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REMOVABLE SINGULARITIES FOR H^p AND FOR ANALYTIC FUNCTIONS WITH BOUNDED DIRICHLET INTEGRAL

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Abstract.

Stochastic calculus, estimates for harmonic measure and the theory of Dirichlet forms are used to give sufficient conditions that a set is a removable singularity set for some H^p space and for the space D_a of analytic functions with bounded Dirichlet integral. For example, a set K situated on the boundary ∂Q of a BMO_1 domain Q in $\mathbf{C}^n$ is a removable singularity for H^p for some $p < \infty$ if K has $2n-1$ dimensional Hausdorff measure 0 and it is a removable singularity set for D_a if $C(\partial Q) = C(\partial Q - K)$, where C denotes the Green capacity.

1. Introduction.

Let U be a bounded open set in $\mathbf{C}^n$. If $\phi: U \to \mathbf{C}$ is an analytic function, $0 < p < \infty$, we say that $\phi \in H^p(U)$ if $|\phi|^p$ has a harmonic majorant in U. If $a \in U$ is fixed we define

$$(1.1) \qquad \|\phi\|_{H^p(U)}^p = \inf\{g(a); g \text{ harmonic majorant of } |\phi|^p\}.$$

We say that $\phi \in D_a(U)$, or that ϕ has a *bounded Dirichlet integral*, if

$$(1.2) \qquad \int_U \sum_{j=1}^n \left|\frac{\partial \phi}{\partial z_j}\right|^2 (y)dm(y) < \infty$$

where dm denotes Lebesgue measure. Since condition (1.2) implies that $\phi(U)$ has finite area, all functions in $D_a(U)$ can be seen to belong to $H^p(U)$ for all $p < \infty$. (See the remark following Theorem 2.2.)

Hence

$$(1.3) \qquad D_a(U) \subset H^{p_2}(U) \subset H^{p_1}(U) \quad \text{for all } 0 < p_1 < p_2 < \infty.$$

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If $K \subset U$ is relatively closed we say that K is a *removable singularity* for $H^p(U \setminus K)$ (respectively $D_a(U \setminus K)$) if every function $\phi \in H^p(U \setminus K)$ (respectively $D_a(U \setminus K)$) extends to an analytic function, denoted by $\tilde{\phi}$, on the whole of U.

In this paper we use stochastic calculus, estimates for harmonic measure and Dirichlet forms to study removable singularities for H^p and D_a. It was proved by Parreau [26] that if $U \subset \mathbf{C}$ and K is a compact subset of U with $\mathrm{cap}(K) = 0$ (where cap denotes logarithmic capacity), then K is removable for H^p for all p. In fact, in this case

$$\|\tilde{\phi}\|_{H^p(U)} = \|\phi\|_{H^p(U \setminus K)} \tag{1.4}$$

(see Yamashita [31]). Järvi extended this result to bounded domains in $\mathbf{C}^n$ [22]. See also Fuglede [15]. Conway and Dudziak [8] proved that the only compacts $K \subset U \subset \mathbf{C}$ with the property that K is a removable singularity for H^p and (1.4) holds are the sets K with $\mathrm{cap}(K) = 0$. In section 3 we give a general estimate of the ratio of the H^p norm of an analytic function on U and the H^p norms of its restriction to $U \setminus K$, where $K \subset U$ is compact (Theorem 3.2). In Theorem 3.1 we prove that if $\Lambda_{2n-1}(K) = 0$ (where Λ_k denotes k-dimensional Hausdorff measure) and K is situated on the boundary ∂Q of a BMO_1 domain $Q \subset \mathbf{C}^n$ then K is a removable singularity for H^p, for some $p < \infty$. (Q is a BMO_1 domain if ∂Q is locally described as the graph of a function ψ with $\nabla\psi \in \mathrm{BMO}$. Thus BMO_1 domains are more general than Lipschitz domains. See Jerison and Kenig [23]). If Q is required to be C^1 then K is removable for H^p for all $p > 1$ and if Q is $C^{1+\varepsilon}$ then K is removable for H^1. These results extend a result of Heins [19], Hejhal [20] which states that if $K \subset \mathbf{C}$ is a subset of an analytic arc and K has zero length, then K is removable for H^1. In view of an example due to Hejhal [20] of a set $K \subset \mathbf{C}$ with $\Lambda_1(K) = 0$ situated on the union of the coordinate axes such that K is not removable for H^1, it is clear that not just the metric size of K but also the geometry of K is important. Therefore it is natural to ask to what extent the conditions on Q in Theorem 3.1 can be relaxed. It is known that H^p and H^q have different removable singularities if $p \neq q$ (See Heins [19], Hasumi [17],)

The following result about removable singularities for D_a is due to Carleson ([6, Theorem VI 3]):

Suppose $K \subset \mathbf{C}$ is situated on a simple, closed curve Γ with continuously varying curvature. Then K is removable for D_a if and only if

$$\mathrm{cap}(\Gamma \setminus K) = \mathrm{cap}(\Gamma). \tag{1.5}$$

In section 5 we extend the if – part of this result to $\mathbf{C}^n$ and to subsets K

of the boundary of BMO_1 domains (Theorem 5.2). The condition (1.5) is replaced by a similar condition involving capacities with respect to the Green kernel. The main ingredients in the proof of Theorem 5.2 is a stochastic interpretation of the condition analogue to (1.5) (Theorem 4.1) and the use of general theory of Dirichlet forms. We also use the general stochastic boundary value result for H^p functions established in section 2. (Theorem 2.2 and Corollary 2.3.)

For a characterization of the removable singularities for D_a and other related spaces in terms of condenser capacities see Hedberg [18].

From now oń U will denote a bounded domain in $\mathbf{C}^n$. Brownian motion in $\mathbf{C}^n$ will be denoted by $(\{B_t\}_{t \geqq 0}, \Omega, \mathscr{F} \cdot P^x)$. If $H \subset \mathbf{C}^n$ we let

$$\tau_H = \inf\{t > 0; B_t \notin H\} \tag{1.6}$$

be the (first) exit time from H of B_t. The Green function of a bounded domain $D \subset \mathbf{C}$, $G_D(x, y)$, can be defined using Brownian motion by

$$\int_F G_D(x, y)dm(y) = E^x\left[\int_0^{\tau_D} \chi_F(B_s)ds\right], \quad F \subset D, \tag{1.7}$$

where E^x denotes expectation with respect to the probability law P^x of Brownian motion starting at x.

We also recall the following version of the Lévy theorem, due to Bernard, Campbell and Davie [3]. See also [9].

Let $\phi: U \to \mathbf{C}$ be analytic, non-constant and let $(\tilde{B}_t, \tilde{P}^x)$ denote Brownian motion in $\mathbf{C}$. Put

$$\lambda = \sum \left|\frac{\partial \phi}{\partial z_j}\right|^2 \quad \text{and} \quad \sigma_t(\omega) = \int_0^t \lambda(B_s(\omega))ds; \quad \omega \in \Omega,\ t \leqq \tau_U. \tag{1.8}$$

Then

$$\phi^*(\omega) = \lim_{t \uparrow \tau_U} \phi(B_t) \quad \text{exists a.s. on } \{\omega; \sigma_{\tau_U} < \infty\} \tag{1.9}$$

and the process

$$\hat{B}_t = \begin{cases} \phi(B_{\sigma_t - 1}) & ; \quad t < \sigma_{\tau_U} \\ \phi^* + \tilde{B}_{t - \sigma_{\tau U}}; & t \geqq \sigma_{\tau_U} \end{cases} \tag{1.10}$$

with a probability law $\mathrm{P}^z \times \tilde{\mathrm{P}}^0$ coincides with Brownian motion in $\mathbf{C}$ starting at $\phi(z)$.

The closure of a set W is denoted by $\bar{W}$, $\subset\subset$ means "compactly contained in" and C_0^2 denotes the C^2 functions with compact support. We put

$$D_k(a, R) = \{x \in \mathbb{R}^k; |x-a| < R\}.$$

2. Boundary values of H^p functions.

We first establish a result (Theorem 2.2) about the existence of "Brownian boundary values" for functions in $H^p(U)$, for any $p > 0$. The case when $p > 1$ is a direct consequence of Doob's martingale convergence theorem (see e.g. Williams [30, p. 60]). The general case follows from Burkholder-Gundy's estimates [5] for exit times of Brownian motions. With the possible exception of statement (iii) Theorem 2.2 is well-known. For completeness we give the details.

LEMMA 2.1. *Let $\phi: U \to \mathbb{C}$ be analytic. Then for all stopping times $T < \tau_U$ and all $p > 0$ we have*

$$E^x[|\phi(B_T)|^p] = |\phi(x)|^p + \frac{p^2}{2} E^x\left[\int_0^T |\phi(B_t)|^{p-2} \sum_{j=1}^n \left|\frac{\partial\phi}{\partial z_j}(B_t)\right|^2 dt\right]. \tag{2.1}$$

PROOF. Let $Z_k = Z_t^{(k)} = B_t^{(2k-1)} + iB_t^{(2k)}$; $1 \leqq k \leqq n$, denote complex Brownian motion, $Z_t = (Z_1, \ldots, Z_n)$. Put $Y_t = \phi(Z_t)$.

Then by the complex version of the Ito formula

$$\begin{aligned} dY_t &= \sum_j \frac{\partial\phi}{\partial z_j} dz_j + \sum_j \frac{\partial\phi}{\partial \bar{z}_j} d\bar{Z}_j + \frac{1}{2}\sum_{j,k} \frac{\partial^2\phi}{\partial z_j \partial z_k} dZ_j dZ_k \\ &\quad + \frac{1}{2}\sum_{j,k} \frac{\partial^2\phi}{\partial \bar{z}_j \partial \bar{z}_k} d\bar{Z}_j d\bar{Z}_k + \sum_{j,k} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k} dZ_j d\bar{Z}_k \\ &= \sum_j \frac{\partial\phi}{\partial z_j} dZ_j, \quad \text{since } \phi \text{ is analytic and } dZ_j dZ_k = 0. \end{aligned}$$

Hence

$$\phi(Z_t) = \phi(Z_0) + \int_0^t \sum_j \frac{\partial\phi}{\partial z_j}(z) dZ_j \quad \text{for } t < \tau_U. \tag{2.2}$$

So if we put $f(z) = |z|^p$ and $W_t = f(Y_t)$ then

$$\begin{aligned} dW_t &= \frac{\partial f}{\partial z} dY_t + \frac{\partial f}{\partial \bar z} d\bar Y_t + \frac{\partial^2 f}{\partial z \partial \bar z} dY_t d\bar Y_t \\ &= \frac{p}{2} |Y_t|^{p-2} \bar Y_t dY_t + \frac{p}{2} |Y_t|^{p-2} Y_t d\bar Y_t + \frac{p^2}{4} |Y_t|^{p-2} dY_t d\bar Y_t. \end{aligned}$$

Since

$$dY_t d\bar Y_t = 2 \sum_j \left| \frac{\partial \phi}{\partial z_j} \right|^2 dt,$$

we get

$$\begin{aligned} E^x[|\phi(Z_T)|^p] &= E^x[W_T] \\ &= W_0 + \frac{p^2}{2} E^x\left[\int_0^T |\phi(Z_t)|^{p-2} \sum_j \left| \frac{\partial \phi}{\partial z_j}(B_t) \right|^2 dt \right], \end{aligned}$$

as claimed.

THEOREM 2.2. *Let $0 < p < \infty$. The following are equivalent:*

(i) $\phi \in H^p(U)$.

(ii) *For all $x \in U$ there exists $M < \infty$ such that $E^x[|\phi(B_T)|^p] \leqq M$ for all stopping times $T < \tau_U$.*

(iii) $E^x[\sigma_{\tau_U}^{p/2}] < \infty$ *for all $x \in U$.*

(iv) $E^x\left[\int_0^{\tau_U} |\phi(B_s)|^{p-2} \sum_j \left| \frac{\partial \phi}{\partial z_j}(B_s) \right|^2 ds \right] < \infty$ for all $x \in U$

(v) $\int_U |\phi(y)|^{p-2} \sum_j \left| \frac{\partial \phi}{\partial z_j}(y) \right|^2 G(x, y) dm(y) < \infty$ for all $x \in U$.

PROOF. (iv) and (v) are equivalent by the stochastic interpretation of the Green function:

$$\int_U f(y) G(x, y) dm(y) = E^x\left[\int_0^{\tau_U} f(B_t) dt \right].$$

The equivalence of (ii) and (iv) follows by Lemma 2.1. By Lévy's theorem we get

$$E^x[|\phi(B_T)|^p] = \tilde E^{\phi(x)}[|\tilde B_{\sigma_T}|^p]. \tag{2.3}$$

As noted by B. Davis ([11, p. 924]) the Burkholder-Gundy estimate for stopping times for Brownian motion ([5]) applies to σ_T as well, so that

$$(2.4)\quad \tilde{E}^{\phi(x)}[|\tilde{B}_{\sigma_T}|^p] \sim \tilde{E}^{\phi(x)}[(|\phi_{(x)}|^2+2n\sigma_T)^{p/2}] = E^x[(|\phi(x)|^2+2n\sigma_T)^{p/2}],$$

where $a \sim b$ mens $1/c\, a \leqq b \leqq ca$ for some constant c. Combining (2.3) and (2.4) we get (ii)$\Leftrightarrow$(iii).

(i)$\Rightarrow$(ii): If $\phi \in H^p(U)$, let $h(x)$ denote a harmonic majorant of $|\phi|^p$. Then for all stopping times $T < \tau_U$ we have

$$(2.5)\qquad E^x[|\phi(B_T)|^p] \leqq E^x[h(B_T)] = h(x).$$

(ii)$\Rightarrow$(i): Suppose (ii) holds. Let $\{U_k\}_{k=1}^{\infty}$ be an increasing sequence of open sets such that $\bar{U}_k \subset U$ and $U = \cup_{k=1}^{\infty} U_k$. Put $\tau_k = \tau_{U_k}$ and define

$$(2.6)\qquad \phi_k(\omega) = \phi(B_{\tau_k}(\omega)),\quad k = 1,2,\dots.$$

Then by the strong Markov property we have (θ_t denotes the time shift operator $\theta_t(B_s) = B_{t+s}$)

$$\begin{aligned}
E^x[|\phi_m-\phi_k|^p] &= E^x[|\phi(B_{\tau_m})-\phi(B_{\tau_k})|^p]\\
&= E^x[E^x[|\phi(B_{\tau_m})-\phi(B_{\tau_k})|^p|\mathscr{F}_{\tau_k}]]\\
&= E^x[E^{B_{\tau_k}}[|\phi(B_{\tau_m}-\phi(B_0)|^p]\\
&\sim E^x[E^{B_{\tau_k}}[\sigma_{\tau_m}^{p/2}]] = E^x\left[E^x\left[\theta_{\tau_k}\left(\int_0^{\tau_m}\lambda(B_s)ds\right)^{p/2}|\mathscr{F}_{\tau_k}\right]\right]\\
&= E^x\left[E^x\left[\left(\int_{\tau_k}^{\tau_m}\lambda(B_s)ds\right)^{p/2}|\mathscr{F}_{\tau_k}\right]\right] = E^x\left[\left(\int_{\tau_k}^{\tau_m}\lambda(B_s)ds\right)^{p/2}\right] \to 0.
\end{aligned}$$

as $k,m \to \infty$. Therefore $\{\phi_k\}$ is a Cauchy sequence in $L^p(\Omega, P^x)$. (If $0 < p < 1$, the metric is given by the distance

$$d_p(f,g) = E^x[|f-g|^p].)$$

By completeness of $L^p(\Omega, P^x)$ there exists $\phi^* \in L^p(\Omega, P^x)$ such that

$$E^x[|\phi_k-\phi^*|^p] \to 0.$$

By Harnack's inequalities $\phi^* \in L^p(\Omega, P^x)$ for all $x \in U$. Put

$$g(x) = E^x[|\phi^*|^p].$$

Then if V is open, $V \subset\subset U$ we have $\bar{V} \subset U_k$ for k large enough, and so

$$\theta_{\tau_V}(|\phi^*|^p) = \lim_{k \to \infty} \theta_{\tau_V}(|\phi_k|^p) = \lim_{k \to \infty} \theta_{\tau_V}(|\phi(B_{\tau_k})|^p)$$

$$= \lim_{k \to \infty} |\phi(B_{\tau_k})|^p = |\phi^*|^p.$$

Therefore, by the strong Markov property

$$E^x[(g(B_{\tau_V})] = E^x[E^{B_{\tau_V}}[|\phi^*|^p]] = E^x[E^x[\theta_{\tau_V}(|\phi^*|^p)|\mathscr{F}_{\tau_V}]]$$

$$= E^x[E^x[|\phi^*|^p|\mathscr{F}_{\tau_V}]] = E^x[|\phi^*|^p] = g(x),$$

and hence g is harmonic. Moreover, by Lemma 2.1 we have

$$g(x) = E^x[|\phi^*|^p] = \lim_k E^x[|\phi(B_{\tau_k})|^k] \geqq |\phi(x)|^p,$$

and we conclude that g is a harmonic majorant of $|\phi|^p$. Moreover, g is the least harmonic majorant of $|\phi|^p$, because if h is any harmonic majorant of $|\phi|^p$ we have

$$g(x) = E^x[|\phi^*|^p] = \lim_k E^x[|\phi(B_{\tau_k})|^p] \leqq \lim_k E^x[h(B_{\tau_k})] = h(x).$$

That completes the proof of Theorem 2.2.

REMARK. It is a consequence of (iii), Theorem 2.2 that if $\phi(U)$ has finite area, then $\phi \in H^p(U)$ for all $p < \infty$. This is seen as follows: Since $\sigma_{\tau_U} \leqq \tau_{\phi(U)}$ (by the Lévy theorem) it is enough to prove that $E^y[\tau_{\phi(U)}^{p/2}] < \infty$ for all $p < \infty$. For this it suffices to prove that $E^0[\tau_D^{p/2}] < \infty$ where $D = D_2(0, R)$ with $\pi R^2 = \text{Area}(\phi(U))$, by a result due to Aizenman and Simon [2]. And this last inequality can be verified directly using the law of Brownian motion.

The last part of the proof of Theorem 2.2 also proves the following:

COROLLARY 2.3. *Let $\phi \in H^p(U)$, $0 < p < \infty$. Let ϕ_k be as defined in* (2.6) *above. Then there exists a "stochastic boundary value function" $\phi^*(\omega)$ given by*

$$\phi^*(\omega) = \lim_{t \to \tau_U} \phi(B_t). \tag{2.7}$$

We have $\phi^* \in L^p(\Omega, P^x)$ and

(2.8) $$E^x[|\phi_k - \phi^*|^p] \to 0 \quad as \quad k \to \infty$$

for all $x \in U$. *The function*

(2.9) $$g(x) = E^x[|\phi^*|^p]$$

is the least harmonic majorant of $|\phi|^p$ *and*

(2.10) $$\|\phi\|_{H^p}^p = E^a[|\phi^*|^p] = \sup\{E^a[|\phi(B_T)|^p]; T \text{ stopping time } < \tau_U\}$$
$$= |\phi(a)|^p + \frac{p^2}{2} E^a\left[\int_0^{\tau_U} |\phi(B_t)|^{p-2} \sum_{j=1}^n \left|\frac{\partial \phi}{\partial z_j}(B_t)\right|^2 dt\right]$$

or $0 < p < \infty$.

REMARK. The existence of the limit in (2.7) follows from the Lévy theorem and from the fact that $\sigma_{\tau_U} < \infty$ a.s. *when* $\phi \in H^p(U)$ (Theorem 2.2 (iii)).

Now assume that U is a BMO_1 domain. Then the Martin boundary of U coincides with the topological boundary of U (Jerison and Kenig [23, Theorem 5.9]). For $1 \leqq p < \infty$ it follows from Corollary 2.3 that the family $\{\phi_k\}$ is uniformly integrable with respect to P^x, for each $x \in U$, so by a result due to Doob [12] we get that there exists a fine boundary value function – also denoted by ϕ – such that

(2.11) $$\phi(x) = E^x[\phi(B_{\tau_U})] \quad \text{for all } x \in U.$$

Moreover,

(2.12) $$\lim_{t \to \tau_U} \phi(B_t(\omega)) = \phi(B_{\tau_U}(\omega)) \text{ a.s. } P^x$$

Thus we have

(2.13) $$\phi^*(\omega) = \phi(B_\tau(\omega))$$

if $1 \leqq p < \infty$ and U is a BMO_1 domain.

3. Removable singularities for H^p functions.

We are now ready to prove the main result about removable singularities for H^p:

THEOREM 3.1. *Let K be a relatively closed subset of $U \subset \mathbf{C}^n$. Suppose K is situated on the boundary ∂Q of a domain Q and that $\Lambda_{2n-1}(K) = 0$.*

(i) *If Q is a $C^{1+\varepsilon}$ domain for some $\varepsilon > 0$, then K is a removable singularity for $H^1(U \setminus K)$.*

(ii) *If Q is a C^1 domain, then K is a removable singularity for $H^p(U \setminus K)$ for all $p > 1$.*

(iii) *If Q is a BMO_1 domain, then there exists $p < \infty$ such that K is a removable singularity for $H^p(U \setminus K)$.*

PROOF. First assume that $n = 1$. (iii): Assume that Q is a BMO_1 domain and let ϕ be analytic on $U \setminus K$. We may assume that U is an open rectangle, so small that both $V = U \cap Q$ and $W = U \setminus \bar{Q}$ are BMO_1 domains and ϕ is analytic on $\bar{U} \setminus K$. Fix $z \in \partial Q \cap U \setminus K$. Choose an open disc $D \subset U$ centered at z such that $\bar{D} \cap K = \emptyset$ and put $\hat{V} = V \cup D$, $\hat{W} = W \cup D$. By modifying ∂Q near ∂D if necessary we may assume that $\hat{V}$ and $\hat{W}$ are BMO_1 domains. Let V_k be the domain obtained by shifting the domain $\hat{V}$ by the distance $1/k$ in the direction of the side of U which meets ∂Q and let z_k denote the corresponding translate of z. If ds_k, $d\lambda_k$ and ds, $d\lambda$ denotes arc length, harmonic measure with respect to z_k on ∂V_k and arc length, harmonic measure with respect to z on $\partial \hat{V}$, we put

$$f_k = \frac{ds_k}{d\lambda_k}, \quad f = \frac{ds}{d\lambda}, \quad d\zeta_k = g_k ds_k, \quad d\zeta = g ds \tag{3.1}$$

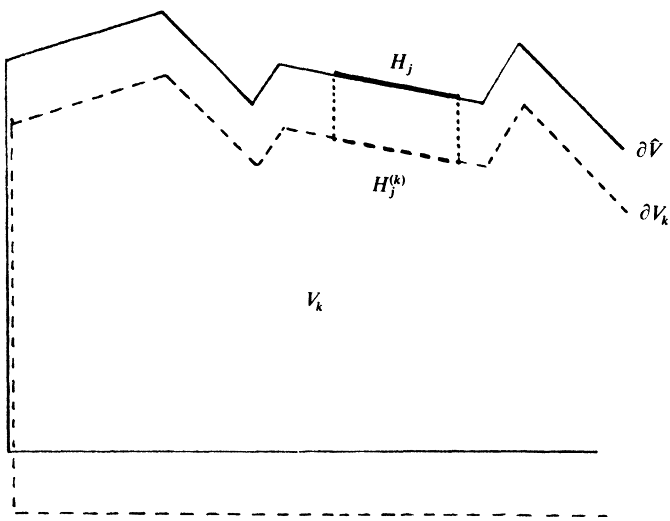

Figure.

(where $d\zeta_k, d\zeta$ is $dx+idy$ on $\partial V_k, \partial \hat{V}$, respectively). Note that $|g_k| \leqq 1$. We claim that

$$g_k(B_{\tau_k}) \to g(B_\tau) \tag{3.2}$$

and

$$f_k(B_{\tau_k}) \to f(B_\tau) \quad \text{for a.a. } \omega \text{ with respect to } P^x \tag{3.3}$$
$$\text{for each } x \in U, \text{ where } \tau_k = \tau_{V_k} \text{ and } \tau = \tau_{\hat{V}}.$$

To prove (3.3) we argue as follows:

For each $j > 0$ there exists a relatively open $H_j \subset \partial \hat{V}$ such that f is continuous outside H_j and $s(H_j) < 1/j$. Let $H_j^{(k)}$ be the set H_j shifted to ∂V_k. Then

$$P^x[B_{\tau_k} \in H_j^{(k)} \text{ for infinitely many } k] \leqq \varepsilon(j) \to 0 \text{ as } j \to \infty.$$

So

$$P^x[B_{\tau_k} \in H_j^{(k)} \text{ for infinitely many } j \text{ and } k] = 0.$$

Hence for a.a. ω there exist $j(\omega)$ and $k(\omega)$ such that for all $k \geqq k(\omega)$, $j \geqq j(\omega)$ we have

$$B_{\tau_k} \notin H_{j(\omega)}^{(k)}.$$

For such ω we have that

$$f_k(B_{\tau_k}) \to f(B_\tau),$$

since f is continuous outside H_j and f_k is obtained by shifting f to ∂V_k. Similarly one obtains (3.2).

Since $\hat{V}$ is a BMO_1 domain we know that $\lambda \in A_\infty(s)$ (Jerison and Kenig [23, Theorem 10.1]). So there exists $\delta > 0$ and $C_1 < \infty$ such that

$$E^x[|f_k(B_{\tau_k})|^{1+\delta}] \leqq C_1 \quad \text{for all } k.$$

(See Coifman and Fefferman [7, p. 248].) Put

$$h_k(\omega) = \phi(B_{\tau_k}) g_k(B_{\tau_k}) f_k(B_{\tau_k}).$$

Then for $\beta > 0$ we have

$$E^x[|h_k|^{1+\beta}] \leqq E^x[|\phi(B_{\tau_k})|^{q(1+\beta)}]^{1/q} E^x[|g_k(B_{\tau_k}) f_k(B_{\tau_k})|^{q'(1+\beta)}]^{1/q'},$$

where $1/q + 1/q' = 1$. So if we choose $\beta = \delta/4$, $q' = (1+\delta/2)/(1+\delta/4)$ and $q = 2+4/\delta$ we see that

$$E^x[|h_k|^{1+\delta/4}] \leqq C_2 \text{ (independent of } k)$$

if $\phi \in H^p(U \setminus K)$ for $p = (2+4/\delta)(1+\delta/4) = 3+4/\delta+\delta/4$. Therefore the sequence $\{h_k\}_k$ is uniformly integrable with respect to P^x (see e.g. [30]) and we conclude that h_k converges in $L^1(P^x)$. This gives that, with $\psi_k = h_k \cdot (2\pi i(B_{\tau_k} - z))^{-1}$,

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial V_k} \frac{\phi(\zeta) d\zeta}{\zeta - z} = E^{z_k}[\psi_k] = E^{z_k}[\psi_k] - E^z[\psi_k] + E^z[\psi_k]$$

$$\to \frac{1}{2\pi i} \int_{\partial \hat{V}} \frac{\phi(\zeta) d\zeta}{\zeta - z} \quad \text{as } k \to \infty.$$

Hence

(3.4) $$\phi(z) = \frac{1}{2\pi i} \int_{\partial \hat{V}} \frac{\phi(\zeta) d\zeta}{\zeta - z}.$$

Similarly we obtain

(3.5) $$\phi(z) = \frac{1}{2\pi i} \int_{\partial \hat{W}} \frac{\phi(\zeta) d\zeta}{\zeta - z}.$$

By adding (3.4) and (3.5) and noting that

(3.6) $$\phi(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\phi(\zeta) d\zeta}{\zeta - z}$$

we obtain

(3.7) $$\phi(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\phi(\zeta) d\zeta}{\zeta - z}.$$

Thus we define

$$\tilde{\phi}(w) = \frac{1}{2\pi i} \int_{\partial U} \frac{\phi(\zeta)d\zeta}{\zeta - w}; \quad w \in U$$

and we have obtained the desired analytic extension of ϕ. This proves part (iii).

The proofs of parts (i) and (ii) are similar. An essential ingredient in the proof of (iii) was that $ds/d\lambda \in L^{1+\delta}(\lambda)$ for some $\delta > 0$. In (i) we use that for $C^{1+\varepsilon}$ domains arc length is boundedly absolutely continuous with respect to harmonic measure (see e.g. Stein [29]) and in (ii) we use that for C^1 domains we have $ds/d\lambda \in L^q(\lambda)$ for all $q < \infty$. (See Dahlberg [10, p. 21].) As before we can then conclude that

$$\int_{\partial V_k} \frac{\phi(\zeta)d\zeta}{\zeta - z} \to \int_{\partial \tilde{V}} \frac{\phi(\zeta)d\zeta}{\zeta - z} \quad \text{as } k \to \infty,$$

and we continue as in case (iii). That completes the proof when $n = 1$.

The proof for the case when $n > 1$ is similar, except that here we use the Bochner-Martinelli integral formula

$$\phi(z) = \frac{1}{nc_n} \int_{\partial V_k} \phi(\zeta) K_b(z, \zeta),$$

where

$$K_b(z, \zeta) = |\zeta - z|^{-2n} \sum_{j=1}^{n} (\bar{\zeta}_j - \bar{z}_j) w_j(\bar{\zeta}) \wedge w(\zeta),$$

$$w(z) = dz_1 \wedge \cdots \wedge dz_n$$

$$w_j(z) = (-1)^{j-1} dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n$$

and

$$c_n = \frac{(-1)^{n(n-1)/2}(2\pi i)^n}{n!}.$$

(See e.g. Rudin [28, p. 347] or Krantz [25, p. 15].) Taking limits as $k \to \infty$ we obtain as for $n = 1$

$$\phi(z) = \frac{1}{nc_n} \int_{\partial \tilde{V}} \phi(\zeta) K_b(z, \zeta)$$

and similarly

$$\phi(z) = \frac{1}{nc_n} \int_{\partial \hat{W}} \phi(\zeta) K_b(z, \xi).$$

By adding these formulas the integrals over $\partial \hat{V} \cap \partial \hat{W}$ cancel and we are left with

$$2\phi(z) = \frac{1}{nc_n} \int_{\partial U} \phi(\zeta) K_b(z, \zeta) + \frac{1}{nc_n} \int_{\partial D} \phi(\zeta) K_b(z, \zeta).$$

Since

$$\phi(z) = \frac{1}{nc_n} \int_{\partial D} \phi(\zeta) K_b(z, \zeta),$$

we conclude that

$$\phi(z) = \frac{1}{nc_n} \int_{\partial U} \phi(\zeta) K_b(z, \zeta).$$

Now define

$$\tilde{\phi}(w) = \frac{1}{nc_n} \int_{\partial U} \phi(\zeta) K_b(w, \zeta) \quad \text{for } w \in U.$$

Then $\tilde{\phi}$ coincides with ϕ in $U \setminus K$ and $\tilde{\phi}$ is smooth in U. This implies that $\tilde{\phi}$ is in fact analytic in U, since $U \setminus K$ is dense in U.

This completes the proof of Theorem 3.1.

Theorem 3.1 gives no information about the $H^p(U)$ norm of the extension $\tilde{\phi}$ of $\phi \in H^p(U \setminus K)$. In the case when K is a *compact* subset of $U \subset \mathbf{C}$ we can estimate the norm of a function $\psi \in H^p(U)$ by its $H^p(U \setminus K)$ norm as follows:

THEOREM 3.2. *Suppose $U \subset \mathbf{C}$ and that K is a compact subset of U. Then for all $p > 0$ there exists a constant $A = A(K)$ such that*

$$\|\tilde{\psi}\|_{H^p(U)} \leqq A \|\psi\|_{H^p(U \setminus K)} \quad \textit{for all } \psi \in H^p(U).$$

PROOF. We may assume that K is not polar. Choose open sets W, $\{U_k\}_{k=1}^{\infty}$ such that $a \in \partial W$, $K \subset W \subset\subset U_1 \subset U_2 \subset \cdots$ and $U = \cup U_k$. It suffices to prove that there exists a constant A independent of k and ψ such that

$$E^a[|\psi(B_{\tau_k})|^p] \leqq AE^a[|\psi(B_{\tau'_k})|^p] \tag{3.8}$$

for all ψ analytic in U and all $k = 1, 2, \ldots$, where $\tau_k = \tau_{U_k}$ and $\tau'_k = \tau_{U_k \setminus K}$. By the strong Markov property we have for all $x \in \partial W$

$$\begin{aligned} E^x[|\psi(B_{\tau_k})|^p] &= E^x[E^x[|\psi(B_{\tau_k})|^p | \mathcal{F}_{\tau'_k}]] = E^x[E^{B_{\tau'_k}}[|\psi(B_{\tau_k})|^p]] \\ &= E^x[E^{B_{\tau'_k}}[|\psi(B_{\tau_k})|^p]\chi_{\{\tau'_k < \tau_k\}}] + E^x[E^{B_{\tau'_k}}[|\psi(B_{\tau'_k})|^p]\chi_{\{\tau'_k = \tau_k\}}] \\ &\leqq \sup_{y \in K} E^y[|\psi(B_{\tau_k})|^p] \cdot P^x[\tau'_k < \tau_k] + E^x[|\psi(B_{\tau'_k})|^p]. \end{aligned} \tag{3.9}$$

So if we put

$$A_k = \sup_{x \in \partial W} E^x[|\psi(B_{\tau_k})|^p],$$

$$A'_k = \sup_{x \in \partial W} E^x[|\psi(B_{\tau'_k})|^p]$$

and

$$\varrho_k = \sup_{x \in \partial W} P^x[\tau'_k < \tau_k] \leqq \sup_{x \in \partial W} P^x[\tau_{U \setminus K} < \tau_U] = \varrho < 1,$$

then by the maximum principle

$$E^y[|\psi(B_{\tau_k})|^p] \leqq A_k \quad \text{for all } y \in K$$

and we have by (3.9)

$$A_k \leqq \varrho_k A_k + A'_k \leqq \varrho A_k + A'_k.$$

Hence

$$A_k \leqq \frac{1}{1-\varrho} A'_k \quad \text{for all } k.$$

By the Harnack inequalities (3.8) follows.

4. A thin set that catches almost all Brownian paths.

We now give a result which describes when a measurable subset of the boundary of a bounded domain in $\mathbf{C}^n$ catches almost all Brownian paths starting from an interior point of the domain. Various versions of this result seem to be known. See Hedberg [18] and Hruščev [21]. Since it is so crucial for the next paragraph we give a proof.

Recall that *the fine topology* on R^k may be described by Brownian motion as follows:

A set $H \subset \mathsf{R}^k$ is finely open and only if $\tau_H > 0$ a.s. P^x for all $x \in H$.

If V is a domain in R^k with a Green function $G = G_V(x, y)$, the Green capacity C_V of a subset F of V is defined as follows:

$$C_V(F) = \sup\{\mu(\mathrm{F})\},$$

the sup being taken over all positive measures μ on F such that

$$\int G_V(x, y) d\mu(y) \leqq 1 \quad \text{for all } x \in \mathsf{R}^k.$$

For information about probabilistic potential theory we refer to [4], [13], and [27].

THEOREM 4.1. *Let*

$$D = D_k(0, R) = \{x \in \mathsf{R}^k; |x| < R\}$$

where $0 < R < \infty$, *let* H *be a relatively closed subset of* D *and let* H_0 *be a (Borel) measurable subset of* H. *Put*

$$H' = \{x \in H;\ \tau_{D \setminus H} = 0 \text{ a.s. } P^x\} = \{x \in H;\ B_t \textit{ hits } H \textit{ immediately} \text{ a.s.}\}.$$

Assume the following holds:

(4.1) *If* B_t *does not hit* H_0 *immediately* a.s. P^x (*that is if* $\tau_{D \setminus H_0} > 0$ a.s. P^x) *then* B_t *hits* ∂D *before* H_0 *with positive* P^x*-probability.*

Then the following are equivalent:

(i) $C_D(H_0) = C_D(H)$.

(ii) B_t *hits* H_0 *immediately*, a.s. P^x *for all* $x \in H'$.

(iii) $\tau_{D \setminus H_0} = \tau_{D \setminus H}$ a.s. P^x *for all* $x \in D$.

(iv) H_0 *is finely dense in* H'.

(v) *For all* $x \in H'$

$$\sum_{m=1}^{\infty} m \cdot C_D(H_0 \cap A_m(x)) = \infty \qquad \textit{if} \quad k = 2$$

$$\sum_{m=1}^{\infty} 2^{m(k-2)} C_D(H_0 \cap A_m(x)) = \infty \quad \textit{if} \quad k > 2$$

where

$$A_m(x) = \{y \in \mathsf{R}^k; 2^{-m-1} < |y - x| \leqq 2^{-m}\}, \quad m = 1, 2, \ldots.$$

PROOF. By considering $H \cap \overline{D_k(0,r)}$ and $H_0 \cap \overline{D_k(0,r)}$ for $r < R$ we see that we may assume that H is compact. The equivalence of (ii), (iii) and (iv) follows directly from the stochastic interpretation of the fine topology. The equivalence of (ii) and (v) follows from the Wiener criterion for hitting a set immediately (see for example Theorem 7.35 in [27])

(i) $\Leftrightarrow$ (ii): The probability of hitting H before ∂H, h_H, may be expressed as

$$h_H(x) = h_H = \int_H G_D(x, y) d\mu_H(y), \tag{4.2}$$

where μ_H is the equilibrium measure on H, i.e.

$$\mu_H(H) = C_D(H),$$

and similarly for H_0. (See [4, p. 285].)

If $C_D(H_0) = C_D(H)$ we conclude that $\mu_H = \mu_{H_0}$ by uniqueness of the equilibrium measure and therefore by (4.2)

$$h_H = h_{H_0}. \tag{4.3}$$

So if B_t does not hit H_0 immediately a.s. P^x, then by (4.1) we have $h_{H_0}(x) < 1$, hence by (4.3), $h_H(x) < 1$ and therefore B_t does not hit H immediately a.s. P^x. Thus (ii) holds.

Conversely, if (ii) holds then (4.3) holds.

Now if $f \in C^2(D)$ with compact support in D then by Green's formula

$$-\frac{1}{2}\int \Delta f(z) G_D(y, z) dm(z) = f(y); \quad x \in D.$$

So by the Fubini theorem, (4.1) and (4.3) we get

$$\begin{aligned}\int f(y) d\mu_H(y) &= -\frac{1}{2}\int \Delta f(z)\left(\int G_D(y, z) d\mu_H(y)\right) dm(z) \\ &= -\frac{1}{2}\int \Delta f(z)\left(\int G_D(y, z) d\mu_{H_0}(y)\right) dm(z) \\ &= \int f(y) d\mu_{H_0}(y), \quad \text{for all such } f.\end{aligned}$$

It follows that $\mu_H = \mu_{H_0}$ and therefore (ii) holds. That completes the proof.

A somewhat surprising consequence of this result is that one can find relatively thin subsets of the boundary of a domain in $\mathbf{C}^n$ such that the subset catches almost all Brownian paths starting from an interior point of the domain before the paths exit from a ball containing the domain:

COROLLARY 4.2. *Let Q be a bounded regular domain in $\mathbf{R}^k$ with $\mathbf{R}^k \setminus \bar{Q}$ connected and choose R such that $D = D_k(0, R) \supset \bar{U}$. Let K be a compact subset of ∂Q. Then the following are equivalent*:

(i) $C_D(\partial Q \setminus K) = C_D(\partial Q)$.

(ii) B_t *hits* $\partial Q \setminus K$ *immediately* a.s. P^x, *for all* $x \in \partial Q$.

(iii) $\tau_{D \setminus (\partial Q \setminus K)} = \tau_{D \setminus \partial Q}$ a.s. P^x *for all* $x \in D$.

(iv) $\partial Q \setminus K$ *is finely dense in* ∂Q.

(v) *For all* $x \in \partial Q$

$$\sum_{m=1}^{\infty} m C_D((\partial Q \setminus K) \cap A_m(x)) = \infty \qquad \text{if } k = 2$$

$$\sum_{m=1}^{\infty} 2^{m(k-2)} C_D((\partial Q \setminus K) \cap A_m(x)) = \infty \qquad \text{if } k > 2,$$

where $A_m(x)$ is as in Theorem 4.1.

Thus, if (i) holds then a.e. Brownian path starting from $x \in Q$ must hit $\partial Q \setminus K$ either before it hits K or immediately after. There are sets K of positive surface area satisfying (i), and thus sets $\partial Q \setminus K$ of surface area less than the area of ∂Q catching a.a. Brownian paths starting from Q. (For an example in the unit circle see Ahlfors and Beurling [1].)

5. Removable singularities for analytic functions with bounded Dirichlet integral.

We now apply the previous results to prove the partial extension of Carleson's result mentioned in the introduction.

THEOREM 5.1. *Let $U \subset \mathbf{C}^n$ be open and K a relatively closed subset of U. Let $D = D_{2n}(0, R) \supset \bar{U}$. Suppose K is situated on the boundary of a* BMO$_1$ *domain Q such that*

$$C_D(\partial Q \cap U \setminus K) = C_D(\partial Q \cap U). \tag{5.1}$$

Then any $\phi \in D_a(U \setminus K)$ extends analytically to U.

PROOF. Let $\phi \in D_a(U \setminus K)$. Then as noted in the introduction $\phi \in H^p(U \setminus K)$ for all $p < \infty$. So we proceed as in the proof of Theorem 3.1 (iii) and in the case $n = 1$ we obtain, using the same notation as there,

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial V_k} \frac{\phi(\zeta)d\zeta}{\zeta - z} \to \frac{1}{2\pi i} \int_{\partial \hat{V}} \frac{\phi_V(\zeta)d\zeta}{\zeta - z} \quad \text{as } k \to \infty,$$

where ϕ_V is the boundary function of $\phi|V$. Similarly

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial \hat{W}} \frac{\phi_W(\zeta)d\zeta}{\zeta - z},$$

where ϕ_W is the boundary function of $\phi|W$. Of course $\phi_V = \phi_W$ on $\partial Q \cap U \setminus K$. The problem here that we did not encounter in Theorem 3.1 is that K may have positive length, so we cannot (yet) conclude that $\phi_V = \phi_W$ a.e. on $\partial Q \cap U \setminus K$. To obtain such a conclusion we proceed as follows:

Since $\hat{V}$ is a BMO_1 domain it follows by a result of P. Jones ([24, Theorem 1]) that $\hat{V}$ is an extension domain for the Sobolev spaces $L_k^p(V)$. In particular, since $\phi|\hat{V}$ has a finite Dirichlet integral it follows from a variant of the Poincaré inequality (see e.g. the proof of Lemma 1.4 in [14]) that $\phi|\hat{V} \in L^2(\hat{V})$ and hence $\phi|\hat{V} \in L_1^2(\hat{V})$, and therefore there exists an extension $\tilde{\phi}$ of $\phi|\hat{V}$ to R^{2n} such that

$$\|\tilde{\phi}\|_{L_1^2(\mathsf{R}^{2n})} = \sum_{|\alpha| \leq 1} \|D^\alpha \tilde{\phi}\|_{L^2(\mathsf{R}^{2n})} < \infty.$$

By Theorem 3.1.3 in [16] there exists a C_D-quasicontinuous modification ϕ^V of $\tilde{\phi}|\bar{\hat{V}}$. Then ϕ^V is finely continuous C_D-quasi-everywhere (q.e.) ([16, Theorem 4.3.2]), so $\phi^V = \phi$ q.e. on $\hat{V}$ and (since ϕ_V is a fine boundary function of $\phi|V$) $\phi^V = \phi_V$ a.e. on ∂V.

Similarly, if we consider $\phi|\hat{W}$ we get a q.e. finely continuous function ϕ^W such that $\phi^W = \phi_W$ a.e. on ∂W. Since $\phi_V = \phi_W$ on $\partial Q \setminus K$ and $\partial Q \setminus K$ is finely dense in ∂Q (Corollary 4.2) we conclude that $\phi^V = \phi^W$ q.e. on $\partial Q \cap U$ and hence $\phi_V = \phi_W$ a.e. on $\partial Q \cap U$.

Now the proof of Theorem 3.1 applies to give the conclusion of the theorem when $n = 1$.

The argument for $n > 1$ is similar. As in the proof of Theorem 3.1 we now use the Bochner-Martinelli kernel instead of the Cauchy kernel. That completes the proof.

REMARK. Using Corollary 4.2 we see that the condition (5.1) in Theorem 5.1 can be replaced by the following (apparently) much weaker condition:

(5.2) For all $x \in \partial Q$ we have

$$\sum_{m=1}^{\infty} m \cdot C_D((\partial Q \setminus K) \cap A_m(x)) = \infty \qquad \text{if } n = 1$$

$$\sum_{m=1}^{\infty} 2^{m(2n-1)} C_D((\partial Q \setminus K) \cap A_m(x)) = \infty \quad \text{if } n > 1.$$

This generalizes one part of Theorem 13 in [18].

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