MATH. SCAND. 60 (1987), 129-147

NEARNESS OF CONTINUED FRACTIONS

LISA JACOBSEN

1. Introduction.

We shall study continued fractions

$$(1.1) \qquad K\frac{a_n}{b_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}, \quad a_n \neq 0, a_n, b_n \in \mathbf{C}.$$

(The restriction to complex elements a_n and b_n is not severe. If a_n and b_n are complex functions of one or several variables, we can investigate $K(a_n/b_n)$ pointwise. Moreover, the considerations in this paper are not altered if we replace $(\mathbf{C}, |\cdot|)$ by some other normed field $(\boldsymbol{F}, \|\cdot\|)$. $\boldsymbol{F}$ could for instance be the field of all memorphic functions.)

The purpose of this paper is to investigate the question: To what extent will "nearness" of two continued fractions $K(a_n/b_n)$ and $K(\tilde{a}_n/\tilde{b}_n)$ imply "nearness" of their properties? By "nearness" of $K(a_n/b_n)$ and $K(\tilde{a}_n/\tilde{b}_n)$ we shall mean $|a_n - \tilde{a}_n|$ and $|b_n - \tilde{b}_n|$ "small" for all n. By "nearness" of properties we think mainly of similarity in convergence behavior and values. (Convergence and value of a continued fraction are defined in the next section.)

Answers to this question are of interest in several situations. In particular if such properties are known for one of these continued fractions, say $K(\tilde{a}_n/\tilde{b}_n)$, then this can be used to gain information on the other one, $K(a_n/b_n)$. Moreover, if these properties are proved to be "sufficiently close", $K(\tilde{a}_n/\tilde{b}_n)$ can be used (as an auxiliary continued fraction) to accelerate the convergence of $K(a_n/b_n)$ or to continue the function $K(a_n(z)/b_n(z))$ analytically, (see [7]).

The said investigations will take place in section 4 (modified convergence), section 5 (ordinary convergence) and section 6 (the special case where $(a_n - \tilde{a}_n) \to 0$ and $(b_n - \tilde{b}_n) \to 0$). In section 2 the basic notation and results used in this paper are given. It is mainly standard notation in accordance with for instance [11]. Section 3 contains some historical remarks.

Received December 11, 1984; in revised form March 4, 1986.

2. Definitions and notation.

The continued fraction (1.1) is said to *converge* and have the *value* f, if its sequence of *approximants*,

$$f_0 = 0 \quad f_n = \frac{a_1}{b_1} \, \frac{}{+} \, \frac{a_2}{b_2} \, \frac{}{+} \cdots \frac{}{+} \, \frac{a_n}{b_n}, \quad (n \geqq 1) \tag{2.1}$$

converges to f in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The analogy to series (and partial sums) is obvious. In forming the nth approximant f_n of (1.1), we "cut off" its nth tail

$$\underset{m=1}{\overset{\infty}{K}} \frac{a_{n+m}}{b_{n+m}} = \frac{a_{n+1}}{b_{n+1}} \, \frac{}{+} \, \frac{a_{n+2}}{b_{n+2}} \, \frac{}{+} \cdots, \quad (n \in \mathbf{N} \cup \{0\}), \tag{2.2}$$

where $\mathbf{N} = \{1, 2, 3, \ldots\}$. If (1.1) converges, then so do all of its tails (2.2). In contrast to convergent series though, the value $f^{(n)}$ of (2.2) will not (normally) approach 0 as $n \to \infty$. (See for instance Theorem 3.1) A *modified approximant* f_n^* of (1.1) is the value we get if we replace the nth tail by a *modifying factor* $w_n \in \hat{\mathbf{C}}$.

For convenience we introduce the sequence

$$s_n(w) = a_n/(b_n + w), \quad (n \in \mathbf{N}) \tag{2.3}$$

of linear fractional transformations. The composition

$$S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w) = \frac{a_1}{b_1} \, \frac{}{+} \, \frac{a_2}{b_2} \, \frac{}{+} \cdots \frac{}{+} \, \frac{a_n}{b_n + w}, \quad (n \in \mathbf{N}) \tag{2.4}$$

is then again a linear fractional transformation.

With this notation we have $f_n = S_n(0)$, $f_n^* = S_n(w_n)$, and $f = S_n(f^{(n)})$. It is easy to prove, (see [11, p. 20]), that

$$S_n(w) = \frac{A_n + A_{n-1} w}{B_n + B_{n-1} w}, \quad (n \in \mathbf{N}) \tag{2.5}$$

where $\{A_n\}$, $\{B_n\}$ are solutions of the linear recurrence relation

$$X_n = b_n X_{n-1} + a_n X_{n-2}, \quad (n \in \mathbf{N}), \tag{2.6}$$

with initial conditions $A_{-1} = B_0 = 1$, $A_0 = B_{-1} = 0$.

Another quantity, which is of importance, is

$$h_n = -S_n^{-1}(\infty) = \frac{B_n}{B_{n-1}} = b_n + \frac{a_n}{b_{n-1}} \, \frac{}{+} \, \frac{a_{n-1}}{b_{n-2}} \, \frac{}{+} \cdots \frac{}{+} \, \frac{a_2}{b_1}, \quad (n \in \mathbf{N}). \tag{2.7}$$

Both $\{-h_n\}$ and $\{f^{(n)}\}$ satisfy the recursion

$$g_{n-1} = a_n/(b_n+g_n), \quad (n \in \mathbb{N}), \tag{2.8}$$

with initial conditions $h_0 = \infty$, $f^{(0)} = f$. A sequence $\{g_n\}_{n=0}^{\infty}$ of elements from $\hat{\mathbb{C}}$ satisfying (2.8) is called a *tail sequence* for $K(a_n/b_n)$. It is called a sequence of *right tails* if $K(a_n/b_n)$ converges and $g_0 = f$, otherwise it is called a sequence of *wrong tails.*

In this paper we shall let $S_m^{(n)}$, $f_m^{(n)}$, $h_m^{(n)}$, $f^{(n)}$, $A_m^{(n)}$, $B_m^{(n)}$ denote the same concepts for the nth tail (2.2) of $K(a_n/b_n)$. Furthermore, the above notation shall always refer to $K(a_n/b_n)$, whereas $\tilde{S}_n$, $\tilde{f}_n, \ldots$, $\tilde{A}_m^{(n)}$, $\tilde{B}_m^{(n)}$ shall always refer to $K(\tilde{a}_n/\tilde{b}_n)$.

Convergence criteria for continued fractions are often stated in terms of *convergence regions* $\{\Omega_n\}_{n=1}^{\infty}$; that is $K(a_n/b_n)$ converges if $(a_n, b_n) \in \Omega_n \subseteq \mathbb{C} \times \mathbb{C}$ for all n. If in addition

$$|f - f_n| \leqq \lambda_n \to 0 \quad \text{if } (a_n, b_n) \in \Omega_n \text{ for all } n, \tag{2.9}$$

where $\{\lambda_n\}$ only depends on $\{\Omega_n\}$, then $\{\Omega_n\}$ is a *uniform sequence of convergence regions.* In order to find such convergence criteria, it has proved useful to apply the following concepts:

DEFINITION 2.1. $\{\Omega_n\}_{n=1}^{\infty}$ is called a sequence of *element regions* for continued fractions $K(a_n/b_n)$, if

$$\Omega_n \subseteq \mathbb{C} \times \mathbb{C} \quad \text{and} \quad \Omega_n \setminus (\{0\} \times \mathbb{C}) \neq \emptyset, \quad (n \in \mathbb{N}). \tag{2.10}$$

$\{V_n\}_{n=0}^{\infty}$ is called a sequence of *pre-value regions* corresponding to a sequence $\{\Omega_n\}$ of element regions, if $V_n \subseteq \hat{\mathbb{C}}$, $V_n \neq \emptyset$ for all n and

$$(a, b) \in \Omega_n,\ w \in V_n \Rightarrow a/(b+w) \in V_{n-1}, \quad (n \in \mathbb{N}). \tag{2.11}$$

If, in addition, $a/b \in V_{n-1}$ for every pair $(a, b) \in \Omega_n$, $(n \in \mathbb{N})$, then $\{V_n\}$ is called a sequence of *value regions* corresponding to $\{\Omega_n\}$.

Value and pre-value regions are also interesting from another aspect: If $\{V_n\}$ is a sequence of value regions corresponding to $\{\Omega_n\}$, then it is easy to prove that

$$(a_n, b_n) \in \Omega_n \text{ for all } n \Rightarrow f_m^{(n)} \in V_n, \quad (m \in \mathbb{N},\ n \in \mathbb{N} \cup \{0\}). \tag{2.12}$$

This means that if $K(a_n/b_n)$ converges, then $f^{(n)} \in \mathrm{Cl}(V_n)$ (the closure of V_n

in $\hat{\mathbb{C}}$) for all n. That is, we have a rough estimate for the location of $f^{(n)}$ or $f = f^{(0)}$ if V_n or V_0 is not too big. Moreover, since $f^{(n)} \in \mathrm{Cl}(V_n)$ and $f_1^{(n)} \in V_n$, we get the following truncation error bound

$$|f - f_{n+1}| = |S_n(f^{(n)}) - S_n(f_1^{(n)})| \leqq \operatorname{diam}(S_n(V_n)). \tag{2.13}$$

If we consider modified approximants, and $\{V_n\}$ is a sequence of pre-value regions corresponding to $\{\Omega_n\}$, then

$$(a_n, b_n) \in \Omega_n,\ w_n \in V_n,\ (n \in \mathbf{N}) \Rightarrow S_m^{(n)}(w_{n+m}) \in V_n, \quad (n \in \mathbf{N} \cup \{0\},\ m \in \mathbf{N}). \tag{2.14}$$

Hence, pre-value regions act as value regions for these modified approximants. If $S_n(w_n)$ converges, then

$$F = \lim S_n(w_n) \in \mathrm{Cl}(V_0)$$

and

$$F^{(n)} = \lim_{m \to \infty} S_m^{(n)}(w_{n+m}) \in \mathrm{Cl}(V_n) \quad \text{for all } n.$$

Furthermore, if $K(a_n/b_n)$ also converges in the ordinary sense to the same value F, then pre-value regions often give better estimates for the location of $f = F$ and for truncation errors, since they frequently can be chosen smaller than ordinary value regions.

With this application of pre-value regions in mind, we introduce the following:

DEFINITION 2.2. Let $\{\Omega_n\}$ be a sequence of element regions, and $\{V_n\}$ a corresponding sequence of pre-value regions. If

$$(a_n, b_n) \in \Omega_n,\ w_n \in V_n,\ (n \in \mathbf{N}) \Rightarrow S_n(w_n) \to F, \quad F \in \mathbf{C} \text{ independent of } \{w_n\}, \tag{2.15}$$

then $\{\Omega_n\}$ is called a sequence of *modified convergence regions* with respect to $\{V_n\}$.

If, in addition, there exists a sequence $\{\lambda_n\}$ of positive numbers converging to 0, such that

$$(a_n, b_n) \in \Omega_n,\ w_n \in V_n,\ (n \in \mathbf{N}) \Rightarrow |F - S_n(w_n)| \leqq \lambda_n, \quad (n \in \mathbf{N}), \tag{2.16}$$

then $\{\Omega_n\}$ is called a *uniform sequence of modified convergence regions* with respect to $\{V_n\}$.

Another question is: Will $K(a_n/b_n)$ converge to F in the ordinary sense if (2.15) holds? Fortunately, the answer is very often yes. For instance, if $K(a_n/b_n)$ converges to f and $\liminf_{n\to\infty}\operatorname{diam}(V_n) > 0$, then $f = F$, (see [10]), Other sufficient conditions are given in section 5.

We are dealing with continued fractions $K(a_n/b_n)$, the most general form. However, $K(a_n/b_n)$ is equivalent to $K(1/b_n^*)$ (that is $K(a_n/b_n)$ and $K(1/b_n^*)$ have the same sequence of approximants), where

$$b_{2n-1}^* = b_{2n-1}\frac{a_2a_4\cdots a_{2n-2}}{a_1a_3\cdots a_{2n-1}}, \tag{2.17}$$

$$b_{2n}^* = b_{2n}\frac{a_1a_3\cdots a_{2n-1}}{a_2a_4\cdots a_{2n}}$$

for all $n \geqq 1$. For such continued fractions, we have $\Omega_n = \{1\} \times G_n$, $G_n \subseteqq \mathbf{C}$ for all n, and the convergence criteria and their development get simpler. (We then say that $\{G_n\}$ is a sequence of element or convergence regions.) On the other hand the continued fraction may get considerably more complicated by this transformation.

If $b_n \neq 0$ for all n, then $K(a_n/b_n)$ is equivalent to $K(a_n^*/1)$, where

$$a_1^* = a_1/b_1, \quad a_n^* = a_n/b_nb_{n-1}, \quad (n \geqq 2). \tag{2.18}$$

This transformation can be very useful, if it can be applied. Then $\Omega_n = E_n \times \{1\}$, and again the convergence criteria and their development get simpler. (We then say that $\{E_n\}$ is a sequence of element or convergence regions.)

3. Historical remarks.

As mentioned in the introduction, the type of "nearness"-results that we are considering, are interesting if properties of $K(\tilde{a}_n/\tilde{b}_n)$ are known. Since the behavior of periodic continued fractions $K(\tilde{a}_n/\tilde{b}_n)$, where

$$\tilde{a}_{kn+p} = \tilde{a}_p \quad \text{and } \tilde{b}_{kn+p} = \tilde{b}_p, \quad (n \geqq 1, p \in \{1, \ldots, k\}) \tag{3.1}$$

for some $k \in \mathbf{N}$, are particularly well known, these have been a natural reference, in particular in the study of limit periodic continued fractions $K(a_n/b_n)$, where

$$\lim_{n\to\infty} a_{kn+p} = \tilde{a}_p, \quad \lim_{n\to\infty} b_{kn+p} = \tilde{b}_p, \quad (p = 1, 2, \ldots, k). \tag{3.2}$$

The convergence behavior of a periodic continued fraction $K(\tilde{a}_n/\tilde{b}_n)$ is governed by the character of the linear fractional transformation $\tilde{S}_k(w)$. If $\tilde{S}_k$ is hyperbolic or loxodromic (i.e. it has two distinct fixed points, one attractive and one repulsive), then $K(\tilde{a}_n/\tilde{b}_n)$ converges to the attractive fixed point, $\tilde{f}$, except in special cases. (Thiele oscillation, (see [12, Satz 2.39]).) If $\tilde{S}_k$ is parabolic (i.e. has only one fixed point in $\hat{\mathbb{C}}$), then $K(\tilde{a}_n/\tilde{b}_n)$ converges to the fixed point $\tilde{f}$. Perron, [12, Satz 2.41], proved that for the case $k = 1$, the "nearness" of a corresponding limit periodic continued fraction $K(a_n/b_n)$ implied that also $K(a_n/b_n)$ converges, under some conditions, and that the "closer" $K(a_n/b_n)$ is to $K(\tilde{a}_n/\tilde{b}_n)$, the closer its value f is to $\tilde{f}$:

THEOREM 3.1 (Perron). *If $\tilde{S}_1(w) = \tilde{a}_1/(\tilde{b}_1 + w)$ is non-parabolic, then*

$$\left.\begin{array}{l} K(\tilde{a}_1/\tilde{b}_1) \text{ converges to } \tilde{f} \\ 0 \neq a_n \to \tilde{a}_1, \quad b_n \to \tilde{b}_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} K(a_n/b_n) \text{ converges}, \\ f^{(n)} \to \tilde{f}. \end{array}\right.$$

From Worpitzky's theorem, (see [11, Corollary 4.36]), follows that $K(a_n/b_n)$ also converges if $k = 1$ and $\tilde{a}_1 = 0$ or $\tilde{b}_1 = \infty$.) Perron also gave truncation error estimates for this case. Later, Szasz [15] and von Pidoll [13] generalized this to $k > 1$.

Theorem 3.1 inspired Gill [3] and Thron and Waadeland [16] to introduce their method for convergence acceleration of limit periodic continued fractions with $k = 1$:

THEOREM 3.2 (Thron and Waadeland). *If $\tilde{S}_1(w) = \tilde{a}_1/(1 + w)$ is non-parabolic, then*

$$\left.\begin{array}{l} K(\tilde{a}_1/1) \text{ converges to } \tilde{f} \\ f = K(a_n/1) \neq \infty \\ \quad 0 \neq a_n \to \tilde{a}_1 \end{array}\right\} \Rightarrow \frac{f - S_n(\tilde{f})}{f - S_n(0)} \to 0.$$

This property of $K(a_n/b_n)$ can also be regarded as an inheritance from $K(\tilde{a}_1/1)$. Clearly, using modifying factors $\tilde{f}$ on $K(\tilde{a}_1/1)$ leads to an extreme convergence acceleration, since $\tilde{S}_n(\tilde{f}) = \tilde{f}$ for all n. Theorem 3.2 is also generalized to $k > 1$, (see [4], [7]).

If $\tilde{S}_k$ is parabolic, extra conditions on $K(a_n/b_n)$ are needed to obtain conclusions like Theorems 3.1 and 3.2. In 1905, Prigshim [14] laid the foundation for part A of the following result (see [16]) for limit periodic continued fractions $K(a_n/1)$ with $k = 1$:

THEOREM 3.3. (Thron and Waadeland). *If $\tilde{S}_1(w) = \tilde{a}_1/(1 + w)$ is parabolic* (i.e. $\tilde{a}_1 = -1/4$), *then*

A. $\quad |a_n + \frac{1}{4}| \leq \dfrac{1}{4(4n^2 - 1)}, \ (n \in \mathbb{N}) \Rightarrow K(a_n/1)$ *converges to* f.

$$\left.\begin{array}{ll}\text{B.} & |a_n+\tfrac{1}{4}| \leqq \dfrac{C}{n^\alpha},\ (n\in\mathbf{N}) \\ & \\ & \textit{where } C>0,\ \alpha>2,\end{array}\right\} \Rightarrow \frac{f-S_n(-1/2)}{f-S_n(0)} \to 0 \textit{ if } f \neq \infty.$$

A similar result is also valid for $K(a_n/b_n)$ and $k > 1$, (see [5], [7]), although the sufficient upper bounds for $|a_n - \tilde{a}_n|$ and $b_n - \tilde{b}_n|$ are more complicated and more restrictive. Some improvement can be obtained for this situation by the main result in section 4 of this paper.

Limit periodicity is a very strong type of nearness between two continued fractions. Since so many useful continued fraction expansions of known functions are limit periodic, this is the type of nearness which mostly has been studied. However, also weaker forms of nearness can induce inheritance of properties. Perron, [12, Satz 2.40] proved that $K(a_n/b_n)$ converges if all pairs (a_n, b_n) are contained in a certain neighborhood of $(\tilde{a}_1, \tilde{b}_1)$, where $\tilde{a}_1$, $\tilde{b}_1$ are as in Theorem 3.1. (This result is vastly improved for continued fractions $K(a_n/1)$ by the uniform parabola theorem, (see [11, Theorem 4.40]).) This result is also generalized to $k > 1$, (see [7]).

Again this is a case of inheritance from $K(\tilde{a}_n/\tilde{b}_n)$ to $K(a_n/b_n)$ of ordinary convergence, modified convergence with modifying factors $w_n \in V_n$, and the location of the values of the tails $f^{(n)}$.

4. Nearness and modified convergence.

As in the previous section, we shall regard $K(\tilde{a}_n/\tilde{b}_n)$ as a given continued fraction, and find sufficient conditions on $\{(a_n, b_n)\}_{n=1}^{\infty}$ for $K(a_n/b_n)$ to inherit certain properties from $K(\tilde{a}_n/\tilde{b}_n)$. The properties we are concerned with here, are convergence of modified approximants and their limit value. Let $\{\tilde{f}^{(n)}\}$ be a tail sequence of $K(\tilde{a}_n/\tilde{b}_n)$. (If $K(\tilde{a}_n/\tilde{b}_n)$ converges, then the values of its tails represent one among several choices.) Then $\tilde{S}_n(\tilde{f}^{(n)} = \tilde{f}^{(0)}$ for all n. This means in particular that the modified approximants $\tilde{S}_n(\tilde{f}^{(n)})$ converge to $\tilde{f}^{(0)}$. When will $\{S_n(\tilde{f}^{(n)})\}$ converge? Or when will $\{S_n(w_n)\}$, with w_n "close to" $\tilde{f}^{(n)}$, converge? The first theorem in this section gives an answer to these questions. It is derived by the following method:

1). Assume that $\tilde{f}^{(n)} \neq \infty$ for all n. Then $\tilde{f}^{(n)} \neq 0, -\tilde{b}_n$ for all n, by (2.8), and we can always find a sequence $\{t_n\}$ of positive numbers such that for a given $D > 0$

$$D_n = t_n|\tilde{b}_n + \tilde{f}^{(n)}| - t_{n-1}|\tilde{f}^{(n-1)}| \geqq D, \quad (n \geqq 1). \tag{4.1}$$

The constant D and the sequence $\{t_n\}$ have proved to be convenient tools for carrying through the next step:

2). Match expressions for r_n, q_n and R_n to obtain that $\{E_n \times G_n\}_{n=1}^{\infty}$ given by

$$(4.2)\qquad E_n = \{z \in \mathbf{C}; |z - \tilde{a}_n| \leqq r_n\}, \quad G_n = \{z \in \mathbf{C}; |z - \tilde{b}_n| \leqq q_n\},$$

is a sequence of element regions corresponding to the sequence $\{V_n\}_{n=0}^{\infty}$ of pre-value regions, V_n given by

$$(4.3)\qquad V_n = \{z \in \mathbf{C}; |z - \tilde{f}^{(n)}| \leqq R_n\}, \quad (n \geqq 0).$$

Thereby we gain control over the modified approximants $S_n(w_n)$, where $w_n \in V_n$ and $(a_n, b_n) \in E_n \times G_n$ for all n.

3). Find conditions for $\{S_n(V_n)\}_{n=1}^{\infty}$ to shrink to a point if $(a_n, b_n) \in E_n \times G_n$ for all n.

With the notation as introduced, we get:

THEOREM 4.1. *Given $K(\tilde{a}_n/\tilde{b}_n)$ with a tail sequence $\{\tilde{f}^{(n)}\}_{n=0}^{\infty}, \tilde{f}^{(n)} \neq \infty$. Let*

$$(4.4)\qquad R_n = \frac{D-\mu}{2t_n}, \quad r_n = R_n \frac{c_n}{t_{n-1}}, \quad q_n = R_n \frac{2D_n - (D-\mu) - 2c_n}{2t_{n-1}|\tilde{f}^{(n-1)}| + D - \mu},$$

where $0 \leqq c_n \leqq D_n - (D-\mu)/2$ for all n and $-D \leqq \mu \leqq D$. Then the following hold:

A. *$\{V_n\}$ is a sequence of pre-value regions corresponding to the element regions $\{E_n \times G_n\}$.*

B. *If, in addition, $\prod_{n=1}^{\infty} Q_n$ diverges to 0, where*

$$(4.5)\qquad Q_n = \frac{(t_{n-1}|\tilde{f}^{(n-1)}| + (D-\mu)/2)^2}{t_n|\tilde{b}_n + \tilde{f}^{(n)}|t_{n-1}|\tilde{f}^{(n-1)}| + c_n(D-\mu)/2}, \quad (n \in \mathbf{N}),$$

then $\{E_n \times G_n\}$ is a uniform sequence of modified convergence regions with respect to $\{V_n\}$. Indeed, if $(a_n, b_n) \in E_n \times G_n$ and $w_n \in V_n$ for all $n \in \mathbf{N}$, then

$$(4.6)\qquad |\lim_{m\to\infty} S_m(w_m) - S_n(w_n)| \leqq \frac{D-\mu}{t_0} \prod_{j=1}^{n} Q_j, \quad (n \in \mathbf{N}).$$

C. *If, in particular,*

$$(4.7)\qquad \delta = \liminf_{n\to\infty} t_{n-1}|\bar{f}^{(n-1)}| > 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} (t_n|\bar{b}_n + \bar{f}^{(n)}|)^{-1} = \infty,$$

then $\prod Q_n = 0$ *if* $(D-\mu)/2 < \sqrt{D\delta+\delta^2}-\delta$.

PROOF. A. Let $(a_n, b_n) \in E_n \times G_n$ and $w_n \in V_n$ be chosen arbitrarily for arbitrary $n \in \mathbb{N}$. Then

$$\left|\frac{a_n}{b_n+w_n} - \bar{f}^{(n-1)}\right| = \left|\frac{(a_n-\tilde{a}_n) - \bar{f}^{(n-1)}(b_n-\bar{b}_n) - \bar{f}^{(n-1)}(w_n-\bar{f}^{(n)})}{(\bar{b}_n+\bar{f}^{(n)}) + (b_n-\bar{b}_n) + (w_n-\bar{f}^{(n)})}\right|$$

$$\leqq \frac{r_n + |\bar{f}^{(n-1)}|(q_n+R_n)}{|\bar{b}_n+\bar{f}^{(n)}| - (q_n+R_n)} = R_{n-1}.$$

B. Let $(a_n, b_n) \in E_n \times G_n$ and $w_n \in V_n$ for all $n \in \mathbb{N}$. Then for $n, m \in \mathbb{N}$,

$$(4.8)\qquad \left|S_n(w_n) - S_{n+m}(w_{n+m})\right| = \left|\frac{a_1}{b_1+S_{n-1}^{(1)}(w_n)} - \frac{a_1}{b_1+S_{n+m-1}^{(1)}(w_{n+m})}\right|$$

$$= \frac{|a_1||S_{n-1}^{(1)}(w_n) - S_{n+m-1}^{(1)}(w_{n+m})|}{|b_1+S_{n-1}^{(1)}(w_n)||b_1+S_{n+m-1}^{(1)}(w_{n+m})|}$$

$$= \left|\frac{S_{n+m}(w_{n+m})}{b_1+S_{n-1}^{(1)}(w_n)}\right| |S_{n-1}^{(1)}(w_n) - S_{n+m-1}^{(1)}(w_{n+m})|$$

$$= |S_0^{(n)}(w_n) - S_m^{(n)}(w_{n+m})| \prod_{j=1}^{n} \left|\frac{S_{n+m-j+1}^{(j-1)}(w_{n+m})}{b_j+S_{n-j}^{(j)}(w_n)}\right|$$

$$\leqq 2R_n \prod_{j=1}^{n} \frac{|\bar{f}^{(j-1)}| + R_{j-1}}{|\bar{b}_j+\bar{f}^{(j)}| - R_j - q_j}$$

$$= \frac{D-\mu}{t_n} \prod_{j=1}^{n} \frac{t_j}{t_{j-1}} \frac{(t_{j-1}|\bar{f}^{(j-1)}| + D-\mu)/2)^2}{t_j|\bar{b}_j+\bar{f}^{(j)}|t_{j-1}|\bar{f}^{(j-1)}| + c_j(D-\mu)/2}$$

$$= \frac{D-\mu}{t_0} \prod_{j=1}^{n} Q_j \to 0 \quad \text{as } n \to \infty.$$

Hence, $\{S_n(w_n)\}$ is a Cauchy sequence on the compact set V_0, and thereby converges to a value $F \in V_0$. Furthermore,

(4.9)
$$|F-S_n(w_n)| \leqq \frac{D-\mu}{t_0}\prod_{j=1}^{n} Q_j.$$

C. Assume that (4.7) holds and that $(D-\mu)/2 < \sqrt{D\delta+\delta^2}-\delta$. (We always have $(D-\mu)/2 \leqq D$. Since $\sqrt{D\delta+\delta^2}-\delta < D/2$, this condition represents a restriction.) Then, with $\varepsilon = \sqrt{D\delta+\delta^2}-\delta-(D-\mu)/2 > 0$, we get

$$Q_n \leqq \frac{(t_{n-1}|\mathfrak{f}^{(n-1)}|+\sqrt{D\delta+\delta^2}-\delta-\varepsilon)^2}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|t_{n-1}|\mathfrak{f}^{(n-1)}|}$$

$$= 1-\frac{D_n-2\sqrt{D\delta+\delta^2}+2\delta+2\varepsilon}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|}+$$

$$+\frac{D\delta+\delta^2+\delta^2+2\delta\varepsilon+\varepsilon^2-2\sqrt{D\delta+\delta^2}\,\delta-2\sqrt{D\delta+\delta^2}\,\varepsilon}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|t_{n-1}|\mathfrak{f}^{(n-1)}|}$$

$$\leqq 1-\frac{D-2\sqrt{D\delta+\delta^2}+2\delta+2\varepsilon-(D+2\delta-2\sqrt{D\delta+\delta^2})}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|}+$$

$$+\varepsilon\frac{2\delta+\varepsilon-2\sqrt{D\delta+\delta^2}}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|t_{n-1}|\mathfrak{f}^{(n-1)}|}.$$

Since $2\delta+\varepsilon-2\sqrt{D\delta+\delta^2} < 2\delta+\sqrt{D\delta+\delta^2}-\delta-2\sqrt{D\delta+\delta^2} < 0$, we therefore get

(4.10)
$$Q_n \leqq 1-\frac{2\varepsilon}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|},$$

which proves part C.

REMARKS 4.2.(i) If we consider continued fractions of the form $K(a_n/1)$, we can chose $c_n = D_n-(D-\mu)/2$ which gives

(4.11)
$$r_n = \frac{(D-\mu)(D_n-\frac{1}{2}(D-\mu))}{2t_nt_{n-1}},\quad q_n = 0,\quad R_n = \frac{D-\mu}{2t_n}$$

and

(4.12)
$$Q_n = \frac{2t_{n-1}|\mathfrak{f}^{(n-1)}|+(D-\mu)}{2t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|-(D-\mu)} \leqq \frac{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|+t_{n-1}|\mathfrak{f}^{(n-1)}|-\mu}{t_n|\tilde{b}_n+\mathfrak{f}^{(n)}|+t_{n-1}|\mathfrak{f}^{(n-1)}|+\mu}.$$

(ii) If we consider continued fractions of the form $(K(1/b_n)$, we can choose

$c_n = 0$ which gives

$$(4.13)\qquad r_n = 0,\quad q_n = \frac{D-\mu}{2t_n}\cdot\frac{2D_n-(D-\mu)}{2t_{n-1}|\mathcal{J}^{(n-1)}|+(D-\mu)},\quad R_n = \frac{D-\mu}{2t_n}.$$

(iii) We can, without difficulty, let the parameter μ vary with n, $\{\mu_n\}$, as long as $\{\mu_n\}$ is non-decreasing. With

$$(4.4')\qquad r_n = r_n(\mu_n),\quad q_n = q_n(\mu_n),\quad R_n = R_n(\mu_n)\quad \text{for all } n$$

we get

$$\frac{r_n(\mu_n)+|\mathcal{J}^{(n-1)}|(q_n(\mu_n)+R_n(\mu_n))}{|\mathcal{b}_n+\mathcal{J}^{(n)}|-(q_n(\mu_n)+R_n(\mu_n))} = R_{n-1}(\mu_n) \leqq R_{n-1}(\mu_{n-1}),$$

which proves A. Furthermore,

$$(4.8')\qquad |S_n(w_n)-S_{n+m}(w_{n+m})| \leqq 2R_n(\mu_n)\prod_{j=1}^{n}\frac{|\mathcal{J}^{(j-1)}|+R_{j-1}(\mu_{j-1})}{|\mathcal{b}_j+\mathcal{J}^{(j)}|-(R_j(\mu_j)-q_j(\mu_j))},$$

which proves B with

$$(4.5')\qquad Q_n = \frac{(t_{n-1}|\mathcal{J}^{(n-1)}|+\frac{1}{2}(D-\mu_{n-1}))(t_{n-1}|\mathcal{J}^{(n-1)}|+\frac{1}{2}(D-\mu_n))}{t_n|\mathcal{b}_n+\mathcal{J}^{(n)}|t_{n-1}|\mathcal{J}^{(n-1)}|+\frac{1}{2}(D-\mu_n)c_n}.$$

In fact, $(D-\mu_n)\prod_{j=1}^{n}Q_j \to 0$ is now sufficient in part B, since the right hand side of (4.8) becomes $((D\text{-}\mu_n)/t_0)\prod_{j=1}^{n}Q_j$. Similarly D can vary with n too, as long as $(D_n-\mu_n)$ is non-increasing.

(iv) The partial derivative $\partial Q_n/\partial(D-\mu) > 0$. This means that

$$Q_n \geqq t_{n-1}|\mathcal{J}^{(n-1)}|/t_n|\mathcal{b}_n+\mathcal{J}^{(n)}| \quad \text{for all } n.$$

Hence,

$$\prod_{n=1}^{\infty}(t_{n-1}|\mathcal{J}^{(n-1)}|/t_n|\mathcal{b}_n+\mathcal{J}^{(n)}|) = 0$$

is necessary for $\prod_{n=1}^{\infty} Q_n = 0$, where

$$\prod_{n=1}^{\infty}\frac{t_{n-1}|\mathcal{J}^{(n-1)}|}{t_n|\mathcal{b}_n+\mathcal{J}^{(n)}|} = \lim_{n\to\infty}\frac{t_0}{t_n}\frac{|\tilde{A}_{n-1}-\tilde{B}_{n-1}\mathcal{J}^{(0)}|}{|\tilde{B}_n+\tilde{B}_{n-1}\mathcal{J}^{(n)}|} = \lim_{n\to\infty}\frac{t_0}{t_n}\frac{|\mathcal{J}_{n-1}-\mathcal{J}^{(0)}|}{|\mathcal{h}_n+\mathcal{J}^{(n)}|}.$$

If $K(\tilde{a}_n/\tilde{b}_n)$ converges to a finite value, and $\{\tilde{f}^{(n)}\}$ is a bounded sequence of wrong tails, then $(\tilde{h}_n + \tilde{f}^{(n)}) \to 0$, and $\prod_{n=1}^{\infty} Q_n = 0$ only if $t_n \to \infty$ fast enough. That is, only if $r_n \to 0, q_n \to 0$ and $R_n \to 0$ fast enough. This is quite in line with [8, Proposition 3.3].

(v) If $K(a_n/b_n)$ converges, $\prod_{n=1}^{\infty} Q_n = 0$ and $\{t_n\}$ does not converge to ∞, then $K(a_n/b_n)$ converges to $\lim S_n(\tilde{f}^{(n)})$, (see [10]).

Application of Theorem 4.1 can give us both *uniform* convergence and truncation error bounds for modified approximants, if $\prod Q_n = 0$. The neighborhoods $E_n \times G_n$ of $(\tilde{a}_n, \tilde{b}_n)$ are often quite small. By use of Stieltjes-Vitali's theorem, [see [11, Theorem 4.30]) we are sometimes able to enlarge these neighborhoods considerably, at the cost of these advantages. See for instance [9].

5. Nearness and ordinary convergence.

Theorem 4.1 gives sufficient conditions for modified convergence. In many cases we then do not need ordinary convergence at all. In particular this is so if the radii R_n of V_n have no limit point at 0. Then $K(a_n/b_n)$ converges *generally* to $F = \lim S_n(\tilde{f}^{(n)})$ if $(a_n, b_n) \in E_n \times G_n$ for all n. (See [10].) This implies the following.

(i) If $K(a_n/b_n)$ converges, then it converges to F.

(ii) $\lim S_n(w_n) = F$ for all sequences $\{w_n\}$ such that $\liminf d(w_n, -h_n) > 0$, where $d(\cdot,\cdot)$ is the chordal metric on the Riemann sphere. (See Remark 6.2 (ii), $F \neq \infty$.)

However, if one wants to prove ordinary convergence, then the following two observations are useful:

(iii) If $0 \in V_n$ for all n, then $K(a_n/b_n)$ converges to the same value F.

(iv) If $\liminf \operatorname{diam}(V_n) > 0$ and $\{h_n\}$ has no limit point at 0, then $K(a_n/b_n)$ converges to the same value F, by (ii).

Clearly,

$$\liminf \operatorname{diam}(V_n) > 0 \text{ if } \limsup t_n < \infty. \quad (\mu \text{ constant.})$$

The following result is a step on the way to find sufficent conditions for $\{t_n\}$ to be bounded:

PROPOSITION 5.1 *Given the sequences $\{X_n\}$ and $\{Y_n\}$ of positive numbers. Then* *there exists a bounded sequence* $\{t_n\}_{n=0}^{\infty}$ *of positive numbers such that*

(5.1) $$t_n X_n - t_{n-1} Y_n \geqq 1, \quad (n \in \mathbb{N}),$$

if and only if

$$\left\{ \frac{1}{X_n} \sum_{m=1}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}} \right\}_{n=2}^{\infty}$$

is bounded.

Proof. Assume that $\{t_n\}$ is bounded and satisfies (5.1). Then

$$t_n \geqq \frac{1+t_{n-1}Y_n}{X_n} \geqq \frac{1}{X_n} + \frac{Y_n}{X_n}\left(\frac{1}{X_{n-1}} + t_{n-2}\frac{Y_{n-1}}{X_{n-1}} \right)$$
$$\geqq \frac{1}{X_n} \sum_{m=2}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}} + t_1 \prod_{j=2}^{n} \frac{Y_j}{X_j} > \frac{1}{X_n} \sum_{m=1}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}},$$

since $t_1 X_1 > 1$. This implies that

$$\limsup \frac{1}{X_n} \sum_{m=1}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}} < \infty.$$

Assume next that

$$\limsup \frac{1}{X_n} \sum_{m=1}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}} < \infty.$$

Then we can choose $\{t_n\}$ such that we have equality in (5.1). We get

$$t_n = \frac{1}{X_n} \sum_{m=2}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}} + t_1 \prod_{j=2}^{n} \frac{Y_j}{X_j} < \frac{t_1 X_1}{X_n} \sum_{m=1}^{n} \prod_{j=m+1}^{n} \frac{Y_j}{X_{j-1}},$$

which proves that $\{t_n\}$ is bounded.

We immediately get:

Corollary 5.2 *With the notation of Theorem* 4.1, *we can find a bounded sequence* $\{t_n\}$ *satisfying* (4.1) *if and only if*

$$\limsup_{n\to\infty} \frac{1}{|\mathfrak{b}_n + \mathfrak{f}^{(n)}|} \sum_{m=1}^{n} \prod_{j=m}^{n-1} \frac{|\mathfrak{f}^{(j)}|}{|\mathfrak{b}_j + \mathfrak{f}^{(j)}|} < \infty.$$

REMARK 5.3. If

$$\limsup_{n\to\infty} \prod_{j=m}^{n} (|\tilde{f}^{(j-1)}|/|\tilde{b}_j+\tilde{f}^{(j)}|) = \infty$$

for an $m \in \mathsf{N}$, then $\{t_n\}$ can never be bounded.

The other question is: when is liminf $|h_n| > 0$. One answer to this problem can be given in terms of *regions W_n for right or wrong tails corresponding to a sequence $\{\Omega_n\}$ of element regions*. That is, a sequence $\{W_n\}_{n=0}^{\infty}$ of non-empty sets from $\hat{\mathsf{C}}$ such that

$$(a_n, b_n) \in \Omega_n,\ w_{n-1} \in W_{n-1} \Rightarrow w_n = -b_n + a_n/w_{n-1} \in W_n \tag{5.2}$$

for all n. Clearly, such a sequence has the property that if $\{f^{(n)}\}$ is a tail sequence of a continued fraction $K(a_n/b_n)$ with $(a_n, b_n) \in \Omega_n$ for all n, then

$$f^{(N)} \in W_N \Rightarrow f^{(n)} \in W_n \text{ for all } n \geqq N, \quad (N \in \mathsf{N} \cup \{0\}). \tag{5.3}$$

THEOREM 5.4. *Let $\{\tilde{g}^{(n)}\}$ be a sequence of finite wrong tails for a continued fraction $K(\tilde{a}_n/\tilde{b}_n)$. Let $\{\tilde{t}_n\}$ be a sequence of positive numbers such that for a $\tilde{D} > 0$*

$$\tilde{D}_n = \tilde{t}_{n-1}|\tilde{g}^{(n-1)}| - \tilde{t}_n|\tilde{b}_n + \tilde{g}^{(n)}| \geqq \tilde{D}, \quad (n \in \mathsf{N}). \tag{5.4}$$

Further, let $\{\tilde{E}_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$ and $\{W_n\}_{n=0}^{\infty}$ be given by

$$W_n = \{z \in \mathsf{C}; |z - \tilde{g}^{(n)}| \leqq \tilde{R}_n\}, \quad \tilde{R}_n = \frac{\tilde{D} - \tilde{\mu}}{2\tilde{t}_n}, \tag{5.5}$$

$$\tilde{E}_n = \{z \in \mathsf{C}; |z - \tilde{a}_n| \leqq \tilde{r}_n\}, \quad \tilde{r}_n = \tilde{R}_n \frac{\tilde{c}_n}{\tilde{t}_{n-1}}, \tag{5.6}$$

and

$$\tilde{G}_n = \{z \in \mathsf{C}; |z - \tilde{b}_n| \leqq \tilde{q}_n\}, \quad \tilde{q}_n = \tilde{R}_n \cdot \frac{2\tilde{D}_n - (\tilde{D} - \tilde{\mu}) - 2\tilde{c}_n}{2\tilde{t}_{n-1}|\tilde{g}^{(n-1)}| - \tilde{D} + \tilde{\mu}}, \tag{5.7}$$

where

$$-\tilde{D} \leqq \tilde{\mu} \leqq \tilde{D}, \quad 0 \leqq \tilde{c}_n \leqq \tilde{D}_n - (\tilde{D} - \tilde{\mu})/2, \quad (n \in \mathsf{N}). \tag{5.8}$$

Then $\{W_n\}$ is a sequence of regions for right or wrong tails corresponding to $\{\tilde{E}_n \times \tilde{G}_n\}$.

PROOF. The proof is a straightforward verification of (5.2).

REMARKS 5.5. (i) A sequence $\{\tilde{t}_n\}$ satisfying (5.4) can not always be found. Indeed, it exists if and only if

$$\sum_{n=0}^{\infty} \prod_{j=1}^{n} \frac{|\tilde{b}_j + \tilde{g}^{(j)}|}{|\tilde{g}^{(j)}|} < \infty. \tag{5.9}$$

This follows since by (5.4)

$$\begin{aligned}\tilde{t}_n &\leqq \frac{t_{n-1}|\tilde{g}^{(n-1)}| - \tilde{D}}{|\tilde{b}_n + \tilde{g}^{(n)}|} \\ &\leqq \left(\tilde{t}_0 - \frac{\tilde{D}}{|\tilde{g}^{(0)}|} \sum_{j=0}^{n-1} \prod_{m=1}^{j} \frac{|\tilde{b}_m + \tilde{g}^{(m)}|}{|\tilde{g}^{(m)}|}\right) \prod_{m=1}^{n} \frac{|\tilde{g}^{(m-1)}|}{|\tilde{b}_m + \tilde{g}^{(m)}|}.\end{aligned}$$

The if part follows similarly.

From [17] follows that no sequence of right tails can satisfy (5.9).

(ii) Theorem 5.4 also holds if $\tilde{g}^{(0)} = \infty$. $\{-\tilde{h}_n\}$ is the tail sequence of $K(\tilde{a}_n/\tilde{b}_n)$ with $-\tilde{h}_0 = \infty$. Theorem 5.4 confirms that $-h_n \in W_n$ for all n if $K(\tilde{a}_n/\tilde{b}_n)$ satisfies the hypotheses with $\tilde{g}^{(n)} = -\tilde{h}_n$, and $(a_n, b_n) \in \tilde{E}_n \times \tilde{G}_n$ for all n, since $\tilde{q}_n < \tilde{R}_n$ and $h_1 = -b_1$, $\tilde{h}_1 = \tilde{b}_1$.

In view of Remark 5.5(ii), we get the following corollary to Theorem 4.1 and Theorem 5.4 (with the notation from these theorems):

COROLLARY 5.6. *Let $K(\tilde{a}_n/\tilde{b}_n)$ be a convergent continued fraction. Let $\{\tilde{f}^{(n)}\}$ denote its sequence of right tails. If*

(5.10) *either* $\liminf_{n\to\infty} t_{n-1}|\tilde{f}^{(n-1)}| > 0$ *and* $\prod_{n=1}^{\infty} (t_n|\tilde{b}_n + \tilde{f}^{(n)}|)^{-1} = \infty$,
or $\prod_{n=1}^{\infty} Q_n = 0$,

(5.11) $\{\tilde{f}^{(n)}\}$ *and* $\{t_n\}$ *are bounded,* $\mu > 0$ *and*

(5.12) *there exists a bounded sequence $\{\tilde{g}^{(n)}\}$ of wrong tails of $K(\tilde{a}_n/\tilde{b}_n)$ and a sequence $\{\tilde{t}_n\}$ bounded away from 0 satisfying* (5.4),

then $\{(E_n \cap \tilde{E}_n) \times (G_n \cap \tilde{G}_n)\}$ is a sequence of convergence regions, and $\{V_n\}$ is a corresponding sequence of limit regions.

PROOF. $\{E_n \times G_n\}$ is a sequence of modified convergence regions with respect to $\{V_n\}$. (Theorem 4.1.)

Let $K(a_n/b_n)$ and $\{w_n\}_{n=1}^{\infty}$ be arbitrarily chosen such that

$$(a_n, b_n) \in (E_n \cap \tilde{E}_n) \times (G_n \cap \tilde{G}_n), \quad w_n \in V_n, \ (n \in \mathbf{N}). \tag{5.13}$$

Assume that $\{\tilde{t}_n\}$ is bounded. (This is no restriction.) Then for some $\tilde{R} > 0$, $\operatorname{diam}(W_n) \geqq \tilde{R}$, $n \in \mathbf{N}$. Since, by [2], $(\tilde{h}_n + \tilde{g}^{(n)}) \to 0$, there exists an $N \in \mathbf{N}$ such that $-\tilde{h}_n \in W_n$ for $n \geqq N$. Define $K(a_n^*/b_n^*)$ by

$$a_n^* = \begin{cases} \tilde{a}_n & \text{for } n \leqq N, \\ a_n & \text{for } n > N, \end{cases} \quad b_n^* = \begin{cases} \tilde{b}_n & \text{for } n \leqq N, \\ b_n & \text{for } n > N. \end{cases}$$

Then $-h_N^* = -\tilde{h}_N \in W_N$ and hence $-h_n^* \in W_n$ for $n \geqq N$. Since $\{W_n\}$ is bounded, $(K(a_n^*/b_n^*)$ converges to the value $F^* = \lim S_n^*(w_n)$. Hence its Nth tail converges to

$$S_N^{*-1}(F^*) = F^{(N)*} = F^{(N)} = S_N^{-1}(F),$$

where $F = \lim S_n(w_n)$. Hence, $K(a_n/b_n)$ converges to $F \in V_0$, which proves the theorem.

REMARK 5.7. Corollary 5.6 generalizes [6, Theorem 2.2A]. $q_n = \tilde{q}_n = 0$ for all n and $K(\tilde{a}_n/\tilde{b}) = K(\tilde{a}_n/1)$ k-periodic in Corollary 5.6 gives [6, Theorem 2.2A], if $\tilde{r}_n$ and r_n are replaced by the (not larger) radii

$$r_n = \frac{D^2 - \mu^2}{4t_n t_{n-1}}, \quad \tilde{r}_n = \frac{\tilde{D}^2 - \tilde{\mu}^2}{4\tilde{t}_n \tilde{t}_{n-1}}.$$

6. Adjacent continued fractions.

We shall first define what we mean by this new concept:

DEFINITION 6.1. Two continued fractions $K(a_n/b_n)$ and $K(\tilde{a}_n/\tilde{b}_n)$ are called *adjacent* (or $K(a_n/b_n)$ is adjacent to $K(\tilde{a}_n/\tilde{b}_n)$) if

$$d(a_n, \tilde{a}_n) \to 0, \quad d(b_n, \tilde{b}_n) \to 0, \tag{6.1}$$

where $d(\cdot, \cdot)$ is the chordal metric on the Riemann sphere.

REMARKS 6.2. (i) The name adjacent was suggested by Prof. H. Waadeland.

(ii) The chordal metric is defined by

$$d(a,b) = \begin{cases} 2|a-b|/\sqrt{(1+|a|^2)(1+|b|^2)} & \text{if } a,b \neq \infty, \\ 2/\sqrt{1+|a|^2} & \text{if } b = \infty,\ a \neq \infty, \\ 0 & \text{if } a = b = \infty, \end{cases}$$

for $a, b \in \hat{\mathbf{C}}$, (see [1, p. 20]).

(iii) This new concept generalizes the relationship between periodic and limit periodic continued fractions.

If μ vary with n (Remark 4.2 (iii)) such that $\mu_n \to D$, then results in the previous sections are on adjacent continued fractions. We get for instance the following corollary to Theorem 4.1:

COROLLARY 6.3. *Given the continued fraction* $K(\tilde{a}_n/1)$ *with a bounded tail sequence* $\{\tilde{f}^{(n)}\}$. *Let* $K(a_n/1)$ *be adjacent to* $K(\tilde{a}_n/1)$. *Assume there exists a bounded sequence* $\{t_n\}$ *satisfying* (4.1). *Then the following holds*:

A. $\{S_n(w_n)\}$ *converges to value* $F \in \hat{\mathbf{C}}$ *for every sequence* $\{w_n\}$ *of modifying factors such that* $w_n - \tilde{f}^{(n)} \to 0$.
B. $F^{(n)} - \tilde{f}^{(n)} \to 0$, *where* $F^{(n)} = S_n^{-1}(F)$.
C. *Let* $d_n = \sup\{t_m t_{m-1}|a_m - \tilde{a}_m|\,;\, m \geqq n\}$ *for all* $n \in \mathbf{N}$. *If* $d_1 < D^2/4$, *then*

$$|F^{(n)} - \tilde{f}^{(n)}| \leqq 2\frac{d_n}{Dt_n}, \tag{6.2}$$

and

$$|F - S_n(\tilde{f}^{(n)})| \leqq 2\frac{d_n}{Dt_0}\prod_{j=1}^{n}\frac{t_j|1+\tilde{f}^{(j)}| + t_{j-1}|\tilde{f}^{(j-1)}| - \sqrt{D^2-4d_{j-1}}}{t_j|1+\tilde{f}^{(j)}| + t_{j-1}|\tilde{f}^{(j-1)}| + \sqrt{D^2-4d_j}}. \tag{6.3}$$

D. *If furthermore there exist a bounded tail sequence* $\{\tilde{g}^{(n)}\}$ *of* $K(\tilde{a}_n/1)$ *and a sequence* $\{\tilde{t}_n\}$ *satisfying* (5.4) *with* $\liminf \tilde{t}_n > 0$, *then* $K(a_n/1)$ *converges to* F.

PROOF. A. From Theorem 4.1B and Remark 4.2(i) it follows that $\{E_n\}$ is a sequence of modified convergence regions with respect to $\{V_n\}$ for any $\mu \in (0, D)$. Since $a_n - \tilde{a}_n \to 0$ and $w_n - \tilde{f}^{(n)} \to 0$, and since $\{r_n\}$ and $\{R_n\}$ are bounded away from 0, there exists an $N \in \mathbf{N}$ such that $a_n \in E_n$ and $w_n \in V_n$ for all $n \geqq N$. This proves A.

B. As in Remark 4.2(iii) we can let μ vary with n. Indeed, since $a_n - \tilde{a}_n \to 0$ we can let $\mu_n \to D$. That is $|F^{(n)} - \tilde{f}^{(n)}| \leqq R_n(\mu_n) \to 0$.

Choose $\mu_n = \sqrt{D^2 - 4d_n}$ for all sufficiently large n ($d_n < D^2/4$). Then $\mu_n > 0$ and $\mu_n \to D$ monotonely. Furthermore,

$$|a_n - \tilde{a}_n| \leqq \frac{d_n}{t_n t_{n-1}} = \frac{D^2 - \mu_n^2}{4t_n t_{n-1}} \leqq \frac{D-\mu_n}{2t_n t_{n-1}} c_n$$

with $c_n = D_n - (D - \mu_n)/2$. Hence, by Theorem 4.1 B and Remark 4.2(i) and (iii), B follows.

C. Similarly, we get

$$|F^{(n)} - \mathcal{f}^{(n)}| \leqq R_n(\mu_n) = \frac{D-\mu_n}{2t_n} = \frac{D - \sqrt{D^2 - 4d_n}}{2t_n} \leqq \frac{2d_n}{t_n D}.$$

and

$$|F - S_n(\mathcal{f}^{(n)})| \leqq \frac{D-\mu_n}{2t_0} \prod_{j=1}^{n} \frac{t_{j-1}|\mathcal{f}^{(j-1)}| + (D-\mu_{j-1})/2}{t_j|\tilde{b}_j + \mathcal{f}^{(j)}| - (D-\mu_j)/2}$$

$$\leqq \frac{2d_n}{Dt_0} \prod_{j=1}^{n} \frac{t_j|\tilde{b}_j + \mathcal{f}^{(j)}| + t_{j-1}|\mathcal{f}^{(j-1)}| - \sqrt{D^2 - 4d_{j-1}}}{t_j|\tilde{b}_j + \mathcal{f}^{(j)}| + t_{j-1}|\mathcal{f}^{(j-1)}| + \sqrt{D^2 - 4d_j}}.$$

D. Follows by an argument similar to the proof of part A.

Corollary 6.3 connects with [4, Theorem 4.1] on convergence acceleration. Under the conditions of Corollary 6.3D, both $K(\tilde{a}_n/1)$ and $K(a_n/1)$ converge. If these conditions still holds with $\tilde{g}^{(n)} = -\tilde{h}_n$ for all n ($\tilde{h}_0 = \infty$ does no harm), then it follows from [4, Theorem 4.1] that

$$\left|\frac{F - S_n(\mathcal{f}^{(n)})}{F - S_n(0)}\right| \leqq \left(1 + \frac{|\mathcal{f}^{(n)}|}{\delta_n(\mathcal{f}^{(n)})}\right)\left(2 + 4\frac{|\mathcal{f}^{(n)}|}{t_{n+1}D}\right)\frac{\mathcal{d}_{n+1}}{|\tilde{a}_{n+1}|}, \tag{6.4}$$

where $\mathcal{d}_n = \sup\{|a_m - \tilde{a}_m|;\, m \geqq n\}$ and $\delta_n(\mathcal{f}^{(n)}) = |\tilde{h}_n + \mathcal{f}^{(n)}| - \tilde{R}_n(\tilde{\mu}_n) > 0$, ($\tilde{g}^{(n)} = -\tilde{h}_n$), by an additional argument, (see [1, Result 12]). Clearly (6.3) also indicates a convergence acceleration of $K(a_n/1)$ by using the modifying factors $\mathcal{f}^{(n)}$ compared to using a sequence $\{w_n\}$, for which $w_n \in V_n(\mu_1)$ and $\liminf|\mathcal{f}^{(n)} - w_n| > 0$.

A result similar to (6.4), but without requiring that $\{t_n\}$ and $\{|\mathcal{f}^{(n)}|\}$ are bounded, was proved in [5]. Results of this type can also be obtained from Theorem 4.1.

REFERENCES

1. L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill Book Company, New York - London, 1979.
2. M. G. De Bruin and L. Jacobsen, *The dominance concept for linear recurrence relations with applications to continued fractions*, Nieuw Arch. Wisk. (4) 3 (1985), 253–266.
3. J. Gill, *The use of attractive fixed points in accelerating the convergence of limit periodic continued fractions*, Proc. Amer. Math. Soc. 47 (1975), 119–126.
4. L. Jacobsen, *Convergence acceleration for continued fractions* $K(a_n/1)$, Trans. Amer. Math. Soc. 275 (1983), 265–285.
5. L. Jacobsen, *Further results on convergence acceleration for continued fractions* $K(a_n/1)$, Trans. Amer. Math. Soc. 281 (1984), 129–146.
6. L. Jacobsen, *Some periodic sequences of circular convergence regions*, in *Analytic Theory of Continued Fractions* (Proc., Loen, Norway, 1981), eds. W. B. Jones, W. J. Thron, H. Waadeland (Lecture Notes in Math. 932), pp. 87–98. Springer-Verlag, Berlin - Heidelberg - New York, 1982.
7. L. Jacobsen, *Modified approximants for continued fractions. Construction and applications*, Norske Vid. Selsk. Skr. (Trondheim) 3 (1983), 1–46.
8. L. Jacobsen, *Functions defined by continued fractions. Meromorphic continuation*, Rocky Mountain J. Math. 15 (1985), 685–703.
9. L. Jacobsen, *A theorem on simple convergence regions for continued fractions*, in *Analytic Theory of Continued Fractions* II (Proc., Pitlochry and Aviemore, Scotland 1985), ed. W. J. Thron (Lecture Notes in Math. 1199), pp. 59–66. Springer-Verlag, Berlin - Heidelberg - New York, 1986.
10. L. Jacobsen, *General convergence of continued fractions*, Trans. Amer. Math. Soc. 294 (1986), 477–485.
11. W. B. Jones and W. J. Thron, *Continued fractions. Analytic theory and applications*, Encyclopedia Math. Appl. 11, Addison-Wesley, Reading, Mass., 1980
12. O. Perron, *Die Lehre von den Kettenbrüchen*, Band II, 3. Auflage, Teubner Verlagsgesellschaft, Stuttgart, 1957.
13. M. von Pidoll, *Beiträge zur Lehre von der Konvergenz unendlicher Kettenbrüche*, Diss. München (1912).
14. A. Pringsheim, *Über einige Konvergenzkriterien für Kettenbrüche mit komplexen Gliedern*, Münch. Ber. 35 (1905), 359–380.
15. O. Szasz, *Über die Erhaltung der Kongvergenz unendlicher Kettenbrüche bei independenter Veränderlichkeit aller ihrer Elemente*. J. für Math. 147 (1917), 132–160.
16. W. J. Thron and H. Waadeland, *Accelerating convergence of limit periodic continued fractions* $K(a_n+1)$, Numer. Math. 34 (1980), 155–170.
17. H. Waadeland, *Tales about tails*, Proc. Amer. Math. Soc. 90 (1984), 57–64.

UNIVERSITETET I TRONDHEIM
INSTITUTT FOR MATEMATIKK OG STATISTIKK
7055 DRAGVOLL
NORWAY