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OPERATOR MEANS, FIXED POINTS, AND THE NORM CONVERGENCE OF MONOTONE APPROXIMANTS

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0. Introduction.

Means of operators have found considerable application in various areas of mathematics and science, particularly in network theory (see [1], [2], [5], [8], [21], [22]). In this paper, we focus on the most fundamental sort of mean of two operators, namely the parallel sum. In practice one must usually compute the parallel sum $A:B$ of A and B by some limiting or iterative technique, e.g., by

$$A:B = \lim_{\varepsilon \downarrow 0} (A+\varepsilon)(A+B+2\varepsilon)^{-1}(B+\varepsilon),$$

where the limit is taken in the strong operator topology. Unfortunately, this net, like most other useful iterative or limiting schemes, does not in general converge in the norm, as was shown by Anderson and Trapp [7]. We prove below a simple proposition (Proposition 2.1) about the algebra $\mathscr{B}(\mathscr{H})$ (all bounded linear operators on a complex Hilbert space $\mathscr{H}$) which will enable us to describe, for many sets of pairs (A,B), precisely when this convergence is in norm. We show in particular that this norm convergence must hold whenever A and B commute, and that if A and B have closed ranges, then it holds if and only if the range of $A+B$ is closed. We further show that for a large class of monotone approximation schemes we get norm convergence of any one of these schemes to $A:B$ if and only if the scheme above converges in norm. Our analysis also establishes analogous results for the geometric mean $A \# B$ of A and B.

Interestingly, the property that $A:B$ is such a norm limit is related to the representation theory of $\mathscr{B}(\mathscr{H})$. Indeed, it turns out (as we show below)

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that this property is equivalent to the property that computing parallel sums commutes with every representation of $\mathscr{B}(\mathscr{H})$, or equivalently that is commutes with the universal representation. This in turn has some interesting implications for projections. It is known (see below) that if E and F are (self-adjoint) projections on $\mathscr{H}$, then $E:F$ is one half the infimum of E and F. Since the results quoted above for operators with closed ranges are clearly applicable to projections, we see that the infimum of E and F, computed in $\mathscr{B}(\mathscr{H})$, agrees with the infimum of E and F, computed in $\mathscr{B}(\mathscr{H})^{dd}$, if and only if $E+F$ has closed range in $\mathscr{H}$. (We use the notation X^d for the Banach space dual of X.) Since norm convergence of $(E+\varepsilon)(E+F+2\varepsilon)^{-1}(F+\varepsilon)$ is preserved by a *-isomorphism of $\mathscr{B}(\mathscr{H})$ into some $\mathscr{B}(\mathscr{K})$, it follows trivially that there can be no formula for the infimum of E and F which is invariant under *-isomorphism. In particular, $\inf(E,F)$ cannot be given by a simple algebraic formula, except in the case where $E+F$ has closed range. Now when $E+F$ has closed range, and hence has a Moore-Penrose inverse $(E+\mathrm{F})^\dagger$, such an algebraic formula was given by Anderson-Schreiber [6]

$$E:F = E(E+F)^\dagger F,$$

so

$$\inf(E,F) = 2E(E+F)^\dagger F.$$

Our results thus show that the Anderson-Schreiber formula for the infimum cannot in fact be generalized in any purely algebraic way to *any* class of pairs of projections which is larger than the one which they treated.

Our last section applies the results of the previous sections to the computation of the geometric mean and to some further iterations, including the ladder iteration which was considered in [5].

In closing this introduction, the authors would like to point out that although their fundamental proposition (Proposition 2.1) is here applied principally to the computation of parallel sums and geometric means, this proposition has potentially a much greater applicability. In particular, it can be used in the following way. Suppose $\mathscr{R}$ is a (perhaps non-linear) operator on $\mathscr{B}(\mathscr{H})$ which is given by some strong operator continuous simple algebraic formula, invariant under *-isomorphisms of $\mathscr{B}(\mathscr{H})$ into other *-algebras of the form $\mathscr{B}(\mathscr{K})$. Suppose too that $\mathscr{R}$ can be shown to have in $\mathscr{B}(\mathscr{H})$ (and in $\mathscr{B}(\mathscr{K})$, by the same token) at most one positive fixed point. Suppose finally that $\mathscr{R}$ is monotone in the sense that $\mathscr{R}(T_1) \geqq \mathscr{R}(T_2)$ whenever $T_1 \geqq T_2 \geqq 0$. Then Proposition 2.1 implies that $\{\mathscr{R}^k(0)\}_{k=1}^{\infty}$ converges in norm in $\mathscr{B}(\mathscr{H})$ (necessarily to a fixed point of $\mathscr{R}$) if and only if $\{\mathscr{R}^k(0)\}_{k=1}^{\infty}$ is bounded. In particular, let $\mathscr{R}(P) = F^*PF+I$, where F is a fixed but arbitrary operator in $\mathscr{B}(\mathscr{H})$, and where I is the identity operator on $\mathscr{B}(\mathscr{H})$. It is relatively easy to show that

$F^*PF+I = \mathscr{R}(P) = P \geqq 0$ can hold for only one P, namely the strong limit of

$$\mathscr{R}^k(0) = \sum_{i=0}^{k-1} (F^*)^i F^i.$$

By Proposition 2.1, we see that $\sum_{i=0}^{\infty} (F^*)^i F^i$ is norm convergent if and only if $\{\sum_{i=1}^{k-1} F(*)^i F^i\}$ is bounded, i.e., if and only if this same series is strong operator convergent. (This is an important result of Zabczyk [26].) More generally, if $\mathscr{R}$ is the Riccati operator, as in Zabczyk [26], then by a similar argument, one can get Zabczyk's result that $\mathscr{R}$ has a positive fixed point in $\mathscr{B}(\mathscr{H})$ if and only if $\{\mathscr{R}^k(0)\}_{k=1}^{\infty}$ is norm bounded in $\mathscr{B}(\mathscr{H})$ if and only if $\{\mathscr{R}^k(0)\}_{k=1}^{\infty}$ converges in norm (to that unique fixed point). These results are important in the study of the stabilization of discrete-time systems; see for example [16], [17], [26].

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1. The parallel sum.

In this section we give a brief survey of the elementary theory of the parallel sum. Our discussion will be limited to those results which we shall actually use, or which place our later results in an appropriate context, and the discussion here is by no means complete. All the results in this section are taken from the papers of Anderson and Schreiber [6], Anderson and Trapp [7], R. G. Douglas [10], and Fillmore and Williams [11], though some of the results had earlier appeared in other sources. For a recent survey, see [25]. Throughout this section, $\mathscr{H}$ will denote a complex Hilbert space, and A and B will be bounded linear operators on $\mathscr{H}$. Recall that $A \geqq B$ means that $A-B$ is positive.

If A and B denote positive operators, and if $A+B$ is invertible in $\mathscr{B}(\mathscr{H})$, we define the parallel sum $A:B$ by

$$A:B = A(A+B)^{-1}B.$$

Note that if A and B are invertible in $\mathscr{B}(\mathscr{H})$, then $A:B = (A^{-1}+B^{-1})^{-1}$. If $A+B$ is not invertible we set

$$A:B = \lim_{\varepsilon \downarrow 0} (A+\varepsilon I):(B+\varepsilon I).$$

This last limit exists when taken in the strong operator topology [7].

The following two theorems provide formulas for the parallel sum.

THEOREM 1.1 (Anderson-Trapp [7], Fillmore-Williams [11]). *Let* $A \geqq 0$ *and* $B \geqq 0$. *Then there are unique operators* C *and* D *which satisfy*

$$A^{1/2} = (A+B)^{1/2}C,$$

$$B^{1/2} = (A+B)^{1/2}D,$$

$$\ker\ C^* \supseteqq \ker (A+B)^{1/2},$$

and

$$\ker\ D^* \supseteqq \ker (A+B)^{1/2}.$$

Moreover

$$A : B = A^{1/2}C^*DB^{1/2}.$$

THEOREM 1.2. *If* $A+B$ *is invertible, then*

$$A : B = (A+B)^{1/2}(\tilde{A} : \tilde{B})(A+B)^{1/2}$$

where

$$\tilde{A} = (A+B)^{-1/2}A(A+B)^{-1/2}$$

and

$$\tilde{B} = (A+B)^{-1/2}B(A+B)^{-1/2}.$$

PROOF. Compute.

PROPOSITION 1.3. *If* $A+B$ *has closed range, then the net* $\{(A+\varepsilon) : (B+\varepsilon)\}_{\varepsilon>0}$ *converges in norm to* $A : B$ *as* $\varepsilon \downarrow 0$.

PROOF. We may assume that the range of $A+B$ is the whole of $\mathscr{H}$, so that $A+B$ is invertible. The result now follows from the norm continuity of addition, multiplication and inversion in $\mathscr{B}(\mathscr{H})$.

REMARK 1.4. Let C and D be as in Theorem 1.1 Clearly $C^*(A+B)^{1/2} = A^{1/2}$, so that for any reasonably well-behaved notion $A \to A^\dagger$ of generalized inverse, we must have

$$C^* = A^{1/2}((A+B)^\dagger)^{1/2}$$

whenever $(A+B)^\dagger$ exists. Similarly,

$$D = ((A+B)^\dagger)^{1/2}B^{1/2}$$

whenever $(A+B)^\dagger$ exists. In particular, when Range $(A+B)$ is closed, i.e., when the Moore-Penrose inverse $(A+B)^\dagger$ of $A+B$ exists, we have

$$A:B = A(A+B)^\dagger B.$$

A routine computation shows that if $A+B$ is invertible, then $CC^* = \tilde{A}$ and $DD^* = \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are as in Theorem 1.2.

The following theorem summarizes the fundamental properties of the parallel sum.

THEOREM 1.5 (Anderson-Trapp [7]). *Let A, B and C be positive operators. Then*

(1) *$A:B$ is monotone in each variable*;
(2) $A:B = B:A$;
(3) $(A:B):C = A:(B:C)$;
(4) $A:B \leqq A$, $A:B \leqq B$;
(5) $\langle A:Bc, c\rangle = \inf_{x+y=c}\langle Ax, x\rangle + \langle By, y\rangle$;
(6) *if A and B are projections, then $2(A:B) = A \wedge B$, the infimum of A and B.*

We shall have need in Sections 3 and 4 for the following results on the range of $A:B$.

THEOREM 1.6 (see [7]). *If A and B are positive operators then*

(1) range $A \cap$ range $B \subseteqq$ rang $A:B$;
(2) range $A^{1/2} \cap$ range $B^{1/2}$ = range $(A:B)^{1/2}$;
(3) *if A and B have closed ranges, then* range $A \cap$ range B = range $A:B$.

2. Representations of $\mathscr{B}(\mathscr{H})$ and invariance of parallel sums.

The purpose of this section is to prove a result (Corollary 2.2 to Proposition 2.1) which will enable us, in principle, to decide when $A:B$ is a *norm* limit of $(A+\varepsilon:B+\varepsilon)$. We shall make use of the fact that $\mathscr{B}(\mathscr{H})$ is a C*-algebra (see for example [9] for the necessary facts about C*-algebras). Throughout this paper we shall write $\mathscr{A}^d$ for the Banach space dual of a C*-algebra (or Banach space) $\mathscr{A}$.

Let $\mathscr{A}$ be any C*-algebra, and let $\{x_\gamma\}$ be a net in $\mathscr{A}$. Let $x \in \mathscr{A}^{dd}$. Then $\{x_\gamma\}$ converges weak* to x if and only if $x_\gamma \to x$ pointwise on the set

$S(\mathscr{A})$ of all states of $\mathscr{A}$. Indeed each ϕ in $\mathscr{A}^d$ can be written in the form $\phi = \phi_1 - \phi_2 + i\phi_3 - i\phi_4$, where each ϕ_i is a positive element of $\mathscr{A}^d$, and where

$$\max\{\|\phi_i\| : 1 \leqq i \leqq 4\} \leqq \|\phi\|.$$

Since moreover

$$\|x - x_\gamma\| = \sup\{|\phi(x - x_\gamma)| : \phi \in \mathscr{A}^d, \|\phi\| = 1\},$$

it follows that $\{x_\gamma\}$ converges in norm to x if and only if $x_\gamma \to x$ uniformly on $S(\mathscr{A})$.

If $\mathscr{A}$ has an identity I, then $S(\mathscr{A})$ is weak * compact in $\mathscr{A}^d$. Suppose now that $x_\gamma \geqq 0$ for each γ, and that for each $\phi \in S(\mathscr{A})$, the net $\{\phi(x_\gamma)\}$ is monotone and bounded. Then clearly $\{\phi(x_\gamma)\}$ converges for each $\phi \in \mathscr{A}^d$, and the limiting function $\phi \to \lim_\gamma \phi(x_\gamma)$ defines an element x of $\mathscr{A}^{dd}$, by the uniform boundedness principle. By Dini's Theorem and the remarks above, we have that $\{x_\gamma\}$ is norm convergent in $\mathscr{A}^{dd}$ whenever this limiting function is weak * continuous on $S(\mathscr{A})$, which is certainly the case if x is an element of $\mathscr{A}$. We thus have the following result.

PROPOSITION 2.1. *Let $\mathscr{A}$ be a C*-algebra with identity, and let $\{x_\gamma\}$ be a monotone norm-bounded net of elements of $\mathscr{A}$. Then $\{x_\gamma\}$ is norm Cauchy in $\mathscr{A}$ if and only if the weak* limit x in $\mathscr{A}^{dd}$ of $\{x_\gamma\}$ is actually an element of $\mathscr{A}$.*

PROOF. We may assume $\{x_\gamma\}$ has an initial index γ_0. Replace each x_γ by $x_\gamma - x_{\gamma_0}$, or by $x_{\gamma_0} - x_\gamma$, and apply the remarks above.

Now the double dual $\mathscr{A}^{dd}$ of a C*-algebra $\mathscr{A}$ is again a C*-algebra. Indeed, one can prove the following result (see [9, Section 12.1]). Let $\mathscr{A}$ be a C*-algebra. Then there exist a Hilbert space $\mathscr{H}_u$ and an isometric *-homomorphism π_u of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})_u$ with the following properties:

(1) π_u extends to an isometric linear map of $\mathscr{A}^{dd}$ onto the weak (or equivalently, strong) operator closure B of $\pi_u(\mathscr{A})$ in $\mathscr{B}(\mathscr{H}_u)$; here we embed $\mathscr{A}$ into $\mathscr{A}^{dd}$ in the usual way;
(2) B is a norm closed *-stable subalgebra of $\mathscr{B}(\mathscr{H}_u)$ containing the identity operator on $\mathscr{H}_u$;
(3) if $\mathscr{A}^{dd}$ is given its weak * topology (as the Banach space dual of $\mathscr{A}^d$), and if B is given the relative topology from the weak operator topology on $\mathscr{B}(\mathscr{H}_u)$, then the extended $\pi_u : \mathscr{A}^{dd} \to B$ is homeomorphism;
(4) every weak * continuous positive linear functional ϕ on $\mathscr{A}^{dd}$ corresponds under π_u to the restriction to B of some vector state $T \to \langle Tx, x\rangle$ of $\mathscr{B}(\mathscr{H}_u)$.

REMARK 2.1. A *-homomorphism of a C*-algebra $\mathscr{A}$ into $\mathscr{B}(\mathscr{K})$, where $\mathscr{K}$ is a complex Hilbert space, is called a *representation of* $\mathscr{A}$ *on* $\mathscr{K}$. The map π_u constructed in [9] is called the *universal representation of* $\mathscr{A}$, and has the universal property that every representation of $\mathscr{A}$ factors through π_u (see [9, Chapter 12]).

When we apply the results above with $\mathscr{A} = \mathscr{B}(\mathscr{H})$, we obtain the following versions of Proposition 2.1.

COROLLARY 2.2. *Let P_γ be a monotone, norm-bounded net of operators in $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a complex Hilbert space. Let P be the weak operator limit in $\mathscr{B}(\mathscr{H})^{dd}$ of P_γ. Then P lies in $\mathscr{B}(\mathscr{H})$ if and only if P_γ is norm Cauchy in $\mathscr{B}(\mathscr{H})$.*

COROLLARY 2.3. *Let P_γ and P be as in Corollary 2.2. Let P' be the weak operator limit in $\mathscr{B}(\mathscr{H})$ of the net $\{P_\gamma\}$. Then $\{P_\gamma\}$ is norm Cauchy in $\mathscr{B}(\mathscr{H})$ if and only if $P' = P$.*

Now consider two positive operators A and B in $\mathscr{B}(\mathscr{H})$. We have

$$A:B = \lim_{\varepsilon\to 0^+} (A+\varepsilon : B+\varepsilon) = \inf_{\varepsilon>0} (A+\varepsilon : B+\varepsilon),$$

and we recall that the net $A+\varepsilon : B+\varepsilon$ is a monotone decreasing net of positive operators. Thus, we have

$$A:B = \operatorname*{norm-limit}_{\varepsilon\to 0^+} (A+\varepsilon, B+\varepsilon)$$

if and only if

$$\pi_u(A):\pi_u(B) = \pi_u(A:B),$$

since for $\varepsilon > 0$, we have

$$\begin{aligned}\pi_u(A+\varepsilon : B+\varepsilon) &= \pi_u((A+\varepsilon)(A+B+2\varepsilon)^{-1}(B+\varepsilon)) \\ &= (\pi_u(A)+\varepsilon)(\pi_u(A)+\pi_u(B)+2\varepsilon)^{-1}(\pi_u(B)+\varepsilon) \\ &= (\pi_u(A)+\varepsilon : \pi_u(B)+\varepsilon).\end{aligned}$$

That is, $A:B$ is a norm limit of $A+\varepsilon : B+\varepsilon$ if and only if we can safely compute the parallel sum of A and B either in $\mathscr{B}(\mathscr{H})$ or in $\mathscr{B}(\mathscr{H}_u)$, without introducing any ambiguity. Since every representation of $\mathscr{B}(\mathscr{H})$ factors through the universal representation, it follows that $A:B$ is a norm limit of $(A+\varepsilon):(B+\varepsilon)$ if and only if the computation of $A:B$ is independent of representation. Now any representation π is order preserving, so $\{\pi(A+\varepsilon):\pi(B+\varepsilon)\}_{\varepsilon>0}$ is also monotone net, and

$$\pi(A:B) \leqq \pi(A+\varepsilon):\pi(B+\varepsilon)$$

for each $\varepsilon > 0$. Thus

$$\pi(A):\pi(B) = \inf_{\varepsilon>0} \pi(A+\varepsilon):\pi(B+\varepsilon) \geqq \pi(A:B).$$

We summarize in the next corollary.

COROLLARY 2.4. *Let A and B be positive operators on a complex Hilbert space $\mathscr{H}$, and let π be a representation (not necessarily normal) of $\mathscr{B}(\mathscr{H})$ (or any of von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ which contains A and B). Then $\pi(A):\pi(B) \geqq \pi(A:B)$. Moreover, if $\pi = \pi_u$, then equality holds if and only if $(A+\varepsilon):(B+\varepsilon) \to_{\varepsilon\downarrow 0} A:B$ in norm.*

DEFINITION. Let A and B be positive operators on a complex Hilbert space $\mathscr{H}$. We say that the pair (A, B) is *uniformly and universally parallelizable* (or UUP for short) if the net $(A+\varepsilon):(B+\varepsilon)$ converges in norm.

We shall show in the next section that if E and F are (self-adjoint) projections on $\mathscr{H}$, then (E, F) is UUP if and only if $(E+F)(\mathscr{H})$ is closed in $\mathscr{H}$. It follows that there are many non-UUP pairs of operators, even when the operators in the pair are required to have closed range. On the other hand, the next result shows that even with no assumptions on the ranges of A and B, there are many UUP-pairs (A, B).

THEOREM 2.5. *Let A and B be positive operators on a complex Hilbert space $\mathscr{H}$. If A and B commute, then (A, B) is* UUP.

PROOF. If A and B commute, then it follows from Theorem 1.1 above that C and D, as in Theorem 1.1, are the unique operators satisfying the system

$$\begin{aligned} A^{1/2} &= (A+B)^{1/2}C, \\ B^{1/2} &= (A+B)^{1/2}D, \\ C &= C^*, \\ D &= D^*. \end{aligned}$$

This system of equations is invariant under the universal representation. Since

$$A : B = A^{1/2}C^{*}DB^{1/2},$$

we have $\pi_u(A : B) = \pi_u(A) : \pi_u(B)$, by uniqueness.

REMARK 2.6. The von Neumann algebra generated by a commuting pair (A, B) is commutative and contains $A : B$, so a straight-forward spectral theoretic proof of Theorem 2.5 can also be given. Note however that by Proposition 2.1, (A, B) is UUP if and only if $A : B$ lies in the (smaller) C*-algebra generated by A, B, and I.

REMARK 2.7. Suppose $\{T_\gamma\}$ is any monotone approximation which converges strong operator to $A : B$, and suppose the computation of T_γ is independent of representation, i.e., $(\pi(T))_\gamma = \pi(T_\gamma)$ for all γ and all π. Then by arguments analogous to those above, we have that T_γ converges in norm if and only if

$$\pi_u(A : B) = \pi_u(A) : \pi_u(B).$$

Thus for any monotone approximation scheme which is independent of representation, we have norm convergence to $A : B$ if and only if (A, B) is UUP.

3. The parallel sum for operators with closed range.

In this section we show that if A and B are positive operators in $\mathscr{B}(\mathscr{H})$, each having closed range, then (A, B) is of class UUP (i.e., the computation of $A : B$ is independent of representation) if and only if $A + B$ has closed range. Observe that if $A + B$ has closed range, then by Proposition 1.3 above, the net $\{(A+\varepsilon I) : (B+\varepsilon I)\}_{\varepsilon > 0}$ converge in norm to $A : B$, so that $A : B$ is of class UUP. (This part of the argument makes no use of the assumption that A and B have closed ranges.) Thus it remains only to show that if A and B have closed ranges, and if (A, B) is of class UUP, then $A + B$ has closed range. For the remainder of this paper we shall denote by $\hat{T}$ the image of $T \in \mathscr{B}(\mathscr{H})$ under the universal representation (see Section 2). We shall also denote by E_T the (orthogonal) projection of $\mathscr{H}$ onto the closure of the range of T.

LEMMA 3.1. *If $A \geqq 0$, then the following four assertions are mutually equivalent.*

(1) *A has closed range;*
(2) *there exist a projection E in $\mathscr{B}(\mathscr{H})$ and strictly positive real numbers α and β such that $\alpha E \leqq A \leqq \beta E$;*

(3) *$\hat{A}$ has closed range;*
(4) $(E_A)\hat{} = E_{\hat{A}}$.

Moreover, if (2) *holds, then the projection E necessarily coincides with E_A.*

PROOF. (1)$\Leftrightarrow$(2) and the coincidence of E with E_A are well-known and easy to prove. To establish the remaining equivalences, it suffices to assume that $0 \leqq A \leqq I$. Now for $0 \leqq A \leqq I$, E_A is the monotone limit in the strong operator topology of the sequence $\{A^{1/n}\}_{n \geq 1}$. Similarly

$$E_{\hat{A}} = \lim_n \hat{A}^{1/n}$$

in the strong operator topology on $\mathscr{B}(\mathscr{H})^{dd}$. By Corollaries 2.2 and 2.3 above, we have $(E_A)\hat{} = E_{\hat{A}}$ if and only if $A^{1/n}$ converges in norm, and it follows also that $(E_A)\hat{} = E_{\hat{A}}$ if and only if $\hat{A}^{1/n}$ converges in norm. By spectral theory and the equivalence of (1) and (2), we have that A has closed range if and only if $A^{1/n}$ converges in norm, and similarly that $\hat{A}$ has closed range if and only if $\hat{A}^{1/n}$ converges in norm. Thus we have (1)$\Leftrightarrow$(4) and (3)$\Leftrightarrow$(4). and the lemma is established.

PROPOSITION 3.2. *Let P and Q be projections in $\mathscr{B}(\mathscr{H})$. Then the following assertions are mutually equivalent.*

(1) *$P+Q$ has closed range;*
(2) $(P \vee Q)\hat{} = \hat{P} \vee \hat{Q}$;
(3) $(P \wedge Q)\hat{} = \hat{P} \wedge \hat{Q}$;
(4) $(P : Q)\hat{} = \hat{P} : \hat{Q}$.

PROOF. We have $E_{P+Q} = P \vee Q$ and $E_{(P+Q)\hat{}} = \hat{P} \vee \hat{Q}$. Thus the equivalence of (1) and (2) follows from Lemma 3.1. The equivalence of (3) and (4) follows from the fact that $P : Q = 1/2(P \wedge Q)$. By Corollary 1.3, (1) implies (4), so it suffices to show (3) implies (2). Now (3) implies

$$((I-P) \vee (I-Q))\hat{} = (I-P)\hat{} \vee (I-Q)\hat{}.$$

Applying the implications (2)$\Rightarrow$(1)$\Rightarrow$(3) to the pair $(I-P, I-Q)$ yields

$$((I-P) \wedge (I-Q))\hat{} = (I-P)\hat{} \wedge (I-Q)\hat{},$$

which is equivalent to (2).

THEOREM 3.3 *Let A and B be positive operators in $\mathscr{B}(\mathscr{H})$, each of which has* *closed range. Then the following properties are mutually equivalent:*

(1) $A+B$ *has closed range*;
(2) $(A:B)\hat{} = \hat{A}:\hat{B}$;
(3) (A,B) *is of class* UUP.

Proof. By the results of Section 2 and the remarks in the first paragraph of this section, it suffices to prove that (2) implies (1). Since A and B have closed ranges, there exist $\alpha, \beta > 0$ such that

$$\alpha E_A \leqq A \leqq \beta E_A$$

and

$$\alpha E_B \leqq B \leqq \beta E_B.$$

It follows from the monotonicity of the parallel sum and form (6) of Theorem 1.5 above that there exist $\alpha', \beta' > 0$ such that

$$\alpha'(E_A \wedge E_B) \leqq A:B \leqq \beta'(E_A \wedge E_B).$$

Hence $A:B$ has closed range. By (1)$\Leftrightarrow$(4) of Lemma 3.1, we have

$$E_{(A:B)\hat{}} = (E_{A:B})\hat{}.$$

By this last and (3) of Theorem 1.6, we have

$$E_{(A:B)\hat{}} = (E_{A:B})\hat{} = (E_A \wedge E_B)\hat{}.$$

But by (1)$\Leftrightarrow$(3) of Lemma 3.1, $\hat{A}$ and $\hat{B}$ also have closed ranges, so in similar fashion Lemma 3.1 and Theorem 1.6 imply

$$E_{\hat{A}:\hat{B}} = E_{\hat{A}} \wedge E_{\hat{B}} = (E_A)\hat{} \wedge (E_B)\hat{}.$$

Thus (2) implies that

$$(E_A \wedge E_B)\hat{} = E_{(A:B)\hat{}} = E_{\hat{A}:\hat{B}} = (E_A)\hat{} \wedge (E_B)\hat{}.$$

Thus by Proposition 3.2, the range of $E_A + E_B$ is closed whenever (2) holds. Now by Corollary 3 to Theorem 2.2 of [11], we have that $A+B$ has closed range $\Leftrightarrow E_A+E_B$ has closed range. Thus (2) implies that $A+B$ has closed range, i.e., (2)$\Rightarrow$(1).

Remark 3.4. Suppose the range of $A+B$ is not closed. Then it is possible to construct more or less explicitly a state of $\mathscr{B}(\mathscr{H})^{dd}$ which separates $E_{(A+B)\hat{}}$

from $(E_{A+B})\hat{}$. Since Range $(A+B)$ is not closed, there exists in Range $(A+B)$ a net $\{x_\gamma\}$ of unit vectors such that $\langle (A+B)x_\gamma, x_\gamma\rangle \to 0$. Let ω_γ denote the state $T \to \langle Tx_\gamma, x_\gamma\rangle$ on $\mathscr{B}(\mathscr{H})$, and let ω be any weak* limit point of $\{\omega_\gamma\}$ in the state space of $\mathscr{B}(\mathscr{H})$. Then $\omega(E_{A+B}) = 1$ since each x_γ lies in $E_{A+B}(\mathscr{H})$. However,

$$\omega(A+B) = \lim_\gamma \omega_\gamma(A+B) = 0.$$

We may regard ω as a vector state on $\mathscr{B}(\mathscr{H})^{dd}$ (see [9, Section 12.1] or Section 2 above). It can be shown, by means of Theorem 1.1 above, that $\omega(A+B) = 0$ implies $\omega(E_{(A+B)\hat{}}) = 0$, so ω separates $(E_{A+B})\hat{}$ from $E_{(A+B)\hat{}}$. Note *that*

$$E_{A+B} = E_A \wedge E_B$$

and that

$$E_{(A+B)\hat{}} = E_{\hat{A}} \vee E_{\hat{B}} = (E_A)\hat{} \vee (E_B)\hat{}$$

whenever A and B have closed ranges. Thus ω separates the image of $E_A \vee E_B$ from the supremum of the images of E_A and E_B whenever A and B have closed ranges. In particular, if $A = P$ and $B = Q$ are projections, then ω separates $(P \vee Q)\hat{}$ from $\hat{P} \vee \hat{Q}$.

4. The geometric mean and other iterations.

The *geometric mean*, $A \# B$, of two invertible positive operators is defined by the formula

$$A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

In the non-invertible case we define

$$A \# B = \lim_{\varepsilon \downarrow 0} (A+\varepsilon I) \# (B+\varepsilon I),$$

where the limit is taken in the strong operator topology. It follows from the transformer inequality (see [4], [8], or [12]) that whenever $A+B$ is invertible, we have

$$A \# B = (A+B)^{1/2}(\tilde{A} \# \tilde{B})(A+B)^{1/2},$$

where $\tilde{A}$ and $\tilde{B}$ are as in Theorem 1.2 above. Note that if A and B commute (*but not otherwise*) then $A \# B = (AB)^{1/2}$, and hence the name geometric mean. The geometric mean of two positive operators has proved useful in a wide variety of contexts (see [5], [8], [24], [27]). Some of the properties of the geometric mean that we shall use are summarized in the following proposition, the proofs of the various parts of which may be found in [4], [8], [24].

PROPOSITION 4.1. *Let A and B be positive operators. Then*

(1) $A \# B = B \# A$.

(2) *If* $A_n \downarrow A$ *and* $B_n \downarrow B$ *strong operator, then* $A_n \# B_n \to A \# B$ *strong operator*

(3) *If* E *and* F *are projections then* $E \wedge F = E \# F$.

(4) *If* A *and* B *have closed ranges*, range $(A \# B) =$ range $A \cap$ range B.

It follows, just as in the case of the parallel sum, that if A and B commute, or if $A+B$ has closed range, then $A \# B$ is the norm limit of $(A+\varepsilon I) \# (B+\varepsilon I)$. Similarly, it follows from Proposition 2.1 and its corollaries that $A \# B$ is the norm limit of this net if and only if $(A \# B)\hat{} = \hat{A} \# \hat{B}$. Since $P \# Q = P \wedge Q$ when P and Q are projections, it also follows that for projections we have $(P:Q)\hat{} = \hat{P}:\hat{Q} \Leftrightarrow (P \# Q)\hat{} = \hat{P} \# \hat{Q} \Leftrightarrow P+Q$ has closed range. An argument very similar to that in Theorem 3.3 shows further that if A and B have closed ranges, then $(A \# B)\hat{} = \hat{A} \# \hat{B}$ implies that $A+B$ has closed range. In particular, we have the following result.

THEOREM 4.2. *Let $A \geqq 0$, $B \geqq 0$, and let A and B have closed ranges. Then*

$$\begin{aligned}(A \# B)\hat{} = \hat{A} \# \hat{B} &\Leftrightarrow (A:B)\hat{} = \hat{A}:\hat{B}\\ &\Leftrightarrow A+B \text{ has closed range}\\ &\Leftrightarrow A \# B \text{ is the norm limit of the net } \{(A+\varepsilon I) \# (B+\varepsilon I)\}_{\varepsilon>0}\\ &\Leftrightarrow (A,B) \text{ is of class UUP.}\end{aligned}$$

In [4] and [12] the following iteration was considered:

$$A_0 = A \quad B_0 = B, \tag{4.1}$$

$$A_{n+1} = (1/2)(A_n+B_n), \quad B_{n+1} = 2(A_n:B_n), \quad n \geqq 1.$$

In [4] and [12], it is also shown that the following result holds, where convergence is taken in the strong operator topology.

PROPOSITION 4.3. *The operators A_n and B_n satisfy*

(a) $A_n \geqq B_n$ *for* $n \geqq 1$;

(b) $A_n \# B_n = A \# B$ *for* $n \geqq 0$;

(c) $\{A_n\}$ *is monotonically decreasing and* $\{B_n\}$ *is monotonically increasing*;
(d) $A \# B = \lim A_n = \lim B_n$.

Thus the A_n and B_n bound $A \# B$ respectively above and below. The effectiveness of this bounding is greatly increased if the A_n and B_n should happen to converge in norm.

THEOREM 4.4 *Let A and B be positive operators with closed ranges, and suppose $A+B$ has closed range. Then A_n and B_n converge in norm to $A \# B$, where A_n and B_n are given by* (4.1) *above.*

PROOF. If Range $((A+B)$ is closed, then A_0 and B_0 are UUP, and we argue by induction. Now $A_1 = 1/2(A_0+B_0)$ has closed range by hypothesis, and $B_1 = 2(A_0 : B_0)$ has closed range by (3) of Theorem 1.6. By Proposition 4.3, $A_1 \geqq B_1$, so A_1+B_1 has closed range. Thus (A_1, B_1) is UUP, and by induction (A_n, B_n) is UUP. Thus we have

$$(A_n)\hat{} = (\hat{A})_n \quad \text{and} \quad (B_n)\hat{} = (\hat{B})_n$$

by Theorem 3.3. (Hence $(A_n)\hat{}$ denotes the iteration computed in $\mathscr{B}(\mathscr{H})$ and then lifted to $\mathscr{B}(\mathscr{H}_u)$, while $(\hat{A})_n$ denotes the iteration computed in $\mathscr{B}(\mathscr{H}_u)$.) By Proposition 4.3,

$$A \# B = \inf\{A_n\} = \sup\{B_n\},$$

and

$$\hat{A} \# \hat{B} = \inf\{\hat{A}_n\} = \sup\{\hat{B}_n\}.$$

It follows that $\hat{A} \# \hat{B} = (A \# B)\hat{}$. The norm convergence of A_n and B_n now follows from Corollary 2.3 to Proposition 2.1.

An analysis similar to that above may be applied to the ladder iteration considered in [5]. This iteration is defined as follows. Let A and B be given positive operators, let Range A be closed, let $\phi(X) = B : (A+X)$, and put

$$X_0 = A+B,$$

and

$$X_{n+1} = B : (A+X_n) = \phi(X_n), \quad n \geqq 0.$$

It was shown in [5] that this iteration converges monotonically down in the strong operator topology to the unique positive fixed point X_∞ of the function ϕ. It was also shown in [5] that

$$X_\infty = \tfrac{1}{2}((B \# (I+4A))-B).$$

REMARK 4.5. Note that we have assumed when defining the ladder iteration that A has closed range. The formula given above for X_∞ is not known to hold without this assumption on the range of A.

We now consider the norm convergence of the ladder iteration. Suppose A and B have closed ranges and that (A, B) is UUP, that is, $A+B$ has closed ranged. Then

$$(\hat{X})_0 = \hat{A}+\hat{B} = (A+B)\hat{} = (X_0)\hat{},$$

and $A+B = X_0$ has closed range.

Now

$$X_1 = B:(A+X_0).$$

But B has closed range, $A+X_0$ has closed range, and $B \leqq A+X_0$, so $B+A+X_0$ has closed range. Thus $(B, A+X_0)$ is UUP, so

$$(X_1)\hat{} = (B:(A+X_0))\hat{} = \hat{B}:(\hat{A}+\hat{X}_0) = (\hat{X})_1.$$

Continuing by induction we have

$$(X_n)\hat{} = \hat{X}_n.$$

Now in the strong operator topology, we have the (monotone) convergence

$$X_n \to \tfrac{1}{2}((B \# (A+4I))-B).$$

Since $A+4I$ is invertible, it follows from the definition of the geometric mean that

$$\tfrac{1}{2}((\hat{B} \# (\hat{A}+4I))-\hat{B}) = \tfrac{1}{2}((B \# (A+4I))-B)\hat{}.$$

Thus by Corollary 2.3 above and the fact that for all n, the sequence $\{X_n\}$ must converge in norm.

REMARK 4.6. As in the case of the parallel sum, we have for any monotone approximation scheme which is independent of representation that the scheme converges in norm to $A \# B$ if and only if (A, B) is of class UUP (see Remark 2.7).

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