical imbeddings $T_i: \mathscr{D}_{K_i} \to \mathscr{D}(\Omega)$. This is the well-known test function space (see [7]). Every $\mathscr{D}_{k_i}$ is a Fréchet space with Heine-Borel property (see [7]), i.e. it is a Fréchet-Montél space, and so is a standard space by [2, V.1.c Corollary 18]. Every standard space F admits $\mathscr{B}(F)$-simple resolution of the identity [2, IV.2 Theorem 1]; thus $\mathscr{D}_{K_i}$ admits $\mathscr{B}(\mathscr{D}_{K_i})$-simple resolution of the identity for every $i \in \mathrm{N}$, and therefore $\mathscr{D}(\Omega)$ admits $\mathscr{B}(\mathscr{D}(\Omega)$-elementary resolution of the identity by Proposition 5.2.

Let

$$\Pi_i = \{P_{K_{i,j}} \mid j \in \mathsf{N}\}$$

where $P_{K_{i,j}}$ is defined by (5.3.1), and let $\psi_i : \mathsf{N}^{\mathsf{N}} \to \Pi_i$ be defined by

$$\psi_i(\sigma) = P_{K_{i,\sigma(1)}} \quad \forall\, i \in \mathsf{N}.$$

Then ψ_i is increasing and cofinal in Π_i. Moreover, for every $f \in \mathscr{D}_{K_i}$ the function $\sigma \to \psi_i(\sigma)(f)$ is continuous since it only depends on the first coordinate $\sigma(1)$. Hence by Theorem 5.1 we have that $\mathscr{D}(\Omega)$ is $\mathscr{B}(\mathscr{D}(\Omega))$-analytically normed.

THEOREM 5.4. *Let F be the projective topology with respect to a family $\{(E_i, f_i) \mid i \in \mathsf{N}\}$, where E_i is a locally convex space and $f_i : F \to E_i$ is a linear map for every $i \in \mathsf{N}$. Suppose that for every $i \in \mathsf{N}$ the space E_i is $\mathscr{F}_i$-analytically normed. If $\mathscr{F}$ is a σ-algebra on F so that*

$$\mathscr{F} \supseteq \sigma\{f_i^{-1}(A_i) \mid i \in \mathsf{N},\ A_i \in \mathscr{F}_i\}, \tag{5.4.1}$$

then F is $\mathscr{F}$-analytically normed.

PROOF. Let Π_i be the set of all continuous seminorms on E_i and let $\varphi_i : \mathsf{N}^{\mathsf{N}} \to \Pi_i$ be an increasing cofinal map such that

$$A_i(a) = \{(\sigma, u) \in \mathsf{N}^{\mathsf{N}} \times E_i \mid \varphi_i(\sigma, u) > a\} \in S(\mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{F}_i) \tag{5.4.2}$$

for all $i \in \mathsf{N}$ and all $a \geqq 0$. Now, put

$$\psi(n, \sigma, x) = \max_{1 \leq i \leq n} \varphi_i(\sigma_i, f_i(x)),$$

for $n \in \mathsf{N}$, $\sigma = (\sigma_i) \in (\mathsf{N}^{\mathsf{N}})^{\mathsf{N}}$ and $x \in F$. If $M = \mathsf{N} \times (\mathsf{N}^{\mathsf{N}})^{\mathsf{N}}$, then $\xi \to \psi(\xi, \cdot)$ is an increasing cofinal map form M into the set of all continuous seminorms on F by definition of the injective limit topology, and $(M, \mathscr{B}(M), \leqq)$ is a smoothly ordered space. Consider the set

$$\begin{aligned} A(a) &= \{(n, \sigma, x) \in M \times F \mid \psi(n, \sigma, x) > a\} \\ &= \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} \{k\} \times \{(\sigma, x) \in (\mathsf{N}^{\mathsf{N}})^{\mathsf{N}} \times F \mid \varphi_i(\sigma_i, f_i(x)) > a\} \\ &= \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} \{k\} \times g_i^{-1}(A_i(a)) \end{aligned}$$

where

$$g_i\colon ((\mathsf{N}^{\mathsf{N}})^{\mathsf{N}} \times F, \mathscr{B}((\mathsf{N}^{\mathsf{N}})^{\mathsf{N}}) \otimes \mathscr{F}) \to (\mathsf{N}^{\mathsf{N}} \times E_i, \mathscr{B}(\mathsf{N}^{\mathsf{N}}) \otimes \mathscr{F}_i)$$

is the map defined by

$$g_i(\sigma, x) = (\sigma_i, f_i(x)).$$

Since g_i is measurable, we see that $A(a) \in S(\mathscr{B}(M) \otimes \mathscr{F})$ by (5.4.1). Hence the Theorem follows.

REMARK. If $\mathscr{F}_i = \mathscr{B}(E_i)$, $i \in \mathsf{N}$, and if $\mathscr{F} = \mathscr{B}(F)$, then

$$\mathscr{F} \supseteqq \sigma\{f_i^{-1}(A_i) \mid i \in \mathsf{N}, \quad A_i \in \mathscr{F}_i\}.$$

EXAMPLE 5.5. Let $\mathscr{D}'(\Omega)$ be the topological dual of $\mathscr{D}(\Omega)$. If $\mathscr{D}'(\Omega)$ has its Mackey topology $\tau(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))$, then we have that

$$\tau(\mathscr{D}'(\Omega), \mathscr{D}(\Omega)) = \prod_{i=1}^{\infty} \tau(\mathscr{D}'_{K_i}, \mathscr{D}_{K_i})$$

by [8, IV.4. Corollary 1], and if $(\mathscr{D}'_{K_i}, \tau(\mathscr{D}'_{K_i}, \mathscr{D}_{K_i}))$ is $\mathscr{F}_i$-analytically normed for every $i \in \mathsf{N}$, then $(\mathscr{D}'(\Omega), \tau(\mathscr{D}'(\Omega), \mathscr{D}(\Omega)))$ is analytically normed with respect to any σ-algebra $\mathscr{F}$ on $\mathscr{D}'(\Omega)$ so that

$$\mathscr{F} \supseteqq \sigma\{f_i^{-1}(A_i) \mid i \in \mathsf{N}, \quad A_i \in \mathscr{F}_i\}. \tag{5.5.1}$$

Every $\mathscr{D}_{K_i}$ is a Montél space, so the Mackey topology $\tau(\mathscr{D}'_{K_i}, \mathscr{D}_{K_i})$, the precompact topology, $\pi(\mathscr{D}'_{K_i}, \mathscr{D}_{K_i})$, and the strong topology $\beta(\mathscr{D}'_{K_i}, \mathscr{D}_{K_i})$ coincide on $\mathscr{D}'_{K_i}$. So if $\mathscr{F}_i$ is the weak*-Baire σ-algebra on $\mathscr{D}'_{K_i}$, then $(\mathscr{D}'_{K_i}, \tau(\mathscr{D}'_{K_i}, \mathscr{D}_{K_i}))$ is $\mathscr{F}_i$-analytically normed by Proposition 4.2. So if $\mathscr{F}$ is any σ-algebra on $\mathscr{D}'(\Omega)$ satisfying (5.5.1), then $\mathscr{D}'(\Omega)$ is $\mathscr{F}$-analytically normed in its Mackey topology which coincides with the strong topology and the precompact topology since $\mathscr{D}(\Omega)$ as the strict inductive limit of Montél spaces $\{\mathscr{D}_{K_i} \mid i \in \mathsf{N}\}$ is a Montél space [8].

At the end, let us mention that we have not used that the elements of Π are subadditive and homogeneous, so some of the results can be extended to topological vector spaces where topology is generated by a family of positive real valued functions.

REFERENCES

1. N. Dunford and J. T. Schwarts, *Linear Operators* I: *General Theory* (Pure and Appl. Math. 7, Interscience Publishers, Inc., New York - London, 1958.
2. J. Hoffmann-Jørgensen, *The Theory of Analytic Spaces*, Various Publications Series 10, Math. Inst., Univ. Aarhus, 1970.
3. J. Hoffmann-Jørgensen, *A generalization of the strict topology*, Math. Scand. 30 (1972), 313–323.
4. J. Hoffmann-Jørgensen, *Stochastic processes on Polish spaces* (to appear).
5. G. Köthe, *Topological Vector Spaces* I (Grundehren Math. Wiss. 159), Springer-Verlag, Berlin - Heidelberg - New York, 1969.
6. C. A. Rogers, *Analytic Sets* (Conf., Univ. London, 1978), Academic Press, Inc., London - New York, 1980.
7. W. Rudin, *Functional Analysis*, McGraw-Hill Book Co., New York - London, 1973.
8. H. H. Schaefer, Topological Vector Spaces (Graduate Texts in Math. 3), Springer-Verlag, New York - Heidelberg - Berlin, 1971.

DEPARTMENT OF MATHEMATICS
FGZ-ZAGREB
UNIVERSITY OF ZAGREB
P. MIŠKINE 129
41000 ZAGREB
YUGOSLAVIA