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LOWER EQUIVARIANT K-THEORY

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0. Introduction.

The algebraic version of equivariant Whitehead torsion was introduced by Rothenberg in [5] using the universal R-extension of the Burnside category $B(G)$.

A redefinition of the K_{-i}-groups of a ring R was given by Pedersen in [3], using the notion of $\mathbf{Z}^i$-graded categories; if $\mathscr{D}$ is the category of finitely generated free R-modules, then $K_{-i}(R) = K_1(\mathscr{D}_{i+1})$, where $\mathscr{D}_{i+1}$ is the $\mathbf{Z}^{i+1}$-graded category associated to $\mathscr{D}$.

We combine these two approaches and define the equivariant K_{-i}-groups of a discrete group G with respect to a ring R and a subset $\mathscr{F} \subset \mathrm{Conj}(G)$, thus obtaining $K_{-i}(R; G; \mathscr{F})$.

The notion of an R-category is reviewed in Section 3. Essentially it is a category with an R-bimodule structure on the hom-sets which behaves well with respect to composition of morphisms. If $\mathscr{D}$ is an R-category, then $R[T] \otimes_R \mathscr{D}$ is an $R[T]$-category ($R[T]$ is the group ring of the infinite cyclic group T). Let $\mathscr{D}_i$ and $(R[T] \otimes_R \mathscr{D})_i$ denote the corresponding $\mathbf{Z}^i$-graded categories and $K_{-i}(\mathscr{D}) = K_1(\mathscr{D}_{i+1})$.

THEOREM A. *If $\mathscr{D}$ is an R-category, then*

$$K_{-i}(R[T] \otimes_R \mathscr{D}) = K_{-i}(\mathscr{D}) \oplus K_{-i-1}(\mathscr{D}) \oplus 2\overline{\mathrm{Nil}}_{-i-1}(\mathscr{D}).$$

Here $\overline{\mathrm{Nil}}_{-i-1}(\mathscr{D})$ is the abelian group which classifies the nilpotent maps in $\mathscr{D}_{i+1}$.

In Section 4 we specialize to universal ring extensions of the restricted Burnside category $B(G; \mathscr{F})$. T^i denotes the direct sum of i-copies of T and $R[T^i]$ its group ring. Using the restriction and induction functors between the categories $B(G \times T; \mathscr{F} \times \{1\})$ and $B(G \times \langle t^n \rangle; \mathscr{F} \times \{1\})$ we construct an action of the monoid $\mathbf{N}^{i+1} = (\mathbf{N} \backslash \{0\}, \cdot)^{i+1}$ on $K_1(R[T^{i+1}]; G; \mathscr{F})$ and prove:

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THEOREM B. $K_{-i}(R;G;\mathscr{F}) = K_1(R[T^{i+1}];G;\mathscr{F})^{\mathrm{inv}\,N_i^{+1}}$.

Finally we have the K_{-i}-analogue of one of the main algebraic results from [5]:

THEOREM C. $K_{-i}(R;G;\mathscr{F}) = \sum_{(H)\in\mathscr{F}}^{\oplus} K_{-i}(R[NH/H])$.

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1. Some functorial constructions.

In this section we outline some functorial constructions on the category of additive categories.

Let $\mathscr{A}$ be an arbitrary category. We define three associated categories, $\mathrm{Aut}(\mathscr{A})$, $\mathrm{Proj}(\mathscr{A})$ and $\mathrm{Nil}(\mathscr{A})$ as follows. The objects of $\mathrm{Aut}(\mathscr{A})$ are pairs (A,a) with $a:A\to A$ on automorphisms. The objects of $\mathrm{Proj}(\mathscr{A})$ are pairs (A,p) with $p:A\to A$ satisfying $p^2=p$, and finally the objects of $\mathrm{Nil}(\mathscr{A})$ are pairs (A,v) with $v:A\to A$ satisfying $v^n=0$ (here we assume that $\mathscr{A}$ has an initial-terminal object). In each case the morphisms are the obvious ones, namely the morphisms of $\mathscr{A}$ which commute with the extra structure. For example

$$\mathrm{Nil}(\mathscr{A})((A,v),(B,u)) = \{f:A\to B \mid fv = uf\}.$$

If $\mathscr{A}$ is an additive category it is easily checked that $\mathrm{Aut}(\mathscr{A})$, $\mathrm{Proj}(\mathscr{A})$ and $\mathrm{Nil}(\mathscr{A})$ all have a natural additive structure. Thus $\mathrm{Aut}(\mathscr{A})$, $\mathrm{Proj}(\mathscr{A})$ and $\mathrm{Nil}(\mathscr{A})$ are endofunctors on the category of additive categories. Henceforth $\mathscr{A}$ denotes a small additive category.

Recall that $K_0(\mathscr{A})$ is defined as the abelian group generated by isomorphism classes of objects in $\mathscr{A}$ modulo the relations $[A\oplus B]=[A]+[B]$. $K_1(\mathscr{A})$ is the abelian group generated by isomorphism classes of objects in $\mathrm{Aut}(\mathscr{A})$ subject to the relations

$$[A,ab] = [A,a]+[A,b] \qquad \text{and} \qquad [A\oplus B, a\oplus b] = [A,a]+[B,b].$$

In particular, $[A,1]=0$, $[A,a^{-1}] = -[A,a]$ and

$$\left[A\oplus B, \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\right] =$$

$$= \left[A \oplus B \oplus B, \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \right] = 0,$$

for any morphism $f: B \to A$. Hence

$$\left[A \oplus B, \begin{pmatrix} a & f \\ 0 & b \end{pmatrix} \right] = \left[B \oplus A, \begin{pmatrix} b & 0 \\ f & a \end{pmatrix} \right] = [A, a] + [B, b].$$

If $\mathscr{A}$ and $\mathscr{A}'$ are equivalent, then $K_0(\mathscr{A}) \cong K_0(\mathscr{A}')$ and $K_1(\mathscr{A}) \cong K_1(\mathscr{A}')$, of course.

In $K_0(\mathrm{Nil}\, \mathscr{A})$ we introduce the relations

$$\left[A \oplus B, \begin{pmatrix} v & f \\ 0 & u \end{pmatrix} \right] = [A, v] + [B, u], \tag{1.1}$$

where $f: B \to A$ is an arbitrary morphism. The quotient group of $K_0(\mathrm{Nil}\, \mathscr{A})$ is $\mathrm{Nil}_0(\mathscr{A})$. Similarly, we introduce

(1.2)

(i) $\overline{K_0}(\mathrm{Proj}\mathscr{A}) = K_0(\mathrm{Proj}\mathscr{A})/[A, 0] = 0.$

(ii) $\widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\, \mathscr{A}) = \mathrm{Nil}_0(\mathrm{Proj}\, \mathscr{A})/[A, 0, v] = 0.$

(iii) $\overline{\mathrm{Nil}_0}(\mathscr{A}) = \mathrm{Nil}_0(\mathscr{A})/[A, 0] = 0.$

For later use we list some obvious relations. In $\bar{K}_0(\mathrm{Proj}\, \mathscr{A})$ we have

$$\left[A \oplus B, \begin{pmatrix} p & f \\ 0 & q \end{pmatrix} \right] = [A, p] + [B, q]. \tag{1.3}$$

Note that $\begin{pmatrix} p & f \\ 0 & q \end{pmatrix}$ is a morphism in $\mathrm{Proj}\, \mathscr{A}$ only if $pf + fq = f$, and (1.3) follows from the equality

$$\begin{pmatrix} 1 & pf - fq \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & f \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & fq - pf \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

In $\widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\, \mathscr{A})$ we have

(1.4)

(i) $\left[A \oplus B, \begin{pmatrix} p & f \\ 0 & q \end{pmatrix}, \begin{pmatrix} v & h \\ 0 & u \end{pmatrix} \right] = [A, p, v] + [B, q, u].$

(ii) $[A, p, v] = [A, p, vp].$

(iii) $[A, p, v] = [A, ,vp] + [A, p, 0] - [A, 1, 0].$

Conjugating by $\begin{pmatrix} 1 & pf-fq \\ 0 & 1 \end{pmatrix}$ yields

$$\left[A \oplus B, \begin{pmatrix} p & f \\ 0 & q \end{pmatrix}, \begin{pmatrix} v & h \\ 0 & u \end{pmatrix}\right] = \left[A \oplus B, \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} v & * \\ 0 & u \end{pmatrix}\right] = [A, p, v] + [B, q, u],$$

proving (i). Note that

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}:$$

$$\left(A \oplus A, \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}\right) \to \left(A \oplus A, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} vp+u(1-p) & 0 \\ 0 & v(1-p)+up \end{pmatrix}\right)$$

is an isomorphism in Nil(Proj $\mathscr{A}$). Thus

$$[A, p, v] + [A, 1-p, u] = [A, 1, vp + u(1-p)].$$

If $(A, p, v) \in \operatorname{Ob}\operatorname{Nil}(\operatorname{Proj}\mathscr{A})$, then

$$(1-p)v = v(1-p),$$

so $(A, 1-p, v)$ and $(A, p, vp) \in \operatorname{Ob}\operatorname{Nil}(\operatorname{Proj}\mathscr{A})$.

Choosing $u = vp$ we get

$$(1.5) \qquad [A, p, v] + [A, 1-p, vp] = [A, 1, vp].$$

Substituting v by vp shows that

$$[A, p, vp] + [A, 1-p, vp] = [A, 1, vp].$$

Thus $[A, p, v] = [A, p, vp]$, proving (ii). Also, by (1.5) (two times)

$$[A, p, v] = [A, 1, vp] - [A, 1-p, vp] = [A, 1, vp] + [A, p, 0] - [A, 1, 0],$$

proving (iii).

Proposition 1.6. $\widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{A}) \cong \overline{\operatorname{Nil}_0}(\mathscr{A}) \oplus \bar{K}_0(\operatorname{Proj}\mathscr{A})$.

Proof. There are homomorphisms (induced by the obvious functors).

$$\overline{\operatorname{Nil}_0}(\mathscr{A}) \xrightarrow{i_1} \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{A}) \xrightarrow{P_1} \overline{\operatorname{Nil}_0}(\mathscr{A})$$

and

$$\bar{K}_0(\operatorname{Proj}\mathscr{A}) \xrightarrow{i_2} \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{A}) \xrightarrow{P_2} \bar{K}_0(\operatorname{Proj}\mathscr{A}),$$

given by

$$i_1[A, v] = [A, 1, v]-[A, 1, 0], \qquad i_2[A, p] = [A, p, 0],$$

$$P_1[A, p, v] = [A, vp] \quad \text{and} \quad P_2[A, p, v] = [A, p].$$

Then $P_2 i_1 = 0$, $P_1 i_2 = 0$, $P_2 i_2 = 1$, $P_1 i_1 = 1$ and by (1.4) (iii)

$$[A, p, v] = [A, 1, vp]+[A, p, 0]-[A, 1, 0]$$

showing that $i_2 P_2 + i_1 P_1 = 1$.

2. $\mathbf{Z}^i$-graded categories.

In this section we review some results of [3] on $\mathbf{Z}^i$-graded categories. We use the terminology from [3]. Let $\mathscr{A}$ be an additive category. For each natural number i we consider the $\mathbf{Z}^i$-graded category $\mathscr{A}_i$. Its object are sets of the form $\{A_J\}_{J \in \mathbf{Z}^i}$ where each A_J is an object in $\mathscr{A}$. An object in $\mathscr{A}_i$ will be denoted by A and $A(J) = A_J$. A morphism $f: A \to B$ in $\mathscr{A}_i$ is a set $\{f_{J,K}\}_{J,K \in \mathbf{Z}^i}$, where $f_{J,K}: A(J) \to B(K)$ and $f_{J,K} = 0$ if

$$|J-K| = \max_{1 \leq s \leq i} |j_s - k_s| > d, \qquad \text{some} \quad d \in \mathbf{N}.$$

A morphism in $\mathscr{A}_i$ will be denoted by a single letter f. It has components $f(J, K) = f_{J,K}$. We say f is bounded by $d = d(f)$. Composition of $f: A \to B$ and $g: B \to C$ is defined by

$$(g \circ f)(J, K) = \sum_L g(L, K) \circ f(J, L).$$

Clearly $\mathscr{A} = \mathscr{A}_0$ and $\mathscr{A}_i$ is an additive category.

A function $F: \mathscr{A} \to \mathscr{B}$ extends to a functor $F_i: \mathscr{A}_i \to \mathscr{B}_i$ and a natural transformation $\eta: F \to G$ extends to $\eta_i: F_i \to G_i$. We have the shift endofunctors $T^{\pm 1}: \mathscr{A}_i \to \mathscr{A}_i$ given by

$$(T^{\pm 1}A)(J) \quad = A(j_i, \ldots, j_{i-1}, j_i \mp 1),$$

$$(T^{\pm 1}f)(J, K) = f((j_1, \ldots, j_{i-1}, j_i \mp 1), (k_1, \ldots, k_{i-1}, k_i \mp 1)).$$

$T^{\pm}$ is naturally isomorphic to $1_{\mathscr{A}_i}$ by

$$\tau^{\pm}(A, T^{\pm}A)(J, K) = \begin{cases} 1 & \text{if} \quad j_1 = k_1, \ldots, j_{i-1} = k_{i-1}, \quad j_i = k_i \mp 1 \\ 0 & \text{otherwise.} \end{cases}$$

Observe the embeddings

$$L\colon \mathscr{A}_i \to \mathscr{A}_{i+1}$$

defined by

$$(2.1)\qquad \begin{aligned} (LA)(J) &= A(j_1, \ldots, j_i, \hat{j}_{i+1}) \\ (Lf)(J,K) &= \begin{cases} f((j_1, \ldots, j_i, \hat{j}_{i+1}), (k_1, \ldots, k_i, \hat{k}_{i+1})) & \text{if } j_{i+1} = k_{i+1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\tau^{\pm 1} \circ Lf = Lf \circ \tau^{\pm 1}$ and $\tau^{\pm 1}\colon T^{\pm 1}LA = LA \to LA$.

Pedersen defines

$$(2.2)\qquad K_{-i}(\mathscr{A}) = K_1(\mathscr{A}_{i+1}).$$

Similarly we define

$$(2.3)\qquad \mathrm{Nil}_{-i}(\mathscr{A}) = \mathrm{Nil}_0(\mathscr{A}_i).$$

For a $\mathbf{Z}^i$-graded object A we let $p_+ : A \to A$ be the projection

$$p_+(J,K) = \begin{cases} 1 & \text{if } J = K \text{ and } j_i \geqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Also, $p_- : A \to A$ denotes the projection

$$p_-(J,K) = \begin{cases} 1 & \text{if } J = K \text{ and } j_i \leqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $(A, a) \in \mathrm{Ob}\,\mathrm{Aut}\,\mathscr{A}_{i+1}$, then $(A, ap_-a^{-1}) \in \mathrm{Ob}\,\mathrm{Proj}\,\mathscr{A}_{i+1}$. Furthermore, since a and a^{-1} are bounded this projection equals the identity on $A(J)$, $j_{i+1} \ll 0$ and zero on $A(K)$, $k_{i+1} \gg 0$. Thus summation in the $i+1$th direction of a certain band around $j_{i+1} = 0$ gives an element $(\bar{A}, ap_-a^{-1}) \in \mathrm{Ob}\,\mathrm{Proj}\,\mathscr{A}_i$.

PROPOSITION 2.4 (Pedersen). *The map* $\mathrm{Aut}(\mathscr{A}_{i+1}) \to \tilde{K}_0(\mathrm{Proj}\,\mathscr{A}_i)$ *which sends the object* (A,a) *to the class* $[\bar{A}, ap_-a^{-1}] - [\bar{A}, p_-]$ *induces an isomorphism from* $K_1(\mathscr{A}_{i+1})$ *to* $\tilde{K}_0(\mathrm{Proj}\,\mathscr{A}_i)$.

The reader is referred to [3] and [4] for a proof. We only remark that the inverse of the maps in (2.4) is given by the functor $\mathrm{Proj}(\mathscr{A}_i) \to \mathrm{Aut}(\mathscr{A}_{i+1})$:

$$(f : (A,p) \to (B,q)) \mapsto (Lf : (LA, 1 - Lp + \tau Lp) \to (LB, 1 - Lq + \tau Lq)).$$

3. $K_{-i}(\cdot)$ of universal R-extensions.

Throughout this section R will be a ring with identity and $\mathscr{C}$ will be a small category with finite coproducts and an initial and terminal object. We consider the groups $K_{-i}(R \otimes \mathscr{C})$, where $R \otimes \mathscr{C}$ is the universal R-extension of $\mathscr{C}$ (cf. [5]). First a category $\mathscr{D}$ is said to be an *R-category* if

(3.1)

(i) for each pair of objects (A, B) the set $\mathscr{D}(A, B)$ is an R-bimodule such that the compositions

$$\mathscr{D}(B, C) \times \mathscr{D}(A, B) \to \mathscr{D}(A, C)$$

are R-linear to the left in the first variable and to the right in the second one and R-balanced,

(ii) there is a 0-object,

(iii) $\mathscr{D}$ has finite coproducts.

A functor $F: \mathscr{D} \to \mathscr{D}'$ is said to be an *R-functor* if $F(rf+gs) = rF(f)+F(g)s$, for all $f, g \in \mathscr{D}(A, B)$ and $r, s \in R$.

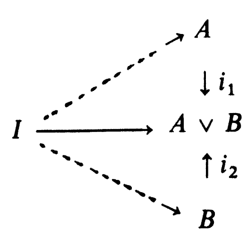
Given a category $\mathscr{C}$, we construct its universal R-extension, $R \otimes \mathscr{C}$, as follows. Let $R(\mathscr{C})$ be the category with the same objects as $\mathscr{C}$ and with morphisms

$$R(\mathscr{C})(A, B) = \{\lambda : \mathscr{C}(A, B) \to R \mid \lambda(f) = 0, \quad \text{almost all } f\}$$

$R(\mathscr{C})(A, B)$ is an R-bimodule; its elements can be written in the form $\sum_{\text{finite}} r_i f_i$. The composition is defined by

$$\left(\sum_j s_j g_j\right) \circ \left(\sum_i r_i f_i\right) = \sum_{i,j} s_j r_i (g_j \circ f_i).$$

$R(\mathscr{C})$ satisfies condition (3.1) (i) but not the two other conditions. An object I of $\mathscr{C}$ is said to be *indecomposable* if given



then at least one of the dotted arrows exists. Let $\mathrm{IND}(\mathscr{C})$ denote the category of indecomposable objects in $\mathscr{C}$. We can now define the universal R-extension $R \otimes \mathscr{C}$; it has the same objects as $\mathscr{C}$ and $R \otimes \mathscr{C}(A, B) = R(\mathscr{C})/K(A, B)$, where $K(A, B)$ is the R-bisubmodule of $R(\mathscr{C})(A, B)$ consisting of all morphisms $\lambda: A \to B$, such that for any morphism $\mu: I \to A$, $I \in \mathrm{IND}(\mathscr{C})$, $\lambda \circ \mu = r_\mu \cdot 0$, some $r_\mu \in R$. One easily deduces that

$$K(A, B) = \{\lambda \in R(\mathscr{C})(A, B) \mid \text{for all } h \in \mathscr{C}(I, A),\ I \in \mathrm{IND}(\mathscr{C}),\ \lambda h = r_\mu \cdot 0\}.$$

The elements of $R \otimes \mathscr{C}(A, B)$ will be written in the form $[\sum r_i f_i]$. The obvious functor $\mathscr{C} \to R \otimes \mathscr{C}$ preserves coproducts, so $R \otimes \mathscr{C}$ is an R-category.

Let T denote the infinite cyclic group with generator t. $R[T]$ is the group algebra of T with coefficients in R. If $\mathscr{D}$ is an R-category, then $R[T] \otimes_R \mathscr{D}$ is the $R[T]$-category with the same objects as $\mathscr{D}$ and morphisms $(R[T] \otimes_R \mathscr{D})(A, B) = R[T] \otimes_R \mathscr{D}(A, B)$.

The R-functor $\mathscr{D} \to R[T] \otimes_R \mathscr{D}$ preserves coproducts and $(R[T] \otimes_R \mathscr{D})(A, B)$ can be identified with $\mathscr{D}(A, B)[T]$.

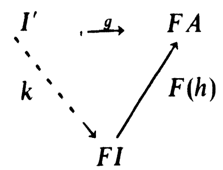
We have the following obvious

Proposition 3.2. *$R[T] \otimes_R (R \otimes \mathscr{C})$ is $R[T]$-isomorphic to $R[T] \otimes \mathscr{C}$. A ring homomorphism $\phi: R_1 \to R_2$ ($\phi(1) = 1$) induces an R_1-functor,*

$$\phi: R_1 \otimes \mathscr{C} \to R_2 \otimes \mathscr{C}.$$

It is the identity on objects and map the morphism $[\sum r_i f_i]$ to $[\sum \phi(r_i) f_i]$.

Proposition 3.3. *Let $F: \mathscr{C} \to \mathscr{C}'$ be a functor preserving initial-terminal objects. Suppose every map $g \in \mathscr{C}'(I', FA)$, $I' \in \mathrm{IND}(\mathscr{C}')$ factorizes as*



for some $h \in \mathscr{C}(I, A)$, $I \in \mathrm{IND}(\mathscr{C})$. Then F extends uniquely to an R-functor $F: R \otimes \mathscr{C} \to R \otimes \mathscr{C}'$, equal to F on objects and with $F([\sum r_i f_i]) = [\sum r_i F(f_i)]$.

Proof. If $F: R \otimes \mathscr{C} \to R \otimes \mathscr{C}'$ is well defined it is obviously an R-functor. It suffices to show that it is well defined. Suppose $\sum r_i f_i h = r_h \cdot 0$ for all $h \in \mathscr{C}(I, A)$, $I \in \mathrm{IND}(\mathscr{C})$. Let $g \in \mathscr{C}'(I', FA)$, $I' \in \mathrm{IND}(\mathscr{C}')$, then

$$\sum r_i F(f_i)g = \sum r_i F(f_i)F(h)k = F(\sum r_i f_i h)k = F(r_h \cdot 0)k = r_h \cdot 0.$$

Note that $F: R \otimes \mathscr{C} \to R \otimes \mathscr{C}'$ preserves coproducts even if $F: \mathscr{C} \to \mathscr{C}'$ does not.

COROLLARY 3.4. *If $\mathscr{C}$ and $\mathscr{C}'$ are equivalent, then $R \otimes \mathscr{C}$ and $R \otimes \mathscr{C}'$ are equivalent.*

PROOF. By definition there are functors $F: \mathscr{C} \to \mathscr{C}'$, $G: \mathscr{C}' \to \mathscr{C}$ and natural equivalences $\eta: 1_{\mathscr{C}} \to GF$, $v: 1_{\mathscr{C}'} \to FG$, such that $F\eta = vF$. It follows that F and G satisfies the condition in Proposition 3.3, so the induced functors $F: R \otimes \mathscr{C} \to R' \otimes \mathscr{C}$ and $G: R \otimes \mathscr{C}' \to R \otimes \mathscr{C}$ exists. Also, the natural equivalences extend.

A class of objects $\mathscr{W}$ in a category $\mathscr{E}$ is said to *generate* $\mathscr{E}$ if, for every $f \in \mathscr{E}(X, Y)$, $f \neq 0$, there exists $W \in \mathscr{W}$ and $j \in \mathscr{E}(W, X)$ such that $fj \neq 0$.

PROPOSITION 3.5. (Rothenberg [5]). *Let $\mathscr{D}$ be an R-category and $F: \mathscr{C} \to \mathscr{D}$ a functor such that $F(\mathrm{IND}(\mathscr{C}))$ generates $F(\mathscr{C})$. Then F extends uniquely to an R-functor $F: R \otimes \mathscr{C} \to \mathscr{D}$.*

$\mathscr{C}$ is said to be a *wedge* of two full subcategories $\mathscr{C}'$ and $\mathscr{C}''$ ($\mathscr{C} = \mathscr{C}' \vee \mathscr{C}''$) if

(3.6)

(i) for all $X \in \mathscr{C}$, $X \cong X_1 \vee X_2$, $X_1 \in \mathscr{C}'$ and $X_2 \in \mathscr{C}''$. If $f: X_1 \vee X_2 \to Y_1 \vee Y_2$ is an isomorphism ($X_1, Y_1 \in \mathscr{C}'$; $X_2, Y_2 \in \mathscr{C}''$), then $f = f_1 \vee f_2$, f_1 and f_2 isomorphisms in $\mathscr{C}'$ and $\mathscr{C}''$, respectively.

(ii) $\mathscr{C}(X_1, X_2) = (0)$ for $X_1 \in \mathscr{C}'$, $X_2 \in \mathscr{C}''$.

(iii) Let $i: \mathscr{C} \to R \otimes \mathscr{C}$ be the functor $(f: A \to B) \mapsto ([f]: A \to B)$. Then $i(\mathrm{IND}(\mathscr{C}'))$ generates $i(\mathscr{C}')$.

PROPOSITION 3.7. *If $\mathscr{C} = \mathscr{C}' \vee \mathscr{C}''$, then*

$$K_{-i}(R \otimes \mathscr{C}) \cong K_{-i}(R \otimes \mathscr{C}') \oplus K_{-i}(R \otimes \mathscr{C}'').$$

The proof of 3.7 is essentially identical to the proof of [5, Theorem 1.17] and will be left to the reader.

Using the Whitehead relation

$$\begin{pmatrix} ab & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

it is easy to prove the following:

LEMMA 3.8. *Let H be an abelian group and $\Phi : \mathrm{ObAut}(\mathscr{A}) \to H$ a map. Then*

$$\begin{array}{ccc} \mathrm{ObAut}(\mathscr{A}) & \xrightarrow{\Phi} & H \\ \downarrow & \nearrow & \\ K_1(\mathscr{A}) & & \end{array}$$

if and only if

(i) $\Phi(A \oplus B, a \oplus b) = \Phi(A, a) + \Phi(B, b)$,
for all $(A, a), (B, b) \in \mathrm{ObAut}(\mathscr{A})$,

(ii) $$\Phi\left(A \oplus B, \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} c\right) = \Phi\left(A \oplus B, \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} c\right) = \Phi(A \oplus B, c),$$

for all $(A \oplus B, c) \in \mathrm{ObAut}(\mathscr{A})$ *and* $h : B \to A$, $g : A \to B$,

(iii) $\Phi(A, 1) = 0$.

We want to describe a map

$$v : K_1(R[T] \otimes_R \mathscr{D}) \to \widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\, \mathscr{D})$$

and begin with giving v on $\mathrm{Aut}(R[T] \otimes_R \mathscr{D})$. It is easy to check that the functor $R[T] \otimes_R \mathscr{D} \to \mathscr{D}_1$ given by

$$\left(\sum t^i f_i : A \to B\right) \mapsto \left(\sum \tau^i L f_i : LA \to LB\right)$$

is well defined. (See section 2 for notation.) We will denote this functor by $(a : A \to B) \mapsto (\tilde{a} : LA \to LB)$. If $(A, a) \in \mathrm{ObAut}(R[T] \otimes_R \mathscr{D})$, then $(LA, \tilde{a}) \in \mathrm{ObAut}(\mathscr{D}_1)$.

Since $\tilde{a}$ and $\tilde{a}^{-1}$ are bounded we can consider the maps (for $k = \max(d(\tilde{a}), d(\tilde{a}^{-1}))$)

$$\tilde{a} p_- \tilde{a}^{-1} \ : \sum_{-k}^{k}{}^{\oplus} LA(j) \to \sum_{-k}^{k}{}^{\oplus} LA(j),$$

$$\tilde{a} p_- \tau \tilde{a}^{-1} : \sum_{-k}^{k}{}^{\oplus} LA(j) \to \sum_{-k}^{k}{}^{\oplus} LA(j),$$

where p_- and τ are as defined in section 2. Write $\sum_{-k}^{k}{}^{\oplus} LA(j) = (2k+1)A$

and set

$$v(A, a) = [(2k+1)A, \tilde{a}p_-\tilde{a}^{-1}, \tilde{a}p_-\tau\tilde{a}^{-1}] - [(2k+1)A, p_-, p_-\tau].$$

If $l > k$, then the restrictions of $\tilde{a}p_-\tilde{a}^{-1}$ and $\tilde{a}p_-\tau\tilde{a}^{-1}$ to the band $|j| \leqq l$ give the same element since $\tilde{a}p_-\tilde{a}^{-1}(j,i) = p_-(j,i)$ and $\tilde{a}p_-\tau\tilde{a}^{-1}(j,i) = \tilde{a}p_-\tilde{a}^{-1}\tau(j,i) = p_-\tau(j,i)$ if $|j| > l$.

PROPOSITION 3.9. *The map v factors over $K_1(R[T] \otimes_R \mathscr{D})$;*

$$v: K_1(R[T] \otimes_R \mathscr{D}) \to \widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\, \mathscr{D}).$$

PROOF. We check the conditions in (3.8). We leave conditions (i) and (iii) to the reader and prove (ii). If $d: A \oplus B \to A \oplus B$ is an invertible matrix with entries in $\mathscr{D}$, then $Ld = \tilde{d}$ preserves the degrees. It follows that

$$v(A \oplus B, dc) = v(A \oplus B, c). \tag{3.10}$$

Suppose we have proven

$$v\left(A \oplus B, \begin{pmatrix} 1 & t^{\pm 1}f \\ 0 & 1 \end{pmatrix} c\right) = v\left(A \oplus B, \begin{pmatrix} 1 & 0 \\ t^{\pm 1}f & 1 \end{pmatrix} c\right) = v(A \oplus B, c), \tag{3.11}$$

for every $f \in \mathscr{D}(B, A)$. Then by applying the Whitehead relation to

$$\begin{pmatrix} 1 & 0 \\ 0 & t^{\pm 1} \end{pmatrix} c \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that

$$v\left(A \oplus B, \begin{pmatrix} 1 & 0 \\ 0 & t^{\pm 1} \end{pmatrix} c\right) = v\left(A \oplus B, \begin{pmatrix} 1 & 0 \\ 0 & t^{\pm 1} \end{pmatrix}\right) + v(A \oplus B, c).$$

If $h = \sum t^m f_m$, then

$$\begin{pmatrix} 0 & h \\ 0 & 1 \end{pmatrix} = \prod \begin{pmatrix} 1 & t^m f_m \\ 0 & 1 \end{pmatrix}$$

so it is enough to consider $h = t^m f$. Now

$$\begin{pmatrix} 1 & t^{m+1}f \\ 0 & 1 \end{pmatrix} c = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & t^m f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} c, \quad \text{if } m \geqq 0$$

and

$$\begin{pmatrix} 1 & t^{m-1} \\ 0 & 1 \end{pmatrix} c = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & t^m f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} c, \quad \text{if } m \leqq 0.$$

There are similar formulas in the case $\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} c$. Thus the proposition will follow from (3.11) by induction (using (3.10) to start it).

We now prove (3.11). If $k = \max(d(\tilde{c}), d(\tilde{c}^{-1}))$, then $\tilde{c}p_-\tilde{c}^{-1}$ maps the band $-k \leqq j \leqq k$ into itself; the map is the identity if $j < -k$ and zero if $j > k$. $\tilde{c}p_-\tau\tilde{c}^{-1} = \tilde{c}p_-\tilde{c}^{-1}\tau$ also maps the band $-k \leqq j \leqq k$ into itself; if $j < -k$ it equals τ, and if $j > k$ it equals zero.

Let $l > k$ and define

$$LB'(j) = \begin{cases} B & \text{if } |j| \leqq 2l \\ 0 & \text{if } |j| > 2l \end{cases}$$

and LB'' by $LB' \oplus LB'' = LB$. Let $h = t^{\pm 1}f$,

$$h' = LB \xrightarrow{\text{proj}} LB' \xrightarrow{\text{incl}} LB \xrightarrow{\bar{h}} LB$$

and

$$h'' = LB \xrightarrow{\text{proj}} LB'' \xrightarrow{\text{incl}} LB \xrightarrow{\bar{h}} LB.$$

Then

$$\begin{pmatrix} 1 & \bar{h} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{h}' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \bar{h}'' \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \bar{h}'' \\ 0 & 1 \end{pmatrix} \tilde{c}p_-\tilde{c}^{-1} \begin{pmatrix} 1 & -\bar{h}'' \\ 0 & 1 \end{pmatrix} = \tilde{c}p_-\tilde{c}^{-1},$$

since $\tilde{c}p_-\tilde{c}^{-1}$ is the identity if $j < -l$ and zero if $j > l$. Consider

$$V = \begin{pmatrix} 1 & \bar{h}'' \\ 0 & 1 \end{pmatrix} \tilde{c}p_-\tau\tilde{c}^{-1} \begin{pmatrix} 1 & -\bar{h}'' \\ 0 & 1 \end{pmatrix}.$$

If $j \geqq -2l$ then $V = \tilde{c}p_-\tau\tilde{c}^{-1}$ and if $j \leqq -2l-2$ then $V = \tau$. In the case $h = tf$ we have

$$V(-2l-1, i) = \begin{cases} 1 & \text{if } i = -2l \\ \begin{pmatrix} 0 & -f \\ 0 & 0 \end{pmatrix} & \text{if } i = -2l+1 \\ 0 & \text{otherwise.} \end{cases}$$

If $h = t^{-1}f$, then

$$V(-2l-1, i) = \begin{cases} 1 & \text{if } i = -2l \\ \begin{pmatrix} 0 & -f \\ 0 & 0 \end{pmatrix} & \text{if } i = -2l-1 \\ 0 & \text{otherwise.} \end{cases}$$

Writing down the matrices for V and using (1.1) it follows that

$$\left[(6l+1)(A\oplus B), \begin{pmatrix}1 & \tilde{h}''\\ 0 & 1\end{pmatrix}\tilde{c}p_{-}\tilde{c}^{-1}\begin{pmatrix}1 & -\tilde{h}''\\ 0 & 1\end{pmatrix}, v\right]$$

$$= [(6l+1)(A\oplus B), \tilde{c}p_{-}\tilde{c}^{-1}, \tilde{c}p_{-}\tau\tilde{c}^{-1}].$$

Observing that $\begin{pmatrix}1 & \tilde{h}'\\ 0 & 1\end{pmatrix}$ restricts to an isomorphism of the band $|j| \leqq 3l$ we get

$$\nu\left(A\oplus B, \begin{pmatrix}1 & t^{\pm}f\\ 0 & 1\end{pmatrix}c\right) = \nu(A\oplus B, c).$$

The proof of the other half of (3.11) is completely analogous.

The map ν above is a split epimorphism. Indeed, we can define a homomorphism in the opposite direction by

$$\delta\colon \widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\,\mathscr{D}) \to K_1(R[T]\otimes_R \mathscr{D})$$

$$\delta[A,p,v] = [A,(1-p)+(v-t)p]-[A,(1-p)+(v+1)p]$$

(3.12) $$= [A,1-p-tp]+[A,1-t^{-1}vp]-[A,1-p+(v+1)p]$$

$$= [A,1-p+(v-t)(v+1)^{-1}p].$$

δ is induced by the obvious functor.

PROPOSITION 3.13. *The map δ is a section of ν.*

PROOF. $1-p-t^{-1}p$ is the inverse of $1-p-tp$ and an easy calculation shows that

$$\nu[A,1-p-tp] = \left[3A, \begin{pmatrix}1&0&0\\0&1&0\\0&0&p\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&p&0\end{pmatrix}\right] - \left[3A, \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\right]$$

$$= [A,p,0].$$

The last equality follows from (1.3) (i).

Assume v has nilpotence index $n+1$ (that is $v^{n+1}=0$). Then

$$(1-t^{-1}vp)^{-1} = 1+t^{-1}vp+t^{-2}v^2p+\ldots+t^{-n}v^np.$$

The maximal bound of the maps $((1-t^{-1}vp)\urcorner)^{\pm 1}$ is n. Using the definition of ν we get

$$\nu[A, 1-t^{-1}vp] = \left[(2n+1)A, \begin{pmatrix} I_{n+1} & M \\ 0 & 0_n \end{pmatrix}, \begin{pmatrix} I_{n+1} & M \\ 0 & 0_n \end{pmatrix} J_{2n+1}\right],$$

where M is the $(n+1)\times n$-matrix with entries

$$(M)_{i,j} = \begin{cases} v^j p & \text{if } \; i = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

J_{2n+1} is the matrix for the cyclic permutation $(1, 2, \ldots, 2n, 2n+1)$. An application of (1.4) (i) shows that

$$\nu[A, 1-t^{-1}vp] = [A, 1, vp]-[A, 1, 0] = [A, 1, vp]-[A, 1, 0].$$

Since $1-p+(v+1)^{-1}p$ is a morphism in $\mathscr{D}$ it follows that (cf. the proof of (3.9)), $\nu[A, 1-p+(v+1)^{-1}p] = 0$.

Thus by (3.12) and (1.4) we have

$$\nu\delta[A, p, v] = [A, 1, vp]+[A, p, 0]-[A, 1, 0] = [A, p, v].$$

We can substitute p_- and τ for p_+ and τ^{-1} in the definition of ν and get a new homomorphism

$$\nu_- : K_1(R[T] \otimes_R \mathscr{D}) \to \widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\, \mathscr{D}).$$

Also, there is a homomorphism

$$\delta_- : \widetilde{\mathrm{Nil}}_0(\mathrm{Proj}\, \mathscr{D}) \to K_1(R[T] \otimes_R \mathscr{D})$$

induced by the functor $\mathrm{Nil}\,\mathrm{Proj}\, \mathscr{D} \to \mathrm{Aut}\, R[T] \otimes_R \mathscr{D}$, which sends (A, p, v) to $(A, 1-p+(v-t^{-1})(v+1)^{-1}p)$.

One shows, exactly as in the proof of Proposition 3.13, that δ_- is a section of ν_-. The same type of calculations also show that

$$(3.14) \qquad \nu_-\delta[A, p, v] = \nu\delta_-[A, p, v] = [A, 1-p, 0]-[A, 1, 0].$$

The embedding $\mathscr{D} \to R[T] \otimes_R \mathscr{D}$ induces

$$i_1 : K_1(\mathscr{D}) \to K_1(R[T] \otimes_R \mathscr{D})$$

with left inverse given by the map $R[T] \to R$, $t \mapsto -1$,

$$p_1 : K_1(R[T] \otimes_R \mathscr{D}) \to K_1(\mathscr{D}).$$

We also have

$$i_2 : \bar{K}_0(\operatorname{Proj}\mathscr{D}) \longrightarrow \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{D}) \xrightarrow{\delta} K_1(R[T]\otimes_R \mathscr{D}),$$

$$p_2 : K^1(R[T]\otimes_R \mathscr{D}) \xrightarrow{\nu} \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{D}) \longrightarrow \bar{K}_0(\operatorname{Proj}\mathscr{D}),$$

$$i_3 : \overline{\operatorname{Nil}}_0(\mathscr{D}) \longrightarrow \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{D}) \xrightarrow{\delta} K_1(R[T]\otimes_R \mathscr{D}),$$

$$p_3 : K_1(R[T]\otimes_R \mathscr{D}) \xrightarrow{\nu} \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{D}) \longrightarrow \overline{\operatorname{Nil}}_0(\mathscr{D}),$$

$$i_4 : \operatorname{Nil}_0(\mathscr{D}) \longrightarrow \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{D}) \xrightarrow{\delta^-} K_1(R[T]\otimes_R \mathscr{D}) \quad \text{and}$$

$$p_4 : K_1(R[T]\otimes_R \mathscr{D}) \xrightarrow{\nu^-} \widetilde{\operatorname{Nil}}_0(\operatorname{Proj}\mathscr{D}) \longrightarrow \overline{\operatorname{Nil}}_0(\mathscr{D}),$$

where the unnamed maps are as in 1.6.

It follows from (3.14) that $p_k i_l = \delta(k, l)$.

PROPOSITION 3.15. $K_1(R[T]\otimes_R \mathscr{D}) \cong K_1(\mathscr{D}) \oplus \bar{K}_0(\operatorname{Proj}\mathscr{D}) \oplus 2\overline{\operatorname{Nil}}_0(\mathscr{D})$.

PROOF. By the above the only thing left to check is that $\sum p_k i_k = 1$. Let $(A, \sum t^m f_m) \in \operatorname{ObAut}(R[T]\otimes_R \mathscr{D})$. Thus $\sum t^m f_m$ is a unit in the ring $R[T]\otimes_R \mathscr{D}(A, A) = \mathscr{D}(A, A)[T]$.

There is an obvious homomorphism $K_1(\mathscr{D}(A, A)[T]) \to K_1(R[T]\otimes_R \mathscr{D})$, mapping an $n\times n$-matrix to the torsion of the corresponding map $nA \to nA$, followed by the map induced by $R[T] \to R[T]$, $t \mapsto -t$. By the usual decomposition of K_1 of a ring (see [B]) we have

$$\left[\sum t^m f_m\right] = \left[\sum f_m\right] + [1-p+(t+v)(1+v)^{-1}p] + [1-q+(t+u)(1+u)^{-1}q],$$

where p and q are matrices over $\mathscr{D}(A, A)$ such that $p^2 = p$ and $q^2 = q \cdot u$ and v are nilpotent matrices over $\mathscr{D}(A, A)$. It follows that $\sum i_k p_k = 1$.

THEOREM A.
$K_{-i+1}(R[T]\otimes \mathscr{C}) \cong K_{-i+1}(R\otimes \mathscr{C}) \oplus K_{-i}(R\otimes \mathscr{C}) \oplus 2\overline{\operatorname{Nil}}_{-i}(R\otimes \mathscr{C})$.

PROOF. The case $i = 0$ follows from (3.15) and (2.4). Writing down a diagram corresponding to [3, (2.15), p. 473] one sees by induction that

$$K_{-i}(R[T^r]\otimes \mathscr{C}) \cong K_1(R[T^r]\otimes_R (R\otimes \mathscr{C})_{i+1}).$$

4. The equivariant $K_{-i}(\cdot)$-groups.

Let G be a discrete group and $\mathscr{F}$ a subset of the set of conjugacy classes of subgroups of G. We shall consider G-finite sets, that is G-sets X with X/G

finite. All G-sets will be assumed to have a base point $\{*\}$ wich is a stationary point. Let $B(G;\mathscr{F})$ be the category of G-finite sets X, such that

$$v \in X - \{*\} \Rightarrow (G_x) \in \mathscr{F}.$$

Here (H) denotes the conjugacy class of $H < G$. The morphisms in $B(G:\mathscr{F})$ are base point preserving G-maps. The category $B(G;\mathscr{F})$ has finite coproducts (and products) and $\{*\}$ is both initial and terminal. Let R be any ring (with $1 \in R$). Define *the equivariant* K_{-i}*-groups* ($i \geqq -1$) *of* G *with respect to* R *and* $\mathscr{F}$ *to be*

$$K_{-i}(R;G;\mathscr{F}) = K_1((R \otimes B(G;\mathscr{F}))_{i+1}).$$

Similarly

$$\mathrm{Nil}_{-i}(R;G;\mathscr{F}) = \overline{\mathrm{Nil}}_0((R \otimes B(G;\mathscr{F}))_i).$$

From Theorem A we have

COROLLARY 4.1.

$$K_{-i}(R[T];G;\mathscr{F}) \cong K_{-i}(R;G;\mathscr{F}) \oplus K_{-i-1}(R;G;\mathscr{F}) \oplus 2\mathrm{Nil}_{-i-1}(R;G;\mathscr{F}).$$

PROPOSITION 4.2. *If* Γ *is abelian, then* $R \otimes B(G \times \Gamma;\mathscr{F} \times \{1\})$ *is an* $R[\Gamma]$*-category.*

PROOF. Let $\gamma \in \Gamma$. Since Γ is abelian the map

$$\gamma\colon \bigvee_i (G \times \Gamma/F_i \times 1)^+ \to \bigvee_i (G \times \Gamma/F_i \times 1)^+,$$

which sends $[g_1, \gamma_1]_i$ to $[g_1, \gamma\gamma_1]_i$ and $+$ to $+$ is a $G \times \Gamma$-map. Choose a point x_i in each $G \times \Gamma$ orbit in X, $(G \times \Gamma)_{x_i} = F_i \times 1$. Then we have the usual $G \times \Gamma$-isomorphism

$$\phi\colon \bigvee_i (G \times \Gamma/F_i \times 1) \to X, \qquad [g_1, \gamma_1]_i \mapsto (g_1, \gamma_1)x_i.$$

If we choose another set of orbit points $\{x_j'\}$, we get another isomorphism ψ. However $[\phi\gamma\phi^{-1}] = [\psi\gamma\psi^{-1}]$ in $R \otimes B(G \times \Gamma;\mathscr{F} \times \{1\})$. Indeed, $\mathrm{IND}(B(G \times \Gamma;\mathscr{F} \times \{1\}))$ has skeleton $\{(G \times \Gamma/F \times 1)^+ \mid (F) \in \mathscr{F}\} \cup \{*\}$ so it is enough to show that if

$$h\colon (G \times \Gamma/F_1 \times 1)^+ \xrightarrow{\cong} (G \times \Gamma/F_2 \times 1)^+$$

is a $G \times \Gamma$-isomorphism, then $h\gamma h^{-1} = \gamma$ in $B(G \times \Gamma; \mathscr{F} \times \{1\})$. This follows from the assumption that Γ is abelian.

For $[\sum r_i f_i] \in R \otimes B(G \times \Gamma; \mathscr{F} \times \{1\})(X, Y)$ define

$$(4.3) \qquad [\textstyle\sum r_i f_i]\gamma = [\sum r_i f_i][\phi\gamma\phi^{-1}], \quad \text{for some } \phi\colon \bigvee_i (G \times \Gamma/F_i \times 1)^+ \xrightarrow{\cong} X$$

and

$$(4.4) \qquad \gamma[\textstyle\sum r_i f_i] = [\psi\gamma\psi^{-1}][\sum r_i f_i], \quad \text{for some } \psi\colon \bigvee_j (G \times \Gamma/F_j \times 1)^+ \xrightarrow{\cong} Y.$$

By the above this definition is independent of the choices of ϕ and ψ. It follows immediately from (4.3) and (4.4) that $R \otimes B(G \times \Gamma; \mathscr{F} \times \{1\})$ is an $R[\Gamma]$-category.

Proposition 4.5. *The $R[\Gamma]$-categories $R[\Gamma] \otimes B(G; \mathscr{F})$ and $R \otimes B(G \times \Gamma; \mathscr{F} \times \{1\})$ are equivalent.*

Proof. Consider the functor

$$\bar{\psi}\colon B(G; \mathscr{F}) \to B(G \times \Gamma; \mathscr{F} \times \{1\}) \to R \otimes B(G \times \Gamma; \mathscr{F} \times \{1\}),$$

which sends $f\colon X \to Y$ to $[f \wedge 1]\colon X \wedge \Gamma^+ \to Y \wedge \Gamma^+$.

By Proposition (3.5) and (4.2) and the fact that $\{G/F^+ \wedge \Gamma^+ \mid (F) \in \mathscr{F}\} \cup \{*\}$ is a skeleton in $\mathrm{IND}(R \otimes B(G \times \Gamma; \mathscr{F} \times 1))$, $\bar{\psi}$ extends to

$$\bar{\Psi}\colon R[\Gamma] \otimes B(G; \mathscr{F}) \to R \otimes B(G \times \Gamma; \mathscr{F} \times \{1\})$$

and

$$\bar{\Psi}\left[\sum_\gamma \gamma \sum_i r_{i,\gamma} f_{i,\gamma}\right] = \sum_\gamma \gamma \left[\sum_i r_{i,\gamma}(f_{i,\gamma} \wedge 1)\right] = \sum_\gamma \left[\sum_i r_{i,\gamma}(f_{i,\gamma} \wedge \gamma)\right].$$

We show that $\bar{\Psi}$ is a full embedding and that every $Z \in B(G \times \Gamma; \mathscr{F} \times \{1\})$ is isomorphic to $X \wedge \Gamma^+$ for some $X \in B(G; \mathscr{F})$. The latter is immediate since

$$Z \cong \bigvee_i (G \times \Gamma/F_i \times 1)^+ \cong \left(\bigvee_i G/F_i^+\right) \wedge \Gamma^+.$$

We consider the map

$$\bar{\Psi}\colon R[\Gamma] \otimes B(G; \mathscr{F})(X, Y) \to R \otimes B(G \times \Gamma; \mathscr{F} \times \{1\})(X \wedge \Gamma^+, Y \wedge \Gamma^+).$$

Suppose $[\sum_{i,\gamma} r_{i,\gamma}(f_{i,\gamma} \wedge \gamma)] = 0$. Let $\phi_x : G/F^+ \to X$ be the G-map which sends $[1]$ to $x \in X$, $(F \subseteqq G_x)$. By assumption we have that

$$r \cdot * = \left(\sum_{i,\gamma} r_{i,\gamma}(f_{i,\gamma} \wedge \gamma)\right) \circ (\phi_x \wedge 1) = \sum_{i,\gamma} r_{i,\gamma}(f_{i,\gamma}\phi_x \wedge \gamma), \tag{4.6}$$

where $*$ is the zero (constant) map. Now,

$$f_1 \wedge \gamma_1 = f_2 \wedge \gamma_2 \Leftrightarrow (f_1 = f_2 \quad \text{and} \quad \gamma_1 = \gamma_2) \quad \text{or} \quad f_1 = f_2 = *.$$

Thus (4.6) implies that

$$\sum_{\gamma} \gamma \sum_{i} r_{i,\gamma} f_{i,\gamma}\phi_x = \left(\sum_{i,\gamma} r_{i,\gamma}\gamma\right) \cdot * \ ,$$

proving that $\bar{\Psi}$ is an embedding.

Let $f : X \wedge \Gamma^+ \to Y \wedge \Gamma^+$ be a $G \times \Gamma$-map and $\{x_i\}$ a choice of one point in each G-orbit of X, inducing

$$\phi : X \xrightarrow{\cong} \bigvee_i G/F_i^+ .$$

Denote projection on the jth factor of $\bigvee_i G/F_i^+$ by p_j. Let

$$[\psi] : X \wedge \Gamma^+ \to \bigvee_i (G/F_i^+ \wedge \Gamma^+)$$

be the map induced by the maps $p_j\phi \wedge 1$ (recall that coproducts are products in an R-category),

$$[\chi] : \bigvee_i (G/F_i^+ \wedge \Gamma^+) \to Y \wedge \Gamma^+$$

is induced by the maps

$$\phi_{y_j} \wedge \gamma_j : G/F_j^+ \wedge \Gamma^+ \to Y \wedge \Gamma^+$$

sending $[[g], \gamma]$ to $(g, \gamma)[y_j, \gamma_j]$, where $[y_j, \gamma_j] = f[x_j, 1]$.

Thus

$$[f] = [\chi\psi] = \left[\sum_i (\phi_{y_i} \wedge \gamma_i)(p_i\phi \wedge 1)\right] = \bar{\Psi}\left[\sum_i \gamma_i\phi_{y_i}p_i\phi\right],$$

showing that $\bar{\Psi}$ is full.

COROLLARY 4.7. $K_{-i}(R[\Gamma]; G; \mathscr{F}) \cong K_{-i}(R; G \times \Gamma; \mathscr{F} \times \{1\})$.

In proving Theorem 3 of the introduction it is convenient to introduce the usual restriction and induction functors. We write

$$\mathscr{F}H = \{(H \cap F)_H \mid (F) \in \mathscr{F}\},$$

where $(-)_H$ denotes conjugacy-class in H, and have

$$\mathrm{Res}^H_K : B(K; \mathscr{F}K) \to B(H; \mathscr{F}H)$$

if $H \subset K$ and $\Gamma \backslash K/H$ is finite for $(\Gamma) \in \mathscr{F}K$.

If $(\Gamma)_H \in \mathscr{F}H$ implies $(\Gamma)_K \in \mathscr{F}K$ we also have

$$\mathrm{Ind}^K_H : B(H; \mathscr{F}H) \to B(K; \mathscr{F}K)$$

by $\mathrm{Ind}^K_H(X) = X \wedge {}_HK^+$. The above functors induces

$$\mathrm{Res}^H_K : K_{-i}(R; K; \mathscr{F}K) \to K_{-i}(R; H; \mathscr{F}H)$$

and

$$\mathrm{Ind}^K_H : K_{-i}(R; H; \mathscr{F}H) \to K_{-i}(R; K; \mathscr{F}K).$$

Note if $(F), (F_1) \in \mathscr{F}$ implies $(F \cap F_1) \in \mathscr{F}$ (that is $\mathscr{F}$ is a family), then $K_{-i}(R; - ; \mathscr{F} -)$ is a Mackey functor (cf. [2]).

We can use the above maps to define an action of $\mathbf{N}_\cdot = \{\mathbf{N} - \{0\}, \cdot\}$ on $K_{-i}(R[T]; G; \mathscr{F})$, namely

$$[n]\alpha = \Phi^{-1}\mathrm{Ind}^{G \times T}_{G \times \langle t^n \rangle}\mathrm{Res}^{G \times \langle t^n \rangle}_{G \times T}\Phi(\alpha),$$

where Φ is the isomorphism of Corollary 4.7.

PROPOSITION 4.8. *The map* $\mathbf{N} \to \mathrm{End}(K_{-i}(R[T]; G; \mathscr{F}))$, $n \mapsto [n]$ *is a morphism of monoids* (i.e. $\mathbf{N}$ *acts on* $K_{-i}(R[T]; G; \mathscr{F})$).

PROOF. Only the fact that $[m][n] = [mn]$ needs verification. But a simple computation shows that

$$B(G \times T; \mathscr{F} \times \{1\}) \xrightarrow{\mathrm{Res}} B(G \times \langle t^n \rangle; \mathscr{F} \times \{1\}) \xrightarrow{\mathrm{Ind}} B(G \times T; \mathscr{F} \times \{1)$$

$$\xrightarrow{\mathrm{Res}} B(G \times \langle t^m \rangle; \mathscr{F} \times \{1\}) \xrightarrow{\mathrm{Ind}} B(G \times T; \mathscr{F} \times \{1\})$$

and

$$B(G \times T; \mathscr{F} \times \{1\}) \xrightarrow{\mathrm{Res}} B(G \times \langle t^{mn} \rangle; \mathscr{F} \times \{1\}) \xrightarrow{\mathrm{Ind}} B(G \times T; f \times \{1\})$$

are naturally equivalent. Thus they induce the same homomorphism on $K_{-i}(R; G \times T; \mathscr{F} \times \{1\})$.

THEOREM B. $K_{-i}(R[T]; G; \mathscr{F})^{\text{inv N.}} = K_{-i-1}(R; G; \mathscr{F})$.

PROOF. By Corollary 4.1 we have

$$K_{-i}(R[T]; G; \mathscr{F}) \cong K_{-i}(R; G; \mathscr{F}) \oplus K_{-i-1}(R; G; \mathscr{F}) \oplus 2\mathrm{Nil}_{-i-1}(R; G; \mathscr{F}).$$

We consider the action on each component. First note that

$$\mathrm{Res}_{G \times T}^{G \times \langle t^n \rangle}(X \wedge T^+) \cong \bigvee_{i=1}^{n} (X \wedge \langle t^n \rangle^+)$$

and that

$$(4.9) \qquad \bigvee_{i=1}^{n} \langle t^n \rangle_i \to T, \qquad (t_i^n \mapsto t^{n+i}) \qquad \text{is an } \langle t^n \rangle\text{-isomorphism.}$$

Given an ordered coproduct of n identical objects we will denote by J_n the map wich map the ith component to the $(i+1)$th by the identity map, the nth component is mapped to $*$. $\hat{J}_n$ is the map sending the ith component to the $(i-1)$th by the identity map and the 1st component to 0. Using this notation it follows that

$$\mathrm{Res}(1 \wedge 1) = I_n, \quad \mathrm{Res}(1 \wedge t) = J_n + t^n \hat{J}_n^{n-2} \text{ and } \mathrm{Res}(1 \wedge t^{-1}) = \hat{J}_n + t^{-n} J_n^{n-2}.$$

Also, $\mathrm{Ind}_{G \times \langle t^n \rangle}^{G \times T}$ just means identifying t^n with t.

(i) If $(X, a) \in \mathrm{Aut}(R \otimes B(G; \mathscr{F}))_{i+1}$ consider it as an element in $\mathrm{Aut}(R[T] \otimes B(G; \mathscr{F}))_{i+1}$. Then

$$\begin{aligned}[n][X, a] &= \Phi^{-1}\mathrm{Ind}\,\mathrm{Res}\,\Phi[X, a] = \Phi^{-1}\mathrm{Ind}\,\mathrm{Res}[X \wedge T^+, a \wedge 1] \\ &= \Phi^{-1}\mathrm{Ind}[n(X \wedge \langle t^n \rangle^+), (a \wedge 1)I_n] = \Phi^{-1}(n[X \wedge T^+, a \wedge 1]) \\ &= n[X, a].\end{aligned}$$

(ii) If $(X, p) \in \mathrm{Proj}(R \otimes B(G; \mathscr{F}))_{i+1}$, then its image in $K_{-i}(R[T]; G; \mathscr{F})$ is $[X, 1-p-tp]$ and

$$\begin{aligned}[n][X, 1-p-tp] &= \Phi^{-1}\mathrm{Ind}\,\mathrm{Res}[X \wedge T^+, (1-p) \wedge 1 - p \wedge t] \\ &= \Phi^{-1}\mathrm{Ind}[n(X \wedge \langle t^n \rangle^+), ((1-p) \wedge 1)I_n - (p \wedge 1)J_n - (p \wedge t^n)\hat{J}_n^{n-2}] \\ &= [nX, (1-p)I_n - pJ_n - pt\hat{J}_n^{n-2}).\end{aligned}$$

The automorphisms $I_n - pJ_n$ and $I_n - \hat{J}_n$ have trivial torsion. Multiplying $(1-p)I_n - pJ_n - pt\hat{J}_n^{n-2}$ on the left by $(I_n - pJ_n)^{-1}(I_n - \hat{J}_n)$ produces an upper triangular matrix which is immediately seen to have torsion equal to $[X, 1-p-tp]$.

(iii) If $(X, v) \in \mathrm{Nil}(R \otimes B(G; \mathscr{F}))_{i+1}$, then its images by the two embeddings of Nil is $[X, (1-tv)(1+v)^{-1}]$ and $[X, (1-t^{-1}v)(1+v)^{-1}]$, respectively.

$$[n][X, (1-tv)(1+v)^{-1}] = [n][X, 1-tv] + [X, (1+v)^{-n}],$$

by (ii), since $[X, (1+v)^{-1}] \in K_{-i}(R; G; \mathscr{F})$. As in (ii) we get

$$[n][X, 1-tv] = [nX, I_n - vJ_n - vt\hat{J}_n^{n-2}].$$

The automorphism $I_n - vJ_n$ has trivial torsion and multiplying $I_n - vJ_n - vt\hat{J}_n^{n-2}$ by $(I_n - vJ_n)^{-1}$ yields an upper triangular matrix the torsion of which is immediately seen to be equal to $[X, 1-tv^n]$.

The same type of calculations shows that

$$[n][X, (1-t^{-1}v)(1+v)^{-1}] = [X, (1-t^{-1}v^n)(1+v)^{-n}].$$

Note that this implies that if n is greater than the nilpotence index of v, then $[n][X, (1-t^{\pm 1}v)(1+v)^{-1}]$ is contained in $K_{-i-1}(R; G; \mathscr{F})$. It follows that

$$K_{-i}(R[T]; G; \mathscr{F})^{\mathrm{inv}\ \mathbf{N}} \cong K_{-i-1}(R; G; \mathscr{F}).$$

Let us summarize the action on the components as follows

$$\begin{aligned}
[n]([X, a], 0, 0, 0) &= (n[X, a], 0, 0, 0)\\
[n](0, [X, p], 0, 0) &= (0, [X, p], 0, 0)\\
[n](0, 0, [X, v], 0) &= ([X, (1+v)^{-n}(1+v^n)], 0, [X, v^n], 0)\\
[n](0, 0, 0, [X, v]) &= ([X, (1+v)^{-n}(1+v^n)], 0, 0, [X, v^n]).
\end{aligned}$$

The following corollary is immediate by induction

COROLLARY 4.10. $K_{-i}(R; G; \mathscr{F}) \cong K_1(R[T^{i+1}]; G; \mathscr{F})^{\mathrm{inv}\ \mathbf{N}^{i+1}}$

REMARK. Using the map $R[T] \to R[T], t \mapsto t^n$, $R[T]$ can be conceived as an $R[T]$-module; multiplication by t is given by $t \cdot p(t) = t^n p(t)$. $R[T]$ decomposes into n-copies of $R[T]$. Thus one can construct an action of $\mathbf{N}$ on $K_1(R[T] \otimes_R \mathscr{D})$ and prove that

$$K_{-i}(\mathscr{D}) \cong K_1(R[T^{i+1}] \otimes_R \mathscr{D})^{\text{inv } N^{i+1}},$$

where $\mathscr{D}$ is an R-category.

THEOREM C. $K_{-i}(R;G;\mathscr{F}) \cong \sum_{(H)\in\mathscr{F}}^{\oplus} K_{-i}(R[NH/H])$.

PROOF. (cf. [5, Theorem 1.18]). If $\mathscr{F}_1$ is infinite then

$$B(G;\mathscr{F}_1) = \bigvee_{(H)\in\mathscr{F}_1} B(G;(H)).$$

This follows from the fact that $B(G;\mathscr{F}_1)(G/F_1^+, G/F_2^+) = \{*\}$, unless F_1 is subconjugated to $F_1, F_1 \lesssim F_2$; the wedge is ordered from above by this relation. It follows from Proposition (3.7) that

$$K_{-i}(R;G;\mathscr{F}_1) \cong \sum_{(H)\in\mathscr{F}_1}^{\oplus} K_{-i}(R;G;(H)).$$

But $B(G;(H))$ is equivalent to $B(NH/H;\{1\})$ (by the functor $G^+ \wedge_{NH} -$) and $R \otimes B(NH/H;\{1\})$ is equivalent to the category of finitely generated free $R[NH/H]$-modules. Thus

$$K_{-i}(R;G;\mathscr{F}_1) \cong \sum_{(H)\in\mathscr{F}_1}^{\oplus} K_{-i}(R[NH/H]).$$

As $K_{-i}(R;G;\mathscr{F})$ is a component of $K_1(R[T^{i+1}];G;\mathscr{F})$, it follows that $K_{-i}(R;G;\mathscr{F})$ is generated by objects $(X,a)\in \mathrm{Ob}\,\mathrm{Aut}(R\otimes B(G;\mathscr{F}))_{i+1}$, where the number of orbit types occuring in the $\mathbf{Z}^{i+1}$ graded G-set X is finite. Hence

$$\varinjlim_{\substack{\mathscr{F}_1\subset\mathscr{F}\\ \mathscr{F}_1 \text{ finite}}} K_{-i}(R;G;\mathscr{F}_1) \cong K_{-i}(R;G;\mathscr{F}).$$

This proves the theorem.

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