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ON THE EMBEDDING AND DIAGONALIZATION OF MATRICES OVER $C(X)$

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1. Introduction.

In [2] Effros suggested the study of C*-algebras which arise as direct limits of C*-algebras of the form $C(X) \otimes F$, where $C(X)$ is the algebra of continuous functions on a nice compact topological space and F is a finite-dimensional C*-algebra. The idea is to generalize the almost complete theory of approximately finite-dimensional C*-algebras.

As a first step in the study of such direct limit C*-algebras one must study *-homomorphisms

$$\varphi: C(X) \otimes M_n \to C(Y) \otimes M_m.$$

The difficulties in imitating the theory of AF-algebras start already at this stage since two unital *-homomorphisms between $C(X) \otimes M_n$ and $C(Y) \otimes M_m$ need not be inner equivalent. Hence the question is if there is a canonical way of describing how $C(X) \otimes M_n$ can be embedded into $C(Y) \otimes M_m$. The purpose of this note is to give such a description when Y satisfies certain topological conditions and the dimension of $\varphi(C(X) \otimes M_n)(y) \subseteqq M_m$ is constant over Y. As will become clear, the question is closely related to the question of which abelian C*-subalgebras of $C(Y) \otimes M_m$ can be diagonalized. Unless Y is a Stonean space, such a diagonalization is not automatically possible, see [4].

Although none of our results depend on the results of [4], the paper of Grove and Pedersen has been an indispensable source of inspiration.

2. Notation.

X, Y will denote compact Hausdorff spaces, M_n the $n \times n$ complex matrices, and $U(n)$ the subset of M_n consisting of the unitary elements. We will identify $C(X) \oplus M_n$ with $C(X, M_n)$, the continuous functions on X with values in M_n.

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If B is a C*-subalgebra of $C(X, M_n)$, we write $B(x)$ for the following C*-subalgebra of M_n:

$$B(x) = \{f(x) \mid f \in B\}, \qquad x \in X.$$

S_n will denote the symmetric group of order $n!$, and as in [4] the cohomology sets $H^1(X, S_n)$ and $H^1(X, U(n))$ will play an important role. For a definition of these sets, which will be sufficient, at least for our proofs, we refer to [5, p. 9–10].

3. Results.

LEMMA 1. *Let $\varphi: C(X) \otimes M_n \to C(Y) \otimes M_m$ be a unital *-homomorphism. Then $n|m$, that is n divides m, and*

$$\dim[\varphi(C(X, M_n)](y) = n^2 \dim[\varphi(C(X) \otimes 1))](y), \qquad y \in Y.$$

PROOF. Let $y \in Y$. Then

$$C(X, M_n) \ni a \mapsto \varphi(a)(y)$$

defines a finite-dimensional representation of $C(X, M_n)$. Therefore there is a number $k(y) \in \mathbf{N}$ such that

$$\varphi(C(X, M_n))(y) \simeq \underbrace{M_n \oplus M_n \oplus \cdots \oplus M_n}_{k(y)\text{ times}}.$$

Since $\varphi(C(X) \otimes 1)(y)$ is the center of $\varphi(C(X, M_n))(y)$ by [6, Corollary 1], it is clear that

$$k(y) = \dim \varphi(C(X) \otimes 1)(y).$$

That $n|m$ follows from [1].

LEMMA 2. *Let A be an abelian C*-algebra in $C(X, M_n)$ containing the unit and such that $\dim A(x)$ is constant over X.*

Assume that both $H^1(X, U(k))$ and $H^1(X, S_k)$ are trivial when $k \leqq n$.

Then A is diagonalizable.

PROOF. Let $k = \dim A(x)$, $x \in X$.

By [4, Lemma 5.2] we can find a finite open covering $\{U_i | i = 1, 2, \ldots, N\}$ of X and elements $Q_1^i, Q_2^i, \ldots, Q_k^i$ in A such that $\{Q_j^i(x)\}_{j=1}^k$ are the minimal projections in $A(x)$, $x \in U_i$, $i = 1, 2, \ldots, N$.

For $x \in U_i \cap U_j$, we can define an element $s_{ij}^x \in S_k$ by the requirement

$$\text{(1)} \qquad \operatorname{Tr}(Q^i_{s_{ij}^x(m)}(x) Q^j_l(x)) \neq 0 \quad \text{iff} \quad l = m$$

$m, l \in \{1, 2, \ldots, k\}$. Then $Q^i_{s_{ij}^x(m)}(x) = Q^j_m(x)$ and the defining relation (1) shows that

$$U_i \cap U_j \ni x \to s_{ij}^x \in S_k$$

is continuous. Since $s_{ij}^x s_{jm}^x = s_{im}^x$, $x \in U_i \cap U_j \cap U_m$, we find that (U_i, s_{ij}) represent an element in $H^1(X, S_k)$. This set is trivial by assumption. Therefore there are continuous functions

$$s_i : U_i \to S_k$$

such that

$$s_{ij}^x = s_i^x (s_j^x)^{-1}, \qquad x \in U_i \cap U_j.$$

Then

$$Q^i_{s_i^x(m)}(x) = Q^j_{s_j^x(m)}(x), \qquad m \in \{1, 2, \ldots, k\}, \qquad x \in U_i \cap U_j.$$

For each $m \in \{1, 2, \ldots, k\}$, we define $Q_m \in C(X, M_n)$ by

$$Q_m(x) = Q^i_{s_i^x(m)}(x), \qquad x \in U_i.$$

Then $\{Q_1(x), \ldots, Q_k(x)\}$ are the minimal projections in $A(x)$ for all $x \in X$.

For any sample $d_1, d_2, \ldots, d_k$ of integers satisfying

$$\sum_{i=1}^{k} d_i = n,$$

the set $\{x \in X \mid \operatorname{Tr}(Q_i(x)) = d_i, \quad i = 1, 2, \ldots, m\}$ is open and closed. In order to show that span $\{Q_1, Q_2, \ldots, Q_k\}$ can be diagonalized over X, we can therefore restrict attention to such a set. Or, for simplicity of exposition, assume that

$$\{x \in X \mid \operatorname{Tr}(Q_i(x)) = d_i, \quad i = 1, 2, \ldots, m\} = X.$$

Let $p_1, p_2, \ldots, p_k$ be diagonal projections in M_n such that $\operatorname{Tr}(p_i) = d_i$, $i = 1, 2, \ldots, k$. Fix $x_0 \in X$. There is a unitary $U \in C(X, M_n)$ such that

$$U(x_0) Q_i(x_0) U(x_0)^* = p_i, \qquad i = 1, 2, \ldots, k.$$

Then

$$\sup_i \|U(x)Q_i(x)U(x)^* - p_i\| < \tfrac{1}{2}$$

for all x in a neighbourhood of x_0.

Let $g: [0, 1] \to [0, 2]$ be a continuous function which is zero in a neighbourhood of 0 and $g(t) = 1/t$, $t \geqq \frac{1}{2}$.

Let $W_i \in C(X, M_n)$ be given by

$$W_i(x) = p_i[g(p_i U(x)Q_i(x)U(x)^* p_i)]^{1/2} U(x)Q_i(x)U(x)^*$$

$x \in X$, $i = 1, 2, \ldots, k$. As shown by Glimm in the proof of [3, Lemma 1.8], we have that

$$W_i(x)^* W_i(x) = U(x)Q_i(x)U(x)^*$$

$$W_i(x)W_i(x)^* = p_i, \qquad i = 1, 2, \ldots, k$$

for all x in the same neighbourhood of x_0 as above.

Let $W = \sum_{i=1}^{k} W_i$. Then $W(x)$ is a unitary such that

$$WUQ_i(WU)^*(x) = p_i, \qquad i = 1, 2, \ldots, k$$

for x in this neighbourhood.

We conclude that there is a finite covering $\{U_i | i = 1, 2, \ldots, N\}$ of X and continuous functions

$$W_i: U_i \to U(n)$$

such that

$$W_i(x)Q_j(x)W_i(x)^* = p_j, \qquad j = 1, 2, \ldots, k, \qquad x \in U_i.$$

Especially $W_j(x)W_i(x)^* \in \{p_1, p_2, \ldots, p_k\}'$, $x \in U_i \cap U_j$. Let $\mathscr{W}$ denote the unitary group of $\{p_1, p_2, \ldots, p_k\}'$. Then $(U_i, W_j W_i^*)$ define an element in $H^1(X, \mathscr{W})$.

Since $\{p_1, p_2, \ldots, p_k\}' \simeq M_{d_1} \oplus M_{d_2} \oplus \cdots \oplus M_{d_k}$, we have that

$$\mathscr{W} \simeq U(d_1) \times U(d_2) \times \cdots \times U(d_k).$$

Since $H^1(X, U(d_i)) = 0$ for all i by assumption, we conclude that $H^1(X, \mathscr{W}) = 0$.
Hence there are continuous functions

$$V_i: U_i \to \mathscr{W}$$

such that $W_j(x)W_i^*(x) = V_j^*(x)V_i(x), \quad x \in U_i \cap U_j$. Define $T \in C(X, M_n)$ by

$$T(x) = V_j(x)W_j(x), \qquad x \in U_j.$$

Then T diagonalizes A.

REMARK. If $\dim A(x)$ is constantly equal to n, the above proof works with the assumptions that

$$H^1(X, S_n) = H^1(X, U(1)) = 0.$$

Since $H^1(X, U(1)) \simeq H^2(X, \mathbf{Z})$, the lemma that results follows from some of the arguments used to prove Theorem 5.3 in [4]. In this case the lemma is very close to Theorem 1.4 of [4]. The morale of the lemma is that the worst or most effective obstruction to diagonalization of matrices over $C(X)$ arises from the fact that the number of eigenvalues of a normal element in $C(X, M_n)$ can vary over X. If this number is constant the multiplicites are irrelevant as long as X satisfies the assumptions of the lemma.

THEOREM 3. *Let X, Y be compact Hausdorff spaces, and*

$$\varphi: C(X, M_n) \to C(Y, M_m)$$

*a unital *-homomorphism (such that especially $n|m$). Assume that*

$$\dim \varphi(C(X, M_n))(y)$$

is constant over Y and that $H^1(Y, U(k)) = H^1(Y, S_k)) = 0, \quad k \leqq m/n$. Then there is a unitary $U \in C(Y, M_m)$ and m/n continuous functions $\psi_i: Y \to X$, $i = 1, 2, \ldots, m/n$, such that

$$\varphi(f)(y) = U(y)\begin{bmatrix} f\circ\psi_1(y) & & & & \\ & f\circ\psi_2(y) & & & \\ & & \ddots & 0 & \\ & & & \ddots & \\ & 0 & & & \ddots \\ & & & & f\circ\psi_{m/n}(y) \end{bmatrix}U(y)^* \tag{2}$$

$y \in Y, \quad f \in C(X, M_n)$.

PROOF. Define elements $\tilde{e}_{ij}$ of $C(X, M_n)$ by

$$\tilde{e}_{ij}(x) = e_{ij}, \qquad x \in X$$

where $\{e_{ij}\}$ is the standard system of matrix units in M_n. Since φ is unital,

$$\{\varphi(\tilde{e}_{ij})(y)\}$$

is a system of matrix units in M_m for all $y \in Y$. Let $\{f_{ij}\}$ be the standard system of matrix units in M_m, and define

$$c_{ij} = \sum_{d=1}^{m/n} f_{i+(d-1)n,\, j+(d-1)n}, \qquad i, j = 1, 2, \ldots, n.$$

For each $y_0 \in Y$ there is a unitary $U \in C(Y, M_m)$ such that

$$U(y_0)\varphi(\tilde{e}_{ij})(y_0)U(y_0)^* = c_{ij}, \qquad i, j = 1, 2, \ldots, n.$$

But then

$$\sup_{ij} \|U(y)\varphi(\tilde{e}_{ij})(y)U(y)^* - c_{ij}\| < \tfrac{1}{2}$$

in a neighbourhood of y_0. Take a function g as in the proof of Lemma 2, and define an element $W \in C(Y, M_m)$ by

$$W(y) = \sum_{i=1}^{n} c_{i1}[g(c_{11}U(y)\varphi(\tilde{e}_{11})(y)U(y)^*c_{11})]^{1/2}U(y)\varphi(\tilde{e}_{1i})(y)U(y)^*, \quad y \in Y.$$

Then $W(y)$ is a unitary such that

$$W(y)U(y)\varphi(\tilde{e}_{ij})(y)U(y)^*W(y)^* = c_{ij}, \qquad i, j = 1, 2, \ldots, n$$

for all y in the above neighbourhood of y_0. The details needed to verify this can be found in [3, proof of lemma 1.8] and [1, proof of lemma 2.3].

Thus we can find a finite covering $\{U_i,\ i = 1, 2, \ldots, N\}$ of Y and continuous functions

$$W_i \colon U_i \to U(m)$$

such that

$$W_k(y)\varphi(\tilde{e}_{ij})(y)W_k(y)^* = c_{ij}, \qquad i,j = 1,2,\dots,n, \qquad y \in U_k.$$

Let $\mathscr{W}$ denote the unitary group of $\{c_{ij}\}' \subseteqq M_m$. Then $(U_i, W_iW_j^*)$ defines an element in

$$H_1(Y, \mathscr{W})$$

Since $\{c_{ij}\}' \simeq M_{m/n}$, our assumption on Y assures that there are continuous functions

$$V_i: U_i \to \mathscr{W}$$

such that

$$W_iW_j^* = V_i^*V_j \qquad \text{over} \qquad U_i \cap U_j.$$

Define $S \in C(Y, M_m)$ by

$$S(y) = V_i(y)W_i(y), \qquad y \in U_i, \qquad i = 1,2,\dots,N.$$

Then S is a unitary in $C(Y, M_m)$ such that

$$S(y)\varphi(\tilde{e}_{ij})(y)S(y)^* = c_{ij}, \qquad y \in Y, \qquad i,j = 1,2,\dots,n.$$

Let $f \in C(X)$. $i,j \in \{1,2,\dots,n\}$. Then

$$S\varphi(f \otimes e_{ij})S^*(y) = S\varphi(f \otimes 1)\varphi(\tilde{e}_{ij})S^*(y)$$
$$= S\varphi(f \otimes 1)S^*(y)c_{ij} = c_{ij}S\varphi(f \otimes 1)S^*(y), \qquad y \in Y.$$

Hence $S\varphi(C(X) \otimes 1)S^*(y) \subseteqq \{c_{ij}\}'$ for $y \in Y$.

Since $\{c_{ij}\}' \simeq M_{m/n}$ and $\dim \varphi(C(X) \otimes 1)(y)$ is constant over X by Lemma 1, we conclude from Lemma 2 that there is a unitary $T \in C(Y, M_m)$ such that $T(y) \in \{c_{ij}\}'$ and

$$TS\varphi(f \otimes 1)S^*T^*(y)$$

is diagonal for all $f \in C(X)$, $y \in Y$.

Let

$$p_1 = \sum_{i=1}^{n} f_{ii}, \qquad p_2 = \sum_{i=n+1}^{2n} f_{ii}, \ldots, \qquad p_{m/n} = \sum_{i=m-n+1}^{m} f_{ii}.$$

Since $TS\varphi(C(X) \otimes \mathbf{1})S^*T^*(y) \subset \{c_{ij}\}'$,

$$TS\varphi(C(X) \otimes \mathbf{1})S^*T^*(y) \subset \operatorname{span}\{p_1, \ldots, p_{m/n}\}$$

for all $y \in Y$.

For each $y \in Y$ there are then elements

$$\psi_1(y), \psi_2(y), \ldots, \psi_{m/n}(y) \in X$$

determined by

$$TS\varphi(f \otimes \mathbf{1})S^*T^*(y)p_i = f(\psi_i(y))p_i, \qquad i = 1, 2, \ldots, \frac{m}{n}$$

$f \in C(X)$. Clearly, $\psi_i: Y \to X$, are continuous functions.

The desired unitary, U, is TS and it is a routine matter to check that $U, \psi_1, \psi_2, \ldots, \psi_{m/n}$ have the right property.

It is clear that there is a great freedom in the choice of the unitary U of Theorem 3. But the question is how much freedom there is in the choice of the functions $\psi_1, \psi_2, \ldots, \psi_{m/n}$. This is answered by the following

PROPOSITION 4. *Let X, Y be compact Hausdorff spaces and let*

$$\varphi: C(X, M_n) \to C(Y, M_m)$$

*be a unital *-homomorphism such that* $\dim \varphi(C(X, M_n))(y)$ *is constant over Y.*

Assume $\psi_1, \psi_2, \ldots, \psi_{m/n}$ are continuous functions from Y to X such that (2) *holds for some unitary U. Let $\varphi_1, \varphi_2, \ldots, \varphi_{m/n}$ be continuous functions from Y to X.*

Then there is a unitary W in $C(Y, M_m)$ such that (2) *holds with φ_i substituted for ψ_i, $i = 1, 2, \ldots, m/n$, and W substituted for U if and only if*

$$\{\psi_1(y), \psi_2(y), \ldots, \psi_{m/n}(y)\} = \{\varphi_1(y), \varphi_2(y), \ldots, \varphi_{m/n}(y)\}, \qquad y \in Y.$$

If Y is connected this condition is equivalent to

$$\{\psi_1, \psi_2, \ldots, \psi_{m/n}\} = \{\varphi_1, \varphi_2, \ldots, \varphi_{m/n}\}.$$

PROOF. Assume first that (2) holds for $\varphi_1, \ldots, \varphi_{m/n}$ and W. Taking the trace, it follows that

$$\sum_{i=1}^{m/n} f(\varphi_i(y)) = \sum_{i=1}^{m/n} f(\psi_i(y))$$

for all $y \in Y$, $f \in C(X)$.

Since X is a compact Hausdorff space, this is only possible if

$$\{\varphi_1(y), \varphi_2(y), \ldots, \varphi_{m/n}(y)\} = \{\psi_1(y), \psi_2(y), \ldots, \psi_{m/n}(y)\}, \qquad y \in Y. \tag{3}$$

Conversely, assume that (3) holds.

Let $k = \dim \varphi(C(X) \otimes \mathbf{1})(y)$, $y \in Y$.

Let $N_1, N_2, \ldots, N_k$ and $M_1, M_2, \ldots, M_k$ be two partitions of $\{1, 2, \ldots, m/n\}$ into k disjoint non-empty subsets such that $\# N_i = \# M_i$, $i = 1, 2, \ldots, k$. Consider

$$\bigcap_{l=1}^{k} \{y \in Y \mid \varphi_i(y) = \varphi_j(y) = \psi_n(y) = \psi_m(y), \quad i, j \in N_l, \quad n, m \in M_l\}.$$

Such a subset of Y is called a configuration.

From (3) and the assumption that $\dim(C(X) \otimes \mathbf{1})(y) = k$, it follows that the configurations form a finite covering of Y by mutually disjoint subsets. Since each configuration is obviously closed, we see that they are in fact both closed and open.

Let $\{F_1, F_2, \ldots, F_N\}$ denote the non-empty configurations. For each $i \in \{1, 2, \ldots, N\}$, it is clear that we can find a unitary $\tilde{W}_i \in M_m$ such that

$$\tilde{W}_i \begin{bmatrix} f \circ \psi_1(y) & & & 0 \\ & f \circ \psi_2(y) & & \\ & & \ddots & \\ 0 & & & f \circ \psi_{m/n}(y) \end{bmatrix} \tilde{W}_i^* = \begin{bmatrix} f \circ \varphi_1(y) & & & 0 \\ & f \circ \varphi_2(y) & & \\ & & \ddots & \\ 0 & & & f \circ \varphi_{m/n}(y) \end{bmatrix}$$

$y \in F_i$, $f \in C(X, M_n)$.

Define

$$W_i(y) = \begin{cases} \tilde{W}_i, & y \in F_i \\ 0, & y \notin F_i \end{cases}$$

and

$$V = \sum_{i=1}^{N} W_i.$$

Then V is a unitary in $C(Y, M_m)$ such that

$$V(y)\begin{bmatrix} f\circ\psi_1(y) & & & \\ & f\circ\psi_2(y) & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & f\circ\psi_{m/n}(y)\end{bmatrix} V(y)^* = \begin{bmatrix} f\circ\varphi_1(y) & & & \\ & f\circ\varphi_2(y) & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & f\circ\varphi_{m/n}(y)\end{bmatrix}$$

$y \in Y$. Let $W = UV^*$. Then (2) holds for $\varphi_1, \varphi_2, \ldots, \varphi_{m/n}$ and W.

If Y is connected, there is only one non-empty configuration. Hence (3) implies that

$$\{\psi_1, \psi_2, \ldots, \psi_{m/n}\} = \{\varphi_1, \varphi_2, \ldots, \varphi_{m/n}\}$$

in this case.

REMARK. The conclusion of Theorem 3 is not true if $H^1(Y, U(k)) \neq 0$ or $H^1(Y, S_k) \neq 0$ for some $k \leqq m/n$. This follows from [4, Theorems 4.1 and 4.2], or rather their proofs, since the non-diagonalizable elements constructed there generate abelian C*-subalgebras with constant dimension over X_0.

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