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ON ESTIMATING SOME INITIAL INVERSE COEFFICIENTS FOR MEROMORPHIC UNIVALENT FUNCTIONS OMITTING A DISC

HENRYKA SIEJKA

1. Introduction.

The author wishes to express her gratitude to Professor O. Tammi for the valuable discussions connected with the problem.

In this paper we shall use the same notations as in [4]. In particular $S(b)$, $0 < b < 1$, denotes the class of functions of the form

$$f(z) = b\left(z + \sum_{n=0}^{\infty} a_n z^n\right),$$

holomorphic-univalent in $D = \{z : |z| < 1\}$ and bounded by 1, $|f(z)| < 1$.

Let next Σ_b, $0 < b < 1$, denote the class of functions of the form

$$H(z) = z + \sum_{n=0}^{\infty} A_n z^{-n}, \tag{1}$$

holomorphic-univalent in $\tilde{D} = \{z : |z| > 1\}$ except at the point of infinity where they have a pole and satisfying the condition $|H(z)| > b$ for $z \in \tilde{D}$.

As a limit case $b \to 0$ we obtain from Σ_b the class Σ of holomorphic-univalent functions in $\tilde{D}$ with the expansion (1) and the restriction $H(z) \neq 0$.

There is a one-to-one correspondence between the classes Σ_b and $S(b)$:

$$f(z)H(1/z) = b, \quad z \in D. \tag{2}$$

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In [2] Launonen derived in the class $S(b)$ some sharp integral inequalities which allowed him to obtain sharp estimation of the coefficient of functions inverse to odd $S(b)$-functions and as a consequence the coefficients of functions which are inverse to any $S(b)$-function.

In [4] Launonen's method was applied to estimate all the coefficients of functions inverse to odd Σ_b-functions. In this paper we shall try to estimate the coefficients in the entire class of functions inverse to Σ_b-functions. Unfortunately, in this case Launonen's inequality gives results only for a few odd coefficients. In [3] Netanyahu estimated all the coefficients inverse to Σ-functions by an ingenious use of variational method. In this paper we shall use the same method by aid of which some additional even and hence also odd coefficients can be estimated.

2. The sharp bounds for the coefficients of functions inverse to Σ_b-functions.

In [4] it was proved that for every function

$$z = I(w) = w + \sum_{n=0}^{\infty} E_n w^{-n}, \tag{3}$$

inverse to Σ_b-function, the inequality

$$\left| \int_\gamma \int_\gamma \mu(w)\mu(s) \frac{I(w)-I(s)}{w-s} \, dw ds \right| \leq \int_\gamma \int_\gamma \mu(w)\overline{\mu(s)} \frac{1-b^2/w\bar{s}}{1-1/I(w)\overline{I(s)}} \, dw d\bar{s} \tag{4}$$

holds, where γ is a closed analytic curve and $\mu(w)$ is a continuous weight function on γ. It was also proved that the function

$$I_0(w) = w + \sum_{n=0}^{\infty} D_{2n+1}^0 w^{-(2n+1)},$$

satisfying the equation

$$I_0(w) + \frac{1}{I_0(w)} = \frac{b^2}{w} + w \tag{5}$$

is the extremal function with respect to the functional

$$J(I) = |E_n|, \tag{6}$$

defined in the family of functions inverse to odd Σ_b-functions.

Note that for any function (3) inverse to Σ_b-function, the function

$$z = I^{1/2}(w^2) = w + \sum_{n=0}^{\infty} D_{2n+1} w^{-(2n+1)}$$

is inverse to odd $\Sigma_{b^{1/2}}$-function and that

$$\begin{aligned} (7) \qquad & E_0 = 2D_1, \\ & E_n = D_{2n+1} + D_{2n-1}D_1 + D_{2n-3}D_3 + \ldots + D_1 D_{2n-1} + D_{2n+1}, \quad n = 1, 2, \ldots . \end{aligned}$$

Substituting in (7) the coefficients D^0_{2n+1} of $\Sigma_{b^{1/2}}$-funtions we shall probably obtain the coefficients of the function $z = \tilde{I}(w)$, extremal in the class of inverse Σ_b-functions. From (5) it follows that the function $w = H_0(z)$, inverse to $I_0(w) \in \Sigma_{b^{1/2}}$, satisfies the equation

$$H_0(z) + \frac{b}{H_0(z)} = \frac{1}{z} + z.$$

Thus $w = \tilde{H}(z)$, inverse to $z = \tilde{I}(w) = I_0^2(w^{1/2})$, satisfies the equation

$$\tilde{H}^{1/2}(z) + \frac{b}{\tilde{H}^{1/2}(z)} = z^{1/2} + \frac{1}{z^{1/2}},$$

from where

$$(8) \qquad \tilde{I}(w) + \frac{1}{\tilde{I}(w)} + 2 = w + \frac{b^2}{w} + 2b.$$

The function $z = \tilde{I}(w)$ maps $|w| > b$ minus a slit along the real axis onto $|z| > 1$.

Denote now

$$(9) \qquad z = \tilde{I}(w) + \sum_{n=0}^{\infty} \tilde{E}_n w^{-n}.$$

Proceeding analogously as in [4] one can show that

$$\begin{aligned} \tilde{E}_0 &= 2(b-1), \\ \tilde{E}_1 &= b^2 - 1, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \tilde{E}_n &= -(\gamma_{n0} + \gamma_{n1} b + \gamma_{n2} b^2 + \ldots + \gamma_{n(n-1)} b^{n-1}), \quad n = 2, 3, \ldots, \end{aligned}$$

where

$$\gamma_{nk} = (-1)^k \frac{2}{k!} \frac{n-k}{n-k+1} \frac{(2n-k-1)!}{[(n-k)!]^2},$$

and that the function $\tilde{I}$ yields the equality in (4). Moreover, $\tilde{I}$ and its rotations are the only extremal functions with respect to $|E_0|$ because in view of (7)

$$(10) \qquad |E_0| = 2|D_1| \leqq 2|D_1^0| = 2(1-b) = -\tilde{E}_0,$$

and I_0 was the only extremal function with respect to $|D_1|$.

Let us now put in (4) the weight function $\mu(w) = w^{t-1}$, where k is any fixed natural number. Then the left-hand side of the considered inequality equals $4\pi^2|E_{2k-1}|$. In order to derive the right-hand side let us denote

$$(11) \qquad \frac{1}{I(w)} = w^{-1} + \sum_{k=2}^{\infty} \alpha_k w^{-k},$$

$t = w\bar{s}$, and notice that the effective term yielding the value of the right-hand side of (4) is the coefficient of the power t^{-n} in the development of the function

$$\frac{1-b^2/w\bar{s}}{1-1/I(w)\overline{I(s)}} = (1-b^2t^{-1})[1+(I(w)\overline{I(s)})^{-1}+(I(w)\overline{I(s)})^{-2}+\ldots].$$

This coefficient can be determined from the following table (cf. [4]):

n	-1	-2	-3	-4	
$(I(w)\overline{I(s)})^{-1} =$	$t^{-1}+$	$\lvert\alpha_2\rvert^2t^{-2}+$	$\lvert\alpha_3\rvert^2t^{-3}+$	$\lvert\alpha_4\rvert^2t^{-4}$	$+\ldots,$
$(I(w)\overline{I(s)})^{-2} =$		$t^{-2}+$	$4\lvert\alpha_2\rvert^2t^{-3}+$	$\lvert 2\alpha_3+\alpha_2^2\rvert t^{-4}$	$+\ldots,$
$(I(w)\overline{I(s)})^{-3} =$			$t^{-3}+$	$9\lvert\alpha_2\rvert^2t^{-4}$	$+\ldots,$
$(I(w)\overline{I(s)})^{-4} =$				t^{-4}	$+\ldots.$

Therefore the inequality (4) for any function $z = I(w)$ assumes the form

$$(12) \qquad 4\pi^2|E_{2k-1}| \leqq \int_\gamma\int_\gamma t^{k-1}T(t)dwd\bar{s},$$

where

$$T(t) = (1-b^2)t^{-1}+[|\alpha_2|^2+1-b^2]t^{-2}+[|\alpha_3|^2+(4-b)^2|\alpha_2|^2+1-b^2]t^{-3}+$$
$$+[|\alpha_4|^2+|2\alpha_3+\alpha_2^2|^2-b^2|\alpha_3|^2+(9-4b^2)|\alpha_2|^2+1-b^2]t^{-4}+\dots,$$

$t = w\bar{s}$.

Let us notice besides that from (3) and (11) we obtain

(13) $$\alpha_2 = -E_0,\ \alpha_3 = E_0^2-E_1,\ \alpha_4 = 2E_0E_1-E_0^3-E_2.$$

The inequality (12) for $k = 1$ yields

(14) $$|E_1| \leqq 1-b^2 = -\tilde{E}_1.$$

In this case there exists a one-parametric family of extremal functions. This is seen by expressing E_1 in terms of the $S(b)$-coefficients. From the connection (2) we decide that $E_1 = a_3-a_2^2$. For this functional there holds the result of [5]: $|a_3-a_2^2|$ is maximized by $f \in S(b)$ for which (a_2 is a parameter)

$$f(z)+\frac{1}{f(z)} = \frac{1}{b}(z-a_2+1/z),\quad a_2 \in [-2(1-b), 2(1-b)].$$

This implies for $H \in \Sigma_b$:

$$H(z)+\frac{b^2}{H(z)} = z-a_2+\frac{1}{z},\quad a_2 \in [-2(1-b), 2(1-b)].$$

Thus, to E_1 there belongs a one-parametric family of extremal functions, defined by the above condition as well as by the rotations of these functions.

For $k = 2$ the condition (12) takes the form

$$|E_3| \leqq |\alpha_2|^2+1-b^2,$$

and in view of (13) and (10) we have

$$|E_3| \leqq 5-8b+3b^2 = -\tilde{E}_3,$$

where the equality holds only for $\tilde{I}$ and its rotations.

Taking next $k = 3$ in (12) we obtain

$$|E_5| \leqq |\alpha_3|^2+(4-b^2)|\alpha_2|^2+1-b^2.$$

Because from (13), (10), and (14)

$$|\alpha_2| \leqq |E_0|^2 + |E_1| \leqq 5 - 8b + 3b^2,$$

so

$$|E_5| \leqq 42 - 112b + 105b^2 - 40b^3 + 5b^4 = -\tilde{E}_5.$$

Again the function $z = \tilde{I}$ is the only extremal function. – Observe that the maximum of $|E_0|$ implies those of $|E_3|$ and $|E_5|$.

In the case $k = 4$ from (12) we have

$$(15) \qquad |E_7| \leqq |\alpha_4|^2 + |2a_3 + \alpha_2^2|^2 - b^2|\alpha_3|^2 + (9 - 4b^2)|\alpha_2|^2 + 1 - b^2,$$

therefore form (13) it follows that the estimation of $|E_7|$ requires the maximum of $|E_2|$. Similarly $k = 5$ yields the maximum for $|E_9|$ by aid of $|E_2|$. Because the method considered fails in the case of even coefficients we shall use variational method in estimating $|E_2|$.

For this purpose, let us notice that from the connection (2) between the classes Σ_b and $S(b)$ it follows that

$$(16) \qquad \begin{aligned} E_0 &= a_2, \\ E_1 &= a_3 - a_2^2, \\ E_2 &= a_4 - 3a_2a_3 + 2a_2^3. \end{aligned}$$

Then in the family $S(b)$ we have to consider the functional

$$(17) \qquad J(f) = \operatorname{re}(a_4 - 3a_2a_3 + 2a_2^3).$$

From the Charzyński theorem [1] it follows directly that every function $w = f(z)$ of the family $S(b)$, extremal with respect to the functional (17), satisfies the following differential-functional equation

$$(18) \qquad (zw'/w)^2 M(w) = N(z), \quad 0 < |z| < 1,$$

where

$$M(w) = 2b^2\left[\frac{b^3}{w^3} + \frac{a_2^2 - a_3}{w} + (\overline{a_2^2 - a_3})w + b^2w^3 - \frac{P}{b^2}\right],$$

$$N(z) = 2b\left[\frac{1}{z^3} - \frac{a_2}{z^2} + 3\mathrm{re}(a_4 - 3a_2a_3 + 2a_2^3) - \bar{a}_2 z^2 + z^3 - \frac{P}{b}\right],$$

$$P = 2b^2 \min_{0 < x \leqq 2\pi} \mathrm{re}\ \{(a_2^2 - a_3)e^{ix} + b^2 e^{i3x}\}.$$

The functions $M(w)$ and $N(z)$ have at least one double zero on the circles $|w| = 1$, $|z| = 1$, respectively.

Because

$$\mathrm{re}\{(a_2^2 - a_3)e^{ix} + b^2 e^{i3x}\} \geqq -|a_2^2 - a_3| + b^2 \cos 3x,$$

by virtue of (16) and (14) we get

$$P = 2b^2 \min_{0 < x \leqq 2\pi} \{-1 + b^2 + b^2 \cos 3x\} = -2b^2, \tag{19}$$

and the minimum is attained for the Pick function defined by the equation

$$\frac{f(z)}{(1 - f(z))^2} = b\frac{z}{(1-z)^2}.$$

Using again the connection (2) between the classes Σ_b and $S(b)$ by (16) and (19) we can express the equation (18) in the form

$$\left[\frac{z^2 H'(z)}{H(z)}\right]^2 \left[H^3(z) - E_1 H(z) + 2b - \frac{b^2 \bar{E}_1}{H(z)} + \frac{b^6}{H^3(z)}\right]$$

$$= z^3 - E_0 z^2 + 3\mathrm{re}\, E_2 + 2b - \frac{\bar{E}_0}{z^2} + \frac{1}{z^3}, \qquad |z| > 1,$$

Since there exists a point $z_0 = e^{i\theta}$, $\theta \in \langle 0, 2\pi\rangle$, such that $N(z_0) = 0$, we have

$$e^{i3\theta} - E_0 e^{i2\theta} + 3\mathrm{re}\, E_2 + 2b - \bar{E}_0 e^{-i2\theta} + e^{-i3\theta} = 0,$$

whence

$$\mathrm{re}\, E_2 \leqq \tfrac{1}{3}(2|E_0| + 2 - 2b) \leqq 2(1-b), \tag{20}$$

which implies

$$|E_2| \leqq 2(1-b) = -\tilde{E}_2.$$

Because the function $\tilde{I}$ was the only extremal function with respect to $|E_0|$, it follows from (20) that it is also the only extremal function with respect to $|E_2|$ and consequently in view of (15), (13), (10), and (14) to $|E_7|$ with

$$|E_7| \leqq -\tilde{E}_7.$$

As mentioned before, the maximum of $|E_2|$ implies that of $|E_9|$ for $k = 5$.

Proceeding analogously as in the case E_2 we get the estimation

$$|E_4| \leqq -\tilde{E}_4. \tag{21}$$

This time the right-hand side of respective equation (18) takes the form

$$N(z) = 2b\left[\frac{1}{z^5} - \frac{3E_0}{z^4} + \frac{3E_0^2-2E_1}{z^3} + \frac{2E_0E_1-E_2-E_0^3}{z^2} + 5\operatorname{re} E_4 + \right.$$

$$\left. + (\overline{2E_0E_1-E_2-E_0^3})z^2 + (\overline{3E_0^2-2E_1})z^3 - 3\bar{E}_0z^4 + z^5 - \frac{P}{b}\right], \quad |z| > 1,$$

where

$$\begin{aligned} P &= 2b^2 \min_{0 < x \leqq 2\pi} \operatorname{re}(-3E_3e^{ix} - 2bE_2e^{i2x} - b^2E_1e^{i3x} + b^4e^{i5x}) \\ &= -2b^2 \max \operatorname{re}(3E_3e^{ix} + 2bE_2e^{i2x} + b^2E_1e^{i3x} - b^4e^{i5x}) \\ &\geqq -2b^2(3|E_3| + 2b|E_2| + b^2|E_1| + b^4) \\ &\geqq -2b^2[3(5-8b+3b^2) + 2b\cdot 2(1-b) + b^2(1-b^2) + b^4] \\ &= -2b^2(15-20b+6b^2). \end{aligned}$$

This estimation is sharp exactly for $\tilde{I}$.

From $N(z)$ we decide that for a certain $z_0 = e^{i\theta}$

$$\begin{aligned} \operatorname{re} E_4 &= -\tfrac{2}{5}[\cos 5\theta + \operatorname{re}\{-3E_0e^{-i4\theta} + (3E_0^2-2E_1)e^{-i3\theta} + \\ &\quad + (2E_0E_1 - E_2 + E_0^3)e^{-i2\theta}\}] + \frac{1}{5}\frac{P}{b} \\ &\leqq \tfrac{2}{5}(1 + 3|E_0| + 3|E_0|^2 + 2|E_1| + 2|E_0||E_1| + |E_2| + |E_0|^3) + \frac{1}{5}\frac{P}{b} \\ &\leqq 14 - 30b + 20b^2 - 4b^3. \end{aligned}$$

Thus we proved (21) with the equality for $\tilde{I}$. As before (12) yields for $k = 6$ and 7 the maxima of $|E_{11}|$ and $|E_{13}|$. – Collect the results:

THEOREM. *For every function*

$$z = I(w) = w + \sum_{n=0}^{\infty} E_n w^{-n},$$

inverse to a Σ_b-function the estimations

$$|E_1| \leqq 2(1-b),$$

$$|E_1| \leqq 1-b^2,$$

$$|E_n| \leqq \gamma_{n0} + \gamma_{n1} b + \gamma_{n2} b^2 + \ldots + \gamma_{n(n-1)} b^{n-1}, \quad n = 2, 3, 4, 5, 7, 9, 11, 13$$

hold. Here

$$\gamma_{nk} = (-1)^k \frac{2}{k!} \frac{n-k}{n-k+1} \frac{(2n-k-1)!}{[(n-k)!]^2}.$$

Save for E_1 the only extremal function for all the coefficients is the function satisfying the equation

$$\tilde{I}(w) + \frac{1}{\tilde{I}(w)} + 2 = w + \frac{b^2}{w} + 2b$$

and its rotations. For E_1 there belongs a one-parametric family of extremal functions satisfying

$$\tilde{I}(w) + c + \frac{1}{\tilde{I}(w)} = w + \frac{b^2}{w}$$

where c is a parameter with the limitation $-2(1-b) \leqq c \leqq 2(1-b)$.

The above examples of sharp estimations strongly suggest that the variational method allows the maximilization of all the E_ν-coefficients. However, the previous form of the variational formula does not reveal the general structure of the coefficients of the differential equation. Thus, a technical modification in the variational formula is necessary for further progress.

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UNIVERSITY OF ŁÓDŹ
INSTITUTE OF MATHEMATICS
UL. STEFANA BANACHA 22
90-238 ŁÓDŹ
POLAND