MATH. SCAND. 60 (1987), 239–252

SMOOTHNESS OF VECTOR SUMS OF PLANE CONVEX SETS

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Resumo. Glato de vektoraj sumoj de konveksaj aroj ebenaj.

La vektora sumo $A+B$ de du konveksaj aroj glataj A kaj B montriĝas esti ne ĉiam nefinie glata. Por ebenaj aroj kun analitaj randoj, la preciza rezulto estas ke la rando de $A+B$ apartenas al la Hölder-a klaso $C^{20/3}$. En la artikolo estas difinitaj glatecaj klasoj adaptitaj al la kalkulado de vektoraj sumoj de finia nombro da konveksaj aroj en dudimensia vektora spaco.

1. Introduction.

Let A and B be two convex subsets of R^2 with real-analytic boundaries. Then their vector sum

$$A+B = \{a+b;\ a\in A, b\in B\}$$

does not necessarily have a smooth (C^∞) boundary. However, the boundary $\partial(A+B)$ is always of class $C^{20/3}$, i.e., it is described by a function possessing derivatives up to order six, and the sixth derivative is Hölder continuous with exponent 2/3. More generally, the conclusion holds for the boundary of the vector sum of any finite number of convex sets in R^2 whose boundaries are smooth and do not possess infinitely flat points; see Theorem 5.4.

The purpose of this paper is to prove these and a few related results. We introduce classes of smoothness which are adapted to the operation of vector addition; see Definitions 3.1–3.3 and 5.1.

The crucial questions which led to this investigation were asked by Christer Borell. I am very grateful to him for this. I would also like to thank Jan Boman, Steven G. Krantz and Wang Xiaoqin who, in addition to Christer Borell, have made valuable remarks on this matter. In particular, Jan Boman's approach to the problem of coordinate invariance was much more natural than mine, and consequently the new proof of Theorem 5.2 given here is more direct than the original one.

Received September 13, 1985; in revised form April 5, 1986.

2. Vector addition and infimal convolution.

The vector sum of two convex sets is most conveniently described, at least locally, by means of the infimal convolution which we proceed to define.

Let f and g be two functions on the real line with values in the extended real line $[-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$. Then their *infimal convolution* $f \square g$ is defined by

$$f \square g(x) = \inf_y (f(y) +^* g(x-y))$$
$$= \inf_y (f(y)+g(x-y);\ f(y) < +\infty,\ g(x-y) < +\infty).$$

Here $+^*$ denotes *upper addition*, which is defined as an extension of the usual addition to $[-\infty, +\infty]$, taking $a +^* b$ to be $+\infty$ if one of a and b is equal to $+\infty$, and to be $-\infty$ if both a and b are equal to $-\infty$. (Such precautions are necessary only if both $+\infty$ and $-\infty$ occur as values, and we may leave them aside here, since most functions will be finite at all points of interest. For details concerning these conventions, see e.g. Moreau [3] and Kiselman [1; p. 159].) The origin of the name of the operation $\square$ is made apparent by the following heuristic but sometimes surprisingly accurate formula, where we write $*$ for the usual convolution:

$$(e^{-f} * e^{-g})(x) = \int e^{-f(y)-g(x-y)}dy \approx \sup_y e^{-f(y)-g(x-y)} = (e^{-(f \square g)})(x).$$

Infimal convolution corresponds to vector addition of the strict epigraphs of f and g. Let us denote by epi f the *epigraph of* f

$$\operatorname{epi} f = \{(x,t) \in \mathbf{R}^2;\ t \geqq f(x)\},$$

and by $\operatorname{epi}_s f$ the *strict epigraph*

$$\operatorname{epi}_s f = \{(x,t) \in \mathbf{R}^2;\ t > f(x)\}.$$

Then it is easily verified that

$$\operatorname{epi}_s f + \operatorname{epi}_s g = \operatorname{epi}_s (f \square g).$$

Moreover,

$$\operatorname{epi} f + \operatorname{epi} g \subset \operatorname{epi} (f \square g) \subset \overline{\operatorname{epi}_s f + \operatorname{epi}_s g} \subset \overline{\operatorname{epi} f + \operatorname{epi} g},$$

where the bar denotes closure in $\mathbf{R}^2$. If $f \square g$ is lower semicontinuous, then epi $(f \,\square\, g)$ is closed and we have

$$\overline{\operatorname{epi} f + \operatorname{epi} g} = \operatorname{epi}(f \,\square\, g).$$

If we know that the vector sum epi f + epi g is closed, we can simply state

$$\operatorname{epi} f + \operatorname{epi} g = \operatorname{epi}(f \,\square\, g).$$

Now, given a boundary point c of a closed convex plane set A with interior points, we can choose an affine coordinate system such that A agrees near c with the epigraph of some convex function. If we have three sets A, B and $A+B$, we can reduce questions about their smoothness to the local behavior of infimal convolutions — except in some cases when the boundary of $A+B$ has the best possible regularity. To be precise we state the following result:

PROPOSITION 2.1. *Let A and B be two closed convex sets in* R^2 *with C^1 boundaries, and consider a point $c \in \partial(A+B)$. If c happens to belong to $A+B$, say $c = a+b$ with $a \in A$, $b \in B$, then there exists an affine coordinate system and convex functions f and g such that A, B and $A+B$ agree with* epi f, epi g *and* epi $(f \,\square\, g)$ *near a, b and c, respectively. If, on the other hand, $c \notin A+B$, then $\partial(A+B)$ contains an entire straight line through c, so that $A+B$ is either a half-plane or a strip.*

PROOF. Let us choose an affine coordinate system with c as the origin and such that all points in $A+B$ lie in the upper half-plane $x_2 \geqq 0$. We shall consider first the case $c \notin A+B$. The quantity $\alpha = \inf(x_2; x \in A)$ must be finite, and $\beta = \inf(y_2; y \in B)$ must be equal to $-\alpha$. Since $c \in \partial(A+B)$, there exist points $c^{(j)} = a^{(j)} + b^{(j)} \in A+B$ converging to c. No subsequence of $(a^{(j)})$ can converge, for otherwise we would get $c \in A+B$. However, the second coordinates $a_2^{(j)}$ must converge, for

$$\alpha \leqq a_2^{(j)} = c_2^{(j)} - b_2^{(j)} \leqq c_2^{(j)} - \beta \to -\beta = \alpha.$$

Therefore the first coordinates $a_1^{(j)}$ account for the divergence: we may for example assume that $(a_1^{(j)})$ is unbounded to the right. Then $(b_1^{(j)})$ must be unbounded to the left. Construct two sequences (j_m) and (k_m) with the properties that

$$a_1^{(j_m)} + b_1^{(k_m)} \leqq -m \quad \text{and} \quad a_1^{(j_{m+1})} + b_1^{(k_m)} \geqq m$$

(given j_m, find k_m first and then j_{m+1}). We conclude that $A+B$ contains a segment $[a^{(j_m)}+b^{(k_m)}, a^{(j_{m+1})}+b^{(k_m)}]$ whose projection on the x_1-axis contains $[-m,m]$ and whose x_2-coordinates tend to zero. Therefore $\partial(A+B)$ must contain the whole x_1-axis, as claimed.

Next assume that $c \in A+B$, and take a coordinate system as before. If $c=0=a+b$ with $a\in A$ and $b\in B$, we must have $x_2 \geqq a_2$ for all $x\in A$ and $y_2 \geqq b_2 = -a_2$ for all $b\in B$. Denote by I the ray $\{x; x_1=0, x_2\geqq 0\}$. Then $A+I$ and $B+I$ are the epigraphs of some functions f and g. Now since A lies in the upper half-plane passing through a, the three sets $\overline{A}, \overline{A+I}$ and $\overline{\operatorname{epi} f}$ agree in some neighborhood of a. Similarly for B and $\overline{A+B}$, that is $\overline{A+B}$ agrees near the origin with the epigraph of some function h, which, however, can be nothing but $f \,\square\, g$. If $\overline{A+B}$ and $A+B$ differ at points arbitrarily close to c, then by the first part of the proof $\partial(A+B)$ will contain an entire line, and it is then not difficult to prove that we must also have $c\notin A+B$ contrary to our hypothesis in the present case. (We do not need this, however.) Thus, near c, $\partial(A+B)$ is given by the graph of $f\,\square\, g$, and $\overline{A+B}$ (in fact $A+B$ itself) is given by the epigraph of $f\,\square\, g$. And the behavior of $f\,\square\, g$ near the origin is completely determined by the behavior of f near the interval $\{x_1; f(x_1)=\inf f = a_2\}$ and the behavior of g near a corresponding interval.

EXAMPLE 2.2. Take $f(x)=x^4/4$ and $g(x)=x^6/6$. Then

$$h(x) = f\,\square\, g(x) = \inf_y\left(\frac{y^4}{4}+\frac{(x-y)^6}{6}\right) = \frac{x^6}{6} - \frac{3|x|^{20/3}}{4} + r(x),$$

where $r\in C^7(\mathbf{R})$. To show this, one may use the observation that the infimum is attained at a unique point y which satisfies $f'(y)=g'(x-y)$, that is $y^3=(x-y)^5$, and which therefore allows $y=x^{5/3}$ as a reasonable approximation. Accepting this result for the time being, we see that $h\in C^{20/3}$ but in no smaller Hölder class. The convex sets

$$A = \operatorname{epi} f = \{(x,t)\in\mathbf{R}^2;\ 4t\geqq x^4\}$$

and

$$B = \operatorname{epi} g = \{(x,t)\in\mathbf{R}^2;\ 6t\geqq x^6\}$$

have real-analytic boundaries, but their vector sum

$$A+B = \operatorname{epi} h = \{(x,t)\in\mathbf{R}^n;\ t\geqq h(x)\}$$

has a boundary of class $C^{20/3}$ only. The orders involved in this example are minimal as we shall see.

When is it true that the boundary of $A+B$ is of class C^k, if A and B are convex sets in R^n with boundaries of class C^k? In R^2 the answer is in the affirmative for $k = 1$, 2, 3 and 4, and, as Example 2.2 shows, in the negative for $k = 7$. A recent example due to Jan Boman shows that the answer is in the negative also for $k = 5$ (infinitely flat points are admitted). In R^3 the regularity drops considerably. In fact, there exist compact convex sets A and B in R^3 with C^∞ boundaries such that $\partial(A+B)$ is not of class C^2. An easy modification of Theorem 3.4 in Kiselman [2] shows this.

3. Smoothness classes of germs of functions.

We shall introduce regularity classes of germs of functions which are adapted to the operations of infimal convolution and vector addition.

DEFINITION 3.1. Let k be a non-negative integer, and let p and $\alpha_1,\ldots,\alpha_m$ be some non-negative numbers ($m \geqq 0$). Then we shall denote by $C_p^k(\alpha_1,\ldots,\alpha_m)_+$, or simply $C_p^k(\alpha)_+$, the cone of all germs of functions f defined for $0 \leqq x \leqq \delta$ for some $\delta > 0$ and such that

$$f(x) = x^p g(x^{\alpha_1},\ldots,x^{\alpha_m}), \quad 0 \leqq x \leqq \delta, \tag{3.1}$$

for some function g of class C^k in a neighborhood of the origin in R^m and satisfying $g(0,\ldots,0) > 0$.

We allow $m = 0$: then g is simply a constant on $\mathsf{R}^m = \{0\}$ and f has the form $f(x) = cx^p$ for some positive constant c. We also permit $\alpha_j = 0$: with the convention $x^0 = 1$ we obviously have for instance $C_p^k(\alpha_1,\alpha_2,0)_+ = C_p^k(\alpha_1,\alpha_2)_+$.

The number p is uniquely determined by f thanks to the requirement that $g(0)$ be positive. However, the function g is not necessarily unique if $m \geqq 2$ and neither are the numbers α_j. For example, if β and γ are positive numbers and $\alpha = \beta\gamma$, and if we take

$$f(x) = x^p(1+x^\alpha) = x^p(1+(x^\beta)^\gamma), \quad x \geqq 0,$$

then $f \in C_p^\infty(\alpha)_+$ but also in $C_p^k(\beta)_+$ provided $k \leqq \gamma$. Another example is

$$f(x) = x^p(1+e^{-1/x}) \quad x > 0, \quad f(0) = 0,$$

which belongs to $C_p^\infty(\alpha_1,\ldots,\alpha_m)_+$ for arbitrary $\alpha_j > 0$ if $m \geqq 1$, but not if $m = 0$.

To define double-sided germs we have to agree on a definition of the powers x^p, x^{α_j} for negative x. It is natural to define $x^p = x^{q/r}$ as $(\sqrt[r]{x})^q$ when r is an odd integer, q an integer.

DEFINITION 3.2. If p and $\alpha_1, \dots, \alpha_m$ are rational numbers with odd denominators, we define $C_p^k(\alpha)$ as the cone of all germs of functions f such that (3.1) holds in a double-sided neighborhood of the origin.

DEFINITION 3.3. Let k, p and $\alpha_1, \dots, \alpha_m$ be as in Definition 3.1. We denote by $V_p^k(\alpha)_+$ the cone of all germs of functions defined for $0 \leqq x \leqq \delta$ for some $\delta > 0$ and such that $f' \in C_p^k(\alpha)_+$. Analogously we define $V_p^k(\alpha)$ provided the conditions in Definition 3.2 are satisfied.

LEMMA 3.4. *Let* k, p *and* $\alpha = (\alpha_1, \dots, \alpha_m)$ *be as in Definition* 3.1. *Then*

$$C_{p+1}^{k+1}(\alpha)_+ + \mathsf{R} \subset V_p^k(\alpha)_+ \subset C_{p+1}^k(\alpha)_+ + \mathsf{R},$$

where R *denotes the constant functions.*

PROOF. If f is in $C_{p+1}^{k+1}(\alpha)_+ + \mathsf{R}$ we have a representation

$$f(x) - f(0) = x^{p+1} G(x^{\alpha_1}, \dots, x^{\alpha_m}), \quad 0 \leqq x \leqq \delta,$$

with $G \in C^{k+1}$. Then

$$f'(x) = x^p g(x^{\alpha_1}, \dots, x^{\alpha_m}) \tag{3.2}$$

where

$$g(t) = g(t_1, \dots, t_m) = (p+1)G(t) + \sum \alpha_j t_j \frac{\partial G}{\partial t_j}(t),$$

in particular we see that $g \in C^k$ and that $g(0) = (p+1)G(0) > 0$. This means that $f \in V_p^k(\alpha)_+$. Next, if $f \in V_p^k(\alpha)_+$ we know that (3.2) holds for some $g \in C^k$ satisfying $g(0) > 0$, and we can define

$$G(t) = \int_0^1 s^p g(s^{\alpha_1} t_1, \dots, s^{\alpha_m} t_m) ds$$

which is a function of class C^k, and we see that

$$f(x) - f(0) = x \int_0^1 f'(sx) ds = x^{p+1} G(x^{\alpha_1}, \dots, x^{\alpha_m}).$$

Moreover, $G(0) = g(0)/(p+1) > 0$, so that $f - f(0) \in C^k_{p+1}(\alpha)_+$.

We shall allow $k = \infty$ in the above definitions and write $\mathscr{E}_p(\alpha)_+ = C^\infty_p(\alpha)_+$ etc., and also briefly consider the spaces $\mathscr{A}_p(\alpha)_+$ obtained when g can be taken real-analytic. Lemma 3.4 shows that $V^\infty_p(\alpha)_+ = \mathscr{E}_{p+1}(\alpha)_+ + \mathbf{R}$.

For $k \geqq 1$ and $p > 0$, the elements of $C^k_p(\alpha)_+$ are germs of strictly increasing functions. The same is true of $C^k_p(\alpha)$ if α and p are as in Definition 3.2 and in addition $p = q/r$ with both q and r odd natural numbers. Under the same conditions, $V^k_p(\alpha)_+$ and $V^k_p(\alpha)$ consists of germs of strictly convex functions.

We shall need a few results on how to operate on these regularity classes.

LEMMA 3.5. *If $f \in C^k_1(\alpha)_+$ and $g \in C^k_p(\alpha)_+$, then $g \circ f \in C^k_p(\alpha)_+$.*

PROOF. We have $f(x) = xF(x^{\alpha_1},\ldots,x^{\alpha_m})$ and $g(y) = y^pG(y^{\alpha_1},\ldots,y^{\alpha_m})$ for $x, y \geqq 0$ sufficiently small, and some F, G of class C^k satisfying $F(0) > 0, G(0) > 0$. Therefore

$$g(f(x)) = x^pF(x^{\alpha_1},\ldots,x^{\alpha_m})^pG(x^{\alpha_1}F(x^{\alpha_1},\ldots,x^{\alpha_m})^{\alpha_1},\ldots,x^{\alpha_m}F(x^{\alpha_1},\ldots,x^{\alpha_m})^{\alpha_m})$$

where all powers of $F(x^{\alpha_1},\ldots,x^{\alpha_m})$ are C^k functions of $(x^{\alpha_1},\ldots,x^{\alpha_m})$.

PROPOSITION 3.6. *Let $k \geqq 1$ and $p > 0$, and let $f \in C^k_p(\alpha)_+$. Then the inverse of f belongs to $C^k_{1/p}(\alpha/p)_+$. A similiar statement holds for $C^k_p(\alpha)$ provided α, p and $1/p$ are rational numbers with odd denominators.*

PROOF. We know that $f(x) = x^pg(x^{\alpha_1},\ldots,x^{\alpha_m})$ as in (3.1), in particular $g(0) > 0$. It is no restriction to assume $g(0) = 1$. We shall solve the equation $\xi = f(x)$ by putting $x = \xi^{1/p}(1+y)^{1/p}$, where y is a new unknown. This gives

$$\xi(1+y)g(\xi^{\alpha_1/p}(1+y)^{\alpha_1/p},\ldots,\xi^{\alpha_m/p}(1+y)^{\alpha_m/p}) = \xi. \tag{3.3}$$

Define a function Φ of $m+1$ variables $t_1,\ldots,t_m,y$ by putting

$$\Phi(t_1,\ldots,t_m,y) = (1+y)g(t_1(1+y)^{\alpha_1/p},\ldots,t_m(1+y)^{\alpha_m/p})-1.$$

This is a C^k function in a neighborhood of the origin in $\mathbf{R}^{m+1}$, and $\Phi(0,\ldots,0,y) = y$. Therefore $\Phi(0,\ldots,0) = 0$ and $\partial\Phi/\partial y(0,\ldots,0,0) = 1 \neq 0$. The implicit function theorem can be applied and yields a C^k function φ of m variables such that $\varphi(0,\ldots,0) = 0$ and

$$\Phi(t_1,\ldots,t_m,\varphi(t_1,\ldots,t_m)) = 0;$$

in other words, $y = \varphi(\xi^{\alpha_1/p}, \ldots, \xi^{\alpha_m/p})$ solves (3.3). The solution x to $f(x) = \xi$ is therefore

$$x = \xi^{1/p}(1+y)^{1/p} = \xi^{1/p}(1+\varphi(\xi^{\alpha_1/p}, \ldots, \xi^{\alpha_m/p}))^{1/p}.$$

This means that $f^{-1} \in C^k_{1/p}(\alpha/p)_+$. For $C^k_p(\alpha)$ the same proof holds since all exponents are rationals with odd denominators.

For the classes $V^k_p(\alpha)_+$ the following results will be needed.

LEMMA 3.7. *If $f \in V^k_0(\alpha)_+$ with $f(0) = 0$ and $g \in V^k_p(\alpha)_+$, then $g \circ f \in V^k_p(\alpha)_+$.*

PROOF. By Lemma 3.4, $f \in C^k_1(\alpha)_+$, and by definition $g' \in C^k_p(\alpha)_+$, so Lemma 3.5 shows that $g' \circ f \in C^k_p(\alpha)_+$. Therefore $(g \circ f)' = (g' \circ f)f' \in C^k_p(\alpha)_+$.

PROPOSITION 3.8. *If $f \in V^k_0(\alpha)_+$ with $k \geqq 1$ and $f(0) = 0$, then also its inverse f^{-1} belongs to $V^k_0(\alpha)_+$. More precisely, if $f(x) = ax + g(x)$ with $g \in V^k_p(\alpha)_+$, $a > 0$, $g(0) = 0$, then $f^{-1}(y) = y/a - h(y)$ with $h \in V^k_p(\alpha, p)_+$.*

PROOF. By Lemma 3.4, $f \in C^k_1(\alpha)_+$, so Proposition 3.6 shows that

$$f^{-1} \in C^k_1(\alpha)_+ \subset V^{k-1}_0(\alpha)_+,$$

which is not quite sufficient. However, $(f^{-1})' = 1/(f' \circ f^{-1})$. We know that $f^{-1} \in C^k_1(\alpha)_+$ and $f' \in C^k_0(\alpha)_+$ so the composition $f' \circ f^{-1}$ is in $C^k_0(\alpha)_+$ by Lemma 3.5. Therefore $(f^{-1})' \in C^k_0(\alpha)_+$ which by definition means that $f^{-1} \in V^k_0(\alpha)_+$.

To prove the last assertion, we first note that $f \in V^k_0(\alpha, p)_+$ (not necessarily in $V^k_0(\alpha)_+$), so that the first part of the proposition gives $f^{-1} \in V^k_0(\alpha, p)_+$. We define a function h by putting $f^{-1}(y) = y/a - h(y)$ and then have

$$y = f(f^{-1}(y)) = af^{-1}(y) + g(f^{-1}(y)) = y - ah(y) + g(f^{-1}(y)).$$

Thus $h = (1/a)g \circ f^{-1}$ where we already know that $f^{-1} \in V^k_0(\alpha, p)_+$ and $g \in V^k_p(\alpha)_+$. Therefore Lemma 3.7 yields $g \circ f^{-1} \in V^k_p(\alpha, p)_+$.

4. Smoothness under the Legendre transformation and infimal convolution.

Let $f : \mathbf{R} \to [-\infty, +\infty]$ be a given function on the real line. We define its *Legendre transform* (or *conjugate function*) as

$$\tilde{f}(\xi) = \sup_{x \in \mathbf{R}} (x\xi - f(x)), \quad \xi \in \mathbf{R}. \tag{4.1}$$

For general properties of conjugate functions, see e.g. Rockafellar [4].

Although it is possible to apply the implicit function theorem directly in the problem concerning the smoothness of $f \square g$, we prefer to pass via the Legendre transformation and consider $f \square g$ as $(\tilde{f}+\tilde{g})^{\sim}$. Since addition causes no difficulties, the whole problem is reduced to a study of the smoothness of $\tilde{f}$. The result in terms of the regularity classes introduced in Section 3 is the following.

THEOREM 4.1. *Let k be a positive integer, let p be a positive number, and let $\alpha_1,\dots,\alpha_m$ be non-negative numbers. If $f \in V_p^k(\alpha_1,\dots,\alpha_m)_+$, then*

$$\tilde{f} \in V_{1/p}^k(\alpha_1/p,\dots,\alpha_m/p)_+.$$

Similarly for $V_p^k(\alpha)$ provided α_j, p and $1/p$ are all rational numbers with odd denominators.

PROOF. We shall prove that $\tilde{f}'(\xi)$ has a representation of type (3.1) with the new parameters. However, the number $x = \tilde{f}'(\xi)$ is the solution to the equation $f'(x) = \xi$ for the given $\xi \geqq 0$. In fact the supremum in (4.1) is attained at the unique x satisfying $f'(x) = \xi$, and since $f''(x) > 0$ for $0 < x < \delta$, x is a C^1 function of ξ and

$$\tilde{f}'(\xi) = x + (\xi - f'(x))dx/d\xi = x.$$

So the problem is to investigate the smoothness of the solution x to $f'(x) = \xi$, ξ being given. Now $f' \in C_p^k(\alpha)_+$ by definition, so Proposition 3.6 yields $(f')^{-1} \in C_{1/p}^k(\alpha/p)_+$. Therefore $\tilde{f}'(\xi) = x$ is a function of ξ in the right smoothness class, which shows that $\tilde{f} \in V_{1/p}^k(\alpha/p)_+$.

For $f \in \mathscr{A}_{p+1}(\alpha)_+$ and $f \in \mathscr{A}_{p+1}(\alpha)$ we get the corresponding result since the implicit function theorem (which was used in the proof of Proposition 3.6) preserves analyticity.

It is easy to track what happens to the smoothness classes under addition:

LEMMA 4.2. *Let $f \in V_p^k(\alpha_1,\dots,\alpha_m)_+$ and $g \in V_q^k(\beta_1,\dots,\beta_n)_+$ and assume that $0 \leqq p \leqq q$. Then $f+g \in V_p^k(\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n,q-p)_+$.*

PROOF. We have representations of f' and g' as follows:

$$f'(x) = x^p F(x^{\alpha_1},\dots,x^{\alpha_m}),$$
$$g'(x) = x^q G(x^{\beta_1},\dots,x^{\beta_n}),$$

where F and G are of class C^k and $F(0) > 0$, $G(0) > 0$. This gives

$$f'(x)+g'(x) = x^p(F(x^{\alpha_1},\dots,x^{\alpha_m})+x^{q-p}G(x^{\beta_1},\dots,x^{\beta_n}))$$

where a new exponent is needed if $q > p$.

THEOREM 4.3. *Let $f \in V_p^k(\alpha_1,\dots,\alpha_m)_+$ and $g \in V_q^k(\beta_1,\dots,\beta_n)_+$, and assume that $k \geqq 1$ and $p \geqq q > 0$. Then*

$$f \,\square\, g \in V_p^k\left(\alpha_1,\dots,\alpha_m,\frac{p}{q}\beta_1,\dots,\frac{p}{q}\beta_n,\frac{p}{q}-1\right)_+.$$

For double-sided germs the analogous result holds under the provision that all numbers α_j, β_k, p, $1/p$, q and $1/q$ are rationals with odd denominators.

PROOF. By Theorem 4.1 we have

$$\tilde{f} \in V_{1/p}^k(\alpha/p)_+ \quad \text{and} \quad \tilde{g} \in V_{1/q}^k(\beta/q)_+.$$

Since $1/p \leqq 1/q$, Lemma 4.2 yields

$$\tilde{f}+\tilde{g} \in V_{1/p}^k(\alpha/p,\beta/q,1/q-1/p)_+.$$

By Theorem 4.1 again

$$(\tilde{f}+\tilde{g})^\sim \in V_p^k(\alpha,p\beta/q,p/q-1)_+.$$

Now $(f \,\square\, g)^\sim = \tilde{f}+\tilde{g}$ wherever $\tilde{f}$ and $\tilde{g}$ are finite, so $(\tilde{f}+\tilde{g})^\sim = (f \,\square\, g)^{\approx}$ near the origin. Finally, the functions we consider satisfy $\tilde{\tilde{h}} = h$ in some neighborhood of the origin, so we are done.

COROLLARY 4.4. *Let $f_1,\dots,f_m$ be a finite number of C^∞ convex functions on $\mathbf{R}$ all with the property that, at every point, some derivative of order two or higher does not vanish. The infimal convolution $f = f_1 \,\square\, \dots \,\square\, f_m$ may be the constant $-\infty$ or an affine function. Otherwise, the germ of f at an arbitrary point consists of an affine function plus a germ from one of the regularity classes*

$$S_p = V_p^\infty(1,2/M(p)) = C_{p+1}^\infty(1,2/M(p))+\mathbf{R}, \quad p = 1,3,5,\dots,$$

where $M(p)$ is the smallest common multiple of the numbers $3,5,7,\dots,p-2$. ($M(1) = M(3) = 1$, $M(5) = 3$, etc.)

PROOF. If for some x the infimum in

$$f_1 \square f_2(x) = \inf_y (f_1(y) + f_2(x-y))$$

is not attained, then $f_1 \square f_2$ is either $-\infty$ identically or an affine function. (Cf. the proof of Proposition 2.1.) This property is then inherited by the convolutions $f_1 \square f_2 \square \ldots \square f_k$, $2 \leqq k \leqq m$. Thus the case when the infimum is always attained remains to be considered, and by means of a translation we can reduce the problem to a study of the germs at the origin. Now, after subtraction of a linear function, each factor f_j has a germ in one of the classes $V_p^\infty(1)$ for some odd integer p. We observe that $V_p^\infty(1) \subset S_p$, and only have to prove that $S_p \square V_q^\infty(1) \subset S_p$ if $q \leqq p$. The special provisions needed for calculating with double-sided germs in Theorem 4.3 are satisfied, so this theorem yields

$$S_p \square V_q^\infty(1) \subset V_p^\infty(1, 2/M(p), p/q, p/q-1).$$

However, this class is precisely S_p: first note that $p/q = 1 + (p/q - 1)$ so that the exponent p/q can be eliminated, then observe that the number

$$\frac{p}{q} - 1 = \frac{p-q}{2} \cdot \frac{2}{q}$$

is a multiple of $2/M(p)$ since it is non-zero only if $q \leqq p-2$. This proves the corollary.

For real-analytic convex functions we get the analogous conclusion with S_p replaced by $\mathscr{A}_{p+1}(1, 2/M(p)) + \mathbf{R}$.

5. Smoothness classes of plane convex sets.

Let A be a subset of a two-dimensional vector space and assume that the boundary of A can be described locally by a function f, so that $y = f(x)$ is the equation of ∂A near a given point for some coordinates (x, y). Of course we would like to say that ∂A has a certain smoothness if f does. But then we need to know that this does not depend on the coordinate system.

DEFINITION 5.1. Let A be a convex subset of an oriented two-dimensional vector space. We shall say that ∂A is of class $V_p^k(\alpha_1, \ldots, \alpha_m)_+$ at a point $c \in \partial A$ if there is an affine coordinate system with positive orientation such that A agrees near c with the epigraph of a function f whose germ at c is in $V_p^k(\alpha_1, \ldots, \alpha_m)_+$.

We would like to prove that all coordinate systems with the x-axis along the tangent at c will do.

THEOREM 5.2. *Let A be a closed convex set in $\mathbb{R}^2$ with the origin on its boundary, and let $M(A)$ be its image under an orientation-preserving linear map M. We assume that the interiors of A and $M(A)$ intersect the positive y-axis, so that both sets are epigraphs of some convex functions F and G near the origin. Let F have the form $F(x) = \lambda x + f(x)$ with $f \in V_p^k(\alpha_1, \ldots, \alpha_m)_+$, and assume that $k \geqq 1$ and that $p > 0$. Then $G(x) = \mu x + g(x)$ with $g \in V_p^k(\alpha_1, \ldots, \alpha_m, p)_+$.*

PROOF. We know that A is defined by $y \geqq F(x)$ near the origin, and similarly $M(A)$ by $y \geqq G(x)$. This means that $y = F(x)$ is equivalent to $M_{21}x + M_{22}y = G(M_{11}x + M_{12}y)$, or again that

$$M_{21}x + M_{22}F(x) = G(M_{11}x + M_{12}F(x)). \tag{5.1}$$

Dividing by x and letting x tend to zero we see that

$$M_{21} + M_{22}\lambda = \mu(M_{11} + M_{12}\lambda)$$

where $\lambda = F'(0)$, $\mu = G'(0)$. After subtracting $(M_{21} + M_{22}\lambda)x = \mu(M_{11} + M_{12}\lambda)x$ from (5.1) we get

$$M_{22}f(x) = \mu M_{12}(f(x) + g(M_{11}x + M_{12}\lambda x + M_{12}f(x)).$$

The number $M_{11} + M_{12}\lambda$ is by necessity positive, for it is the derivative at the origin of the abscissa of the point $M(x, F(x))$ which is on $\partial M(A)$. Therefore the function $h(x) = (M_{11} + M_{12}\lambda)x + M_{12}f(x)$ belongs to $V_0^k(\alpha, p)_+$ (cf. Lemma 4.2) but in general not to $V_0^k(\alpha)_+$.

Proposition 3.8 tells us that $h^{-1} \in V_0^k(\alpha, p)_+$, and Lemma 3.7 that $f \circ h^{-1} \in V_p^k(\alpha, p)_+$. Now

$$g(y) = (M_{22} - \mu M_{12})f(h^{-1}(y))$$

where

$$M_{22} - \mu M_{12} = (M_{11} + M_{12}\lambda)^{-1} \det M$$

is positive. This proves our claim.

COROLLARY 5.3. *It makes sense to define smoothness classes of convex sets using Definition 5.1 if $k \geqq 1$, $p > 0$, and $p \in \Sigma \mathbb{N} \alpha_j$.*

PROOF. If $p = n_1\alpha_1 + \ldots + n_m\alpha_m$ for some non-negative integers n_j, then x^p is a monomial in $x^{\alpha_1}, \ldots, x^{\alpha_m}$. Therefore the exponent p is not needed in the representations, and we see that

$$V_p^k(\alpha_1, \ldots, \alpha_m, p)_+ = V_p^k(\alpha_1, \ldots, \alpha_m)_+.$$

THEOREM 5.4. *Let $A_1, \ldots, A_m$ be a finite number of convex sets in the plane with C^∞ boundaries, and without infinitely flat points. Then the vector sum $A_1 + \ldots + A_m$ is either the whole plane, a half-plane, a parallel strip, or a convex set whose boundary at every point belongs to one of the regularity classes of germs of sets corresponding to $S_p = V_p^\infty(1, 2/M(p))$ for some odd natural number p.*

PROOF. First note that the classes S_p do satisfy the criterion of Corollary 5.3. Indeed $p = p\alpha_1 \in \Sigma \mathbf{N}\alpha_j$. So it is meaningful to speak of regularity classes of sets, not just of functions. Given c on the boundary of $A_1 + \ldots + A_m$ we conclude from Proposition 2.1 that either $\partial(A_1 + \ldots + A_m)$ contains an entire straight line through c or else there are $c_j \in \partial A_j$ such that $c_1 + \ldots + c_m = c$. In the first case we are done; in the second we apply Corollary 4.4 to the epigraphs of functions f_j defining A_j near c_j in a coordinate system whose x-axis is tangent to $A_1 + \ldots + A_m$ at c. If $p_j + 1$ is the order of contact at c_j, then $f_j \in V_{p_j}^\infty(1)$, so the proof of Corollary 4.4 even gives the additional information that $f_1 \,\square \ldots \square\, f_m$ belongs to S_p with $p = \max p_j$.

Let us look in more detail at the situation in Corollary 4.4 and Theorem 5.4 when $m = 2$. The convolution of a germ in $V_p^\infty(1)$ and one in $V_q^\infty(1)$ with $q \leqq p$ is in

$$V_p^\infty(1, p/q, p/q - 1) = V_p^\infty(1, (p-q)/q) = C_{p+1}^\infty(1, (p-q)/q) + \mathbf{R} \tag{5.1}$$

by Theorem 4.3. Now if $q = 1$ or $q = p$, this class is equal to $V_p^\infty(1)$ which contains only C^∞ germs. To get a non-smooth germ by convolving in this way, we have to take both $q > 1$ and $q < p$. Therefore, the lowest orders of contact which may give a convolution which is not in C^∞ are four and six, corresponding to $q = 3$ and $p = 5$. The class (5.1) is then $V_5^\infty(1, 2/3)$, whose germs have representations

$$h(x) = h(0) + x^6 H(x, x^{2/3});$$

see Lemma 3.4. We have a Taylor expansion

$$H(x, x^{2/3}) = H(0) + xH_1'(0) + x^{2/3}H_2'(0) + \text{higher order terms},$$

and arrive at the functions already considered in Example 2.2.

The first few classes S_p occurring in Corollary 4.4 and Theorem 5.4 are

$$S_1 = V_1^\infty(1), \quad S_3 = V_3^\infty(1,2) = C_4^\infty(1)+\mathsf{R},$$
$$S_5 = V_5^\infty(1,2/3), \quad S_7 = V_7^\infty(1,2/15).$$

It is clear that S_5 is contained in $C^{20/3}$ and that all other classes are contained in $C^{8+2/15}$. Therefore the vector sums in Theorem 5.4 have C^8 boundaries except when the orders of contact for two of the sets are as in Example 2.2, i.e., four and six respectively, and the orders of contact for the other $m-2$ sets are at most six. In the exceptional case the boundary is of class $C^{20/3}$ but no better, for it turns out that the negative term $-3/4|x|^{20/3}$ in Example 2.2 can never be wiped out by convolving h with convex functions having zeros of order two, four or six.

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