MATH. SCAND. 60 (1987), 273–291

THE LAW OF LARGE NUMBERS, EXAMPLES AND COUNTEREXAMPLES

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In this paper we construct some examples and counterexamples that give answer to some open problems arising in the law of large numbers for non-separable and non-measurable random variables, a theory which has recently been developed by J. Hoffman-Jørgensen [5] and E. Giné and J. Zinn [3].

I would like to thank J. Hoffmann-Jørgensen who has greatly influenced on the present paper.

Throughout all of this paper, $(S, \mathscr{S}, \mu)$ denotes a probability space, B a Banach space, B^* its dual and $\mathrm{LLN}(\mu, B)$ the set of all functions $f: S \to B$ which satisfy the following version of the law of large numbers (see [5])

$$(*) \qquad \exists a \in B\colon \lim_{n \to \infty} \|a - 1/n \sum_{i=1}^{n} f(s_i)\| = 0 \quad \text{for} \quad \mu^\infty - \text{a.s.}\ (s_j) \in S^\infty,$$

where $(S^\infty, \mathscr{S}^\infty, \mu^\infty)$ is the countable product of $(S, \mathscr{S}, \mu)$ with itself.

From [5] we know that if $f \in \mathrm{LLN}(\mu, B)$, then $\int^* \|f\| d\mu < \infty$, f is Gelfand integrable and the a occurring in $(*)$ equals the Gelfand integral of f. M. Telegrand [6] has shown that if $f \in \mathrm{LLN}(\mu, B)$, then the convergence in $(*)$ takes place in probability, too, and if $\int^* \|f\| d\mu < \infty$ and we have convergence in probability in $(*)$, then we also have convergence a. s.

We shall use the notation of [5], where the author considers the following four function spaces which we shall frequently use

$$L^1_w(\mu, B) = \{f: S \to B \mid f \text{ is weakly } \mu\text{-integrable}\}$$

$$L^1_G(\mu, B) = \{f: S \to B \mid f \text{ is Gelfand-integrable}\}$$

$$L^1(\mu, B) = \{f: S \to B \mid f \text{ is Bochner-integrable}\}$$

$$L^1_*(\mu, B) = \{f: S \to B \mid \int^* \|f(s)\| \mu(ds) < \infty\}.$$

Received June 04, 1985.

1. Examples of Banach spaces that satisfy LLN.

First we shall prove two very useful lemmas. We let c denote the cardinal of the continuum, i.e. $c = \operatorname{card} \mathsf{R}$.

LEMMA 1.1. *Let $(S, \mathscr{S}, \mu)$ be a probability space, where S is a Polish space with* $\operatorname{card} S = c$, *$\mathscr{S}$ is the Borel σ-algebra on S and μ is a diffuse probability measure. Then there exists a class $\mathscr{F} = \{F_t | t \in T\}$ of subsets of S such that* $\operatorname{card} T = c$ *and*

$$\operatorname{card} F_t \leqq \aleph_0, \quad t \in T, \tag{1.1.1}$$

$$F_{t'} \cap F_{t''} = \emptyset, \quad t' \neq t'', \quad t', t'' \in T, \tag{1.1.2}$$

$$\forall B \in \mathscr{S}^\infty \text{ with } \mu^\infty(B) > 0, \quad \exists t \in T \text{ such that } B \cap F_t^\infty \neq \emptyset, \tag{1.1.3}$$

$$\bigcup_{t \in T} F_t = S. \tag{1.1.4}$$

PROOF. Let S be a Polish space with $\operatorname{card} S = c$. Then $\operatorname{card} \mathscr{S} = c$ and also $\operatorname{card} \mathscr{S}^\infty = c$. Let $\mathscr{S}_+ = \{B \in \mathscr{S}^\infty | \mu^\infty(B) > 0\}$. Then $\operatorname{card} \mathscr{S}_+^\infty = c$. The set S and the family $\mathscr{S}_+^\infty$ may be enumerated by cardinals less than c: $\mathscr{S}_+^\infty = \{B_t | t < c\}$ and $S = \{s_t | t < c\}$. The set $B_0 \in \mathscr{S}_+^\infty$ has positive measure and therefore $B_0 \neq \emptyset$. Let us take $(s_i^0) \in B_0$ and define the set $F_0 \subseteqq S$ by

$$F_0 = \begin{cases} \{s_1^0, s_2^0, \ldots, \ldots\} & \text{if } s_0 \in \{s_1^0, s_2^0, \ldots, \ldots\} \\ \{s_0, s_1^0, s_2^0, \ldots, \ldots\} & \text{otherwise.} \end{cases}$$

Then $\operatorname{card} F_0 \leqq \aleph_0$, $F_0^\infty \cap B_0 \neq \emptyset$ and $s_0 \in F_0$. The set $(F_0^c)^\infty$ has μ^∞ measure 1 and therefore it intersects every set of positive μ^∞ measure, in particular $(F_0^c)^\infty \cap B_1 \neq \emptyset$. Let us take $(s_i^1) \in (F_0^c)^\infty \cap B_1$ and define the set $F_1 \subseteqq S$ by

$$F_1 = \begin{cases} \{s_1^1, s_2^1, \ldots, \ldots\} & \text{if } s_1 \in \{s_1^1, s_2^1, \ldots, \ldots\} \cup F_0 \\ \{s_1, s_1^1, s_2^1, \ldots, \ldots\} & \text{otherwise.} \end{cases}$$

Then $\operatorname{card} F_1 \leqq \aleph_0$, $F_1^\infty \cap B_1 \neq \emptyset$, $F_1 \cap F_0 = \emptyset$ and $s_1 \in F_0 \cup F_1$. By transfinite induction we shall finish our proof. To this end, suppose that the family $\{F_t | t < \alpha\}$, $\alpha < c$, has been constructed such that it satisfies (1.1.1)–(1.1.4) with α instead of c. Let

$$F^\alpha = \bigcup_{t < \alpha} F_t.$$

We claim that

$$B_\alpha \cap ((F^\alpha)^c)^\infty \neq \phi.$$

To show this, it is enough to prove that

$$(1.1.5) \qquad (\mu^\infty)^*(B_\alpha \cap ((F^\alpha)^c)^\infty > 0.$$

Borel sets in a Polish space are either countable or have cardinality c, and since

$$\operatorname{card} F^\alpha = \operatorname{card} \bigcup_{t<\alpha} F_t \leqq \operatorname{card} \alpha \cdot \aleph_0 < c,$$

we conclude, because μ is diffuse, that

$$\mu_*(F^\alpha) = 0.$$

By Lemma 2.1 in [5], it then follows that

$$(\mu^\infty)^*(((F^\alpha)^c)^\infty) = 1,$$

which implies (1.1.5) since $\mu(B_\alpha) > 0$. Let us take $(s_i^\alpha) \in B_\alpha \cap ((F^\alpha)^c)^\infty$ and define the set $F_\alpha \subseteqq S$ by

$$F_\alpha = \begin{cases} \{s_1^\alpha, s_2^\alpha, \ldots, \ldots\} & \text{if } s_\alpha \in \{s_1^\alpha, s_2^\alpha, \ldots, \ldots\} \cup F^\alpha \\ \{s_\alpha, s_1^\alpha, s_2^\alpha, \ldots, \ldots\} & \text{otherwise.} \end{cases}$$

Then $\operatorname{card} F_\alpha \leqq \aleph_0$, $F_\alpha \cap F_t = \phi$, $\forall t < \alpha$, $s_\alpha \in \bigcup_{t \leq \alpha} F_t$ and $F_\alpha^\infty \cap B_\alpha \neq \phi$. Hence by transfinite induction (1.1.1)–(1.1.4) holds

LEMMA 1.2. *Let $(S, \mathscr{S}, \mu)$ be a probability space, where S is a Polish space with* card $S = c$, *$\mathscr{S}$ is the Borel σ-algebra and μ is a diffuse probability measure. Then for every $n \in \mathbf{N}$ there exists a class $\mathscr{F}_n = \{F_t | t \in T\}$ of subsets of S such that* card $T = c$ *and*

$$(1.2.1) \qquad \operatorname{card} F_t \leqq n+1, \quad \forall t \in T,$$

$$(1.2.2) \qquad F_{t'} \cap F_{t''} = \phi, \quad t' \neq t'', \quad t', t'' \in T,$$

$$(1.2.3) \qquad \forall B \in \mathscr{S}^n \text{ with } \mu^n(B) > 0, \ \exists t \in T \text{ such that } F_t^n \cap B \neq \phi,$$

$$(1.2.4) \qquad \bigcup_{t \in T} F_t = S.$$

PROOF. Replace $\mathscr{S}^\infty$ with $\mathscr{S}^n$ in the proof of Lemma 1.1.

THEOREM 1.3. *Let* $(S, \mathscr{S}, \mu)$ *be a probability space,* T *a set and* $f: S \times T \to \mathsf{R}$ *a stochastic process such that* $f(s) = f(s, \cdot) \in l_2(T)$, *for every* $s \in S$. *If the function* $\eta: S \times S \to \mathsf{R}$ *defined by*

$$\eta(s, v) = \sum_{t \in T} f(s,t) f(v,t) \quad \text{for } (s, v) \in S \times S \tag{1.3.1}$$

is $\mu \times \mu$ *measurable and*

$$\int_S^* \sqrt{\eta(s,s)}\,\mu(ds) < \infty, \tag{1.3.2}$$

then $f \in \mathrm{LLN}(\mu, l_2(T))$.

PROOF. Directly from the definition of η it follows that

$$|\eta(s,v)| \leqq \sqrt{\eta(s,s)}\sqrt{\eta(v,v)}, \quad \forall (s,v) \in S \times S,$$

hence Fubini's theorem and (1.3.2) shows that $\eta \in L_1(S \times S, \mu \times \mu)$. Since

$$T(s) = \{t \mid f(s,t) \neq 0\}$$

is countable and

$$\int_S \sum_{t \in T(s)} |f(s,t) f(v,t)|\,\mu(dv) \leqq \sqrt{\eta(s,s)} \int^* \sqrt{\eta(v,v)}\,\mu(dv) < \infty,$$

we have

$$\int_{S \times S} \eta\, d(\mu \times \mu) = \int_S \mu(ds) \int_S \sum_{t \in T(s)} f(s,t) f(t,v)\,\mu(dv)$$

$$= \int_S \sum_{t \in T} f(s,t) m(t)\,\mu(ds),$$

where $m(t) = \int f(v,t)\mu(dv)$ is the mean of $f(\cdot, t)$. Hence by subtracting from f

its mean we may assume that

$$\int_{S\times S} \eta d(\mu\times\mu) = 0. \tag{1.3.3}$$

We then want to show that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} f(s_i) \right\|_2 = 0 \quad \text{for } \mu^\infty\text{-a.s. } (s_i)\in S^\infty. \tag{1.3.4}$$

The limit in (1.3.4) can be written in the following way

$$\begin{aligned} \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} f(s_i) \right\|_2^2 &= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} f(s_i, t) \right)^2 \\ &= \lim_{n\to\infty} \sum_{t\in T} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} f(s_i, t) f(s_j, t) \\ &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(s_i, s_j) \\ &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \eta(s_i, s_i) + \lim_{n\to\infty} \frac{2}{n^2} \sum_{1\le i<j\le n} \eta(s_i, s_j). \end{aligned} \tag{1.3.5}$$

The last term in (1.3.5) is obtained by symmetry of η. The law of large numbers in [2, p. 122] applied to X_i = the outer μ^∞-envelope of $\sqrt{\eta(\xi_i, \xi_i)}$, and condition (1.3.2) imply that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \eta(s_i, s_i) = 0 \quad \text{for } \mu^\infty\text{-a.s. } (s_i)\in S^\infty.$$

It remains to prove that the last term in (1.3.5) tends to 0 μ^∞-a.s. Let us define

$$U_n = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \eta(s_i, s_j) \quad \text{and} \quad \mathscr{F}_n = \sigma\{U_i \mid i \geqq n\} \quad \text{for } n \geqq 2.$$

Then for $n \geqq 2$ we have

$$\begin{aligned} U_n = E\{U_n \mid \mathscr{F}_n\} &= \frac{2}{n(n-1)} \sum_{1\le i<j\le n} E\{\eta(s_i, s_j) \mid \mathscr{F}_n\} \\ &= E\{\eta(s_1, s_2) \mid \mathscr{F}_n\}. \end{aligned}$$

If we put $\mathscr{T}_{-n} = \mathscr{F}_n$ and $S_{-n} = E\{\eta(s_1, s_2) | \mathscr{F}_n\}$ for $n \geqq 2$, then $\{S_n, \mathscr{T}_n, -\infty < n \leqq -2\}$ is a martingale and $E|S_{-2}| < \infty$ because $\eta \in L^1(\mu \times \mu)$. If

$$\mathscr{T}_{-\infty} = \bigcap_{n=2}^{\infty} \mathscr{F}_n,$$

then Theorem 11.1.1 in [2, p. 376] ensures that

$$\lim_{n \to \infty} S_n \quad (\text{say } S_{-\infty}) \quad \text{exists } \mu \times \mu-\text{a.s.,} \tag{1.3.6}$$

$$S_{-\infty} \text{ is } \mathscr{T}^{-\infty} \text{ measurable,} \tag{1.3.7}$$

$$E|S_{-\infty}| < \infty. \tag{1.3.8}$$

Since $\mathscr{T}_{-\infty}$ is σ-algebra of permutable events, the Hewitt-Savage zero-one law (see [2, Corollary 7.3.8]) ensures that $S_{-\infty}$ is a constant random variable which coincide with $E\eta$. But by (1.3.3) $E\eta = 0$ and therefore

$$\lim_{n \to \infty} U_n = 0.$$

Consequently

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \eta(s_i, s_j) = \lim_{n \to \infty} \frac{n-1}{n} U_n = 0 \quad \text{for } \mu^\infty-\text{a.s. } (s_i) \in S^\infty.$$

Theorem 1.3 gives a sufficient condition for f to belong to $\mathrm{LLN}(\mu, l_2(T))$ and naturally it brings us to the question whether the condition of Theorem 1.3 is also necessary. But in general, as the following result shows, this is not true.

THEOREM 1.4. *Let $(S, \mathscr{S}, \mu)$ be a probability space, where S is a Polish space with card $S = c$, $\mathscr{S}$ is the Borel σ-algebra and μ is a diffuse probability measure. Let $\mathscr{F} = \{F_t | t \in T\}$ be the class of subsets of S that is established by Lemma 1.2 with $n = 2$, and let $f : S \times T \to \mathrm{R}$ be the function defined by*

$$f(s, t) = 1_{F_t}(s). \tag{1.4.1}$$

Then the function $s \mapsto f(s) = f(s, \cdot)$ is an element of $\mathrm{LLN}(\mu, l_2(T))$ but

$$(s, v) \mapsto \eta(s, v) = \sum_{t \in T} f(s, t) f(v, t)$$

is not $\mu \times \mu$ measurable.

PROOF. For every $s \in S$ we have that $f(s) \in l_2(T)$ because

$$\|f(s)\|_2^2 = \sum_{t \in T} |f(s,t)|^2 = \sum_{t \in T} |1_{F_t}(s)|^2 = \sum_{t \in T} 1_{F_t}(s) = 1$$

and this shows that $s \mapsto f(s)$ is a map from S to $l_2(T)$.

We shall prove that

$$\left\|\frac{1}{n} \sum_{i=1}^{n} f(s_i)\right\|_2 \to 0 \quad \mu^\infty-\text{a.s.} \quad (s_i) \in S^\infty \tag{1.4.2}$$

that is $f \in \mathrm{LLN}(\mu, l_2(T))$ with mean 0. We have

$$\begin{aligned}\left\|\frac{1}{n} \sum_{i=1}^{n} f(s_i)\right\|_2^2 &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t \in T} f(s_i,t) f(s_j,t) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t \in T} 1_{F_t}(s_i) 1_{F_t}(s_j) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1_F(s_i, s_j)\end{aligned}$$

where

$$F = \bigcup_{t \in T} F_t \times F_t.$$

If we define

$$L = \{(s_k) \in S^\infty \mid s_i \neq s_j, \forall i,j,\ i \neq j\}, \tag{1.4.3}$$

then $\mu^\infty(L) = 1$ because μ is diffuse. Since card $F_t \leqq 3$, we have for all $(s_i) \in L$

$$\begin{aligned}\left\|\frac{1}{n} \sum_{i=1}^{n} f(s_i)\right\|_2^2 &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1_F(s_i, s_j) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{t \in T} 1_{F_t}(s_i) \sum_{j=1}^{n} 1_{F_t}(s_j) \\ &\leqq \frac{1}{n^2} \sum_{i=1}^{n} 3 \sum_{t \in T} 1_{F_t}(s_i) = \frac{3}{n}\end{aligned}$$

which shows that $f \in \mathrm{LLN}(\mu, l_2(T))$. It remains to prove that the scalar product

$$\eta(s_1, s_2) = (f(s_1), f(s_2)) = \sum_{t \in T} 1_{F_t}(s_1) 1_{F_t}(s_2) = 1_F(s_1, s_2)$$

is not a $\mu \times \mu$ measurable function. To this end, notice that if $s \in F_u$, for the section $F(s)$ we have

$$F(s) = \{v \in S \mid (v, s) \in F\} = \{v \in S \mid (v, s) \in \bigcup_{t \in T} F_t \times F_t\}$$

$$= \{v \in S \mid \exists t \in T: (v, s) \in F_t \times F_t\} = F_u,$$

and so $\mu(F(s)) = 0$ for all $s \in S$. If $A \subseteqq F$ is a $\mu \times \mu$ measurable set, then $\mu(A(s)) = 0$ for every section $A(s)$, $s \in S$, which implies that

$$(\mu \times \mu)(A) = \int \mu(A(s)) \mu(ds) = 0,$$

and consequently $(\mu \times \mu)_*(F) = 0$. If $B \in \mathscr{S}^2$ and $(\mu \times \mu)(B) > 0$, then

$$B \bigcap F = B \bigcap \left(\bigcup_{t \in T} F_t \times F_t \right) = \bigcup_{t \in T} (B \bigcap F_t \times F_t) \neq \emptyset$$

which implies that $(\mu \times \mu)^*(F) = 1$, and so F and thus η is not $\mu \times \mu$ measurable.

REMARK. The family $\{F_t \mid t \in T\}$ has been chosen so that

$$S = \bigcup_{t \in T} F$$

which leads to

$$\left\| \frac{1}{n} \sum_{i=1}^{n} f(s_i) \right\|_2^2 \geqq \frac{1}{n}$$

Therefore, for every $(s_i) \in L$ we have

$$\sqrt{\frac{1}{n}} \leqq \left\| \frac{1}{n} \sum_{i=1}^{n} f(s_i) \right\|_2 \leqq \sqrt{\frac{3}{n}}.$$

which implies that

$$P\left\{1 \leqq \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(s_i) \right\|_2 \leqq \sqrt{3} \right\} = 1.$$

The Portmanteau theorem states that if

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(s_i)$$

converge in law to some Gaussian Radon probability measure γ on $l_2(T)$, then

$$\limsup_{n \to \infty} P\left\{ \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(s_i) \right\|_2 < \alpha \right\} \geqq \gamma(B_\alpha)$$

where B_α is an open ball with radius α. But for $\alpha < 1$

$$P\left\{ \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(s_i) \right\|_2 < \alpha \right\} = 0$$

and thus f does not satisfy CLT (Central Limit Theorem), but it does satisfy LLN.

The following theorem shows that there exist functions that have non-essentially separable valued range, but do satisfy LLN. Several other examples are known (see e.g. [5]).

THEOREM 1.5. *Let* $(S, \mathcal{S}, \mu)$ *be a probability space, where* S *is a Polish space with* card $S = c$, $\mathcal{S}$ *is the Borel* σ*-algebra and* μ *is a diffuse probability measure. Then for any set* T *with* card $T \geqq c$ *we have that* $L^1(\mu, B)$ *is the proper subspace of* LLN(μ, B), *where* B *can be any of the following spaces*: $l_p(T)$ *for* $p \in (1, \infty]$ *or* $c_0(T)$.

PROOF. It is no loss of generality to assume that card $T = c$. Let $F = \{F_t \mid t \in T\}$ be the class of subsets of S that has been established by Lemma 1.2 with $n = 1$, and let $f : S \times T \to \mathbb{R}$ be the function defined by

$$f(s, t) = 1_{F_t}(s). \tag{1.5.1}$$

For every $s \in S$, $f(s) = f(s, \cdot)$ belongs to $l_p(T)$ for all $p \geqq 1$ and also $c_0(T)$.

If we put

$$L = \{(s_k) \in S^\infty \mid s_i \neq s_j,\ \forall i, j, i \neq j\},$$

then $\mu^\infty(L) = 1$ since μ is diffuse. For $(s_i) \in L$ we have

$$\left\|\frac{1}{n} \sum_{i=1}^{n} f(s_i)\right\|_p^p = \sum_{t \in T} \left(\frac{1}{n} \sum_{i=1}^{n} 1_{F_t}(s_i)\right)^p \leqq n\left(\frac{2}{n}\right)^p = \frac{2^p}{n^{p-1}}$$

for $p \geqq 1$ and

$$\left\|\frac{1}{n} \sum_{i=1}^{n} f(s_i)\right\|_\infty = \sup_{t \in T} \left|\frac{1}{n} \sum_{i=1}^{n} 1_{F_t}(s_i)\right| \leqq \frac{2}{n},$$

which shows that $f \in \mathrm{LLN}(\mu, l_p(T))$ for all $1 < p \leqq \infty$, and $f \in \mathrm{LNN}(\mu, c_0(T))$. If $s \in F_t$, $s' \in F_{t'}$, and $1 < p \leqq \infty$, then

(1.5.2) $\quad \|f(s) - f(s')\|_p = 1 \quad \text{if} \quad t \neq t' \quad \text{and} \quad \|f(s) - f(s')\|_p = 0 \quad \text{if} \quad t = t'.$

Let $N \in \mathscr{S}$. Then by (1.5.2), $f(S \setminus N)$ is separable if and only if $f(S \setminus N)$ is countable. By definition of $\{F_t \mid t \in T\}$ f maps at most two elements of $S \setminus N$ to the same element of $f(S \setminus N)$. Hence $f(S \setminus N)$ is countable if and only if $S \setminus N$ is countable, and since μ is diffuse, we have that if $f(S \setminus N)$ is separable, then $\mu(S \setminus N) = 0$. Consequently f does not have essentially separable valued range and therefore f is not Bochner μ-measurable.

Remark. For $p = 1$ the function defined by (1.5.1) does not belong to $\mathrm{LLN}(\mu, l_1(T))$. In [5, Theorem 2.4], its is shown that if $f \in \mathrm{LLN}$, then f is Gelfand integrable. But if $f \in L_G^1(\mu, l_1(T))$, then

$$\int_S f(s)\mu(ds) = 0$$

since $f(\cdot, t) = 0$ μ-a.s. for all $t \in T$. However, if $a_t \equiv 1$, then $a = (a_t) \in l_\infty(T) = (l_1(T))^*$ and

$$\int_S a(f(s))\mu(ds) = \int_S \sum_{t \in T} 1_{F_t}(s)\mu(ds) = 1$$

but

$$a\left(\int_S f\,d\mu\right) = 0 \neq 1,$$

so $f \notin L^1_G(\mu, l_1(T))$ and consequently $f \notin \mathrm{LLN}(\mu, l_1(T))$. With a bit more care one can even show that f is not weakly measurable when we consider f as an $l_1(T)$-valued random element.

2. LLN for ξ-additive probability measures.

In this chapter we shall consider ξ-additive measures as defined in [4]. In [5, Corollary 4.4], it is proved that

$$\mathrm{LLN}(\mu, C(T)) = L^1_*(\mu, C(T)) \cap L^1_w(\mu, C(T))$$

if T is compact, first countable and separable. We shall now see that if μ is ξ-additive for some cardinal $\xi > \aleph_0$, then this result holds provided that the density and the character of T are less than ξ. Recall that

$$\operatorname{dens} T = \min\{\operatorname{card} D \mid D \text{ dense in } T\}$$

$$\operatorname{character}(T, t) = \min\{\operatorname{card} \mathcal{N} \mid \mathcal{N} \text{ is neigbourhood base at } t\}$$

whenever T is a topological space and $t \in T$.

THEOREM 2.1. *Let $(S, \mathcal{S}, \mu)$ be a probability space, ξ a cardinal with $\xi > \aleph_0$, and μ a ξ-additive measure. If T is a compact topological space with* $\operatorname{dens} T < \xi$, $\operatorname{character}(T, t) < \xi$ *for all $t \in T$, then*

$$\mathrm{LLN}(\mu, C(T)) = L^1_*(\mu, C(T)) \cap L^1_w(\mu, C(T)). \tag{2.1.1}$$

PROOF. In [5, Theorem 2.4], it is proved that

$$\mathrm{LLN}(\mu, B) \subseteq L^1_w(\mu, B) \cap L^1_*(\mu, B)$$

for every Banach space B. We shall prove the reverse inclusion. Let $f \in L^1_w(\mu, C(T)) \cap L^1_*(\mu, C(T))$. Then

$$\int_S^* \|f(s)\|\mu(ds) = \int_S^* \sup_{t\in T} |f(s,t)|\mu(ds) < \infty \tag{2.1.2}$$

which implies that f is a first order stochastic process. If we can prove that f is totally bounded in μ-mean, then by Theorem 3.3 in [5], we have finished. Therefore it remains to prove that

$$(2.1.3)\qquad \forall\varepsilon > 0 \;\; \exists\mathscr{A} \in \Gamma(T)\colon \; \int_S^* W_A(f(s))\mu(ds) < \varepsilon, \quad \forall A \in \mathscr{A},$$

where $\Gamma(T) = \{\mathscr{A} | \mathscr{A}$ is finite cover of $T\}$ and

$$W_A(f) = \sup_{t,t' \in A} |f(t) - f(t')|, \quad A \subseteqq T.$$

Since dens $T < \xi$, there exists $D \subseteqq T$ such that card $D < \xi$ and $\bar{D} = T$. The functions $f(s)\colon T \to \mathrm{R}$ are continuous functions on T for every $s \in S$, and therefore every open $U \subseteqq T$ we have:

$$W_U(f(s)) = \sup_{t,t' \in U} |f(s,t) - f(s,t')| = \sup_{t,t' \in U \cap D} |f(s,t) - f(s,t')|$$

$$= W_{U \cap D}(f(s)), \quad \forall s \in S.$$

The family $\{s \mapsto |f(s,t) - f(s,t')| \,|\, t,t' \in T\}$ is a family of μ-measurable functions, so by [4, Corollary 3.2], the functions

$$S \mapsto U_{U \cap D}(f(s)), \quad U \subseteqq T, \quad U \text{ open},$$

are μ-measurable since card$(U \cap D) < \xi$. The inequality

$$W_A(f(s)) \leqq 2\|f(s))\|, \quad \forall s \in S \; \forall A \subseteqq T,$$

and (2.1.2) implies

$$\int_S^* W_A(f(s))\mu(ds) < \infty, \quad \forall A \subseteqq T.$$

Hence

$$\int_S W_U(f(s))\mu(ds) < \infty, \quad \forall U \subseteqq T, \; U \text{ open}.$$

For fixed $t \in T$, we have that character$(T, t) < \xi$, and so there exists a downwards filtering family $\mathcal{N}$ of open neighbourhoods at t such that card $\mathcal{N} < \xi$. Since $f(s, \cdot)$ is continuous at t, we have

$$\inf_{N \in \mathcal{N}} W_N(f(s)) = \inf_{N \in \mathcal{N}} \sup_{t,t' \in N} |f(s,t) - f(s,t')| = 0, \quad \forall s \in S,$$

and by [4, Corollary 3.2], we have

$$\inf_{N \in \mathcal{N}} \int_S W_N(f(s))\mu(ds) = \int_N \inf_{N \in \mathcal{N}} W_N(f(s))\mu(ds) = 0,$$

which implies that for every $\varepsilon > 0$ there exist $N(t) \in \mathcal{N}$ such that

$$\int_N W_{N(t)}(f(s))\mu(ds) < \varepsilon.$$

Since the family $\mathcal{M} = \{N(t) | t \in T\}$ is an open cover of T, the compactness of T ensures that there exists a finite subset $\mathcal{A} = \{N(t_i) | i = 1, \dots, n\}$ of $\mathcal{M}$ such that $\mathcal{A}$ is a cover of T and such that

$$\int_S W_{N(t_i)}(f(s))\mu(ds) < \varepsilon \quad \forall i,\ i = 1, \dots, n.$$

Recall that a function $f\colon (S, \mathcal{S}, \mu) \to B$ is Bochner μ-measurable if f is weakly μ-measurable and f has μ-essentially separable range, that is, if there exists a μ-null set N such that $f(S \setminus N)$ is separable. The ξ-additive measures give us an opportunity to extend the definition of Bochner measurability in the following way: A weakly μ-measurable function $f\colon (S, \mathcal{S}, \mu) \to B$ is ξ-Bochner μ-measurable if there exists a μ-null set N such that dens$f(S \setminus N) < \xi$. But it turns out, as we show below, that we do not get more than the usual Bochner measurability.

THEOREM 2.2. *Let $(S, \mathcal{S}, \mu)$ be a probability space, where μ is a ξ-additive measure with $\xi > \aleph_0$, and let $f\colon (S, \mathcal{S}, \mu) \to B$ be a weakly measurable function. If there exists a μ-null set N such that* dens$f(S \setminus N) < \xi$, *then f is Bochner μ-measurable.*

PROOF. Let $D = \{x_\alpha | \alpha < \Gamma\}$ be a subset of B such that $f(S \setminus N) \subseteqq \bar{D}$ and

$\Gamma < \xi$. Since $\xi > \aleph_0$, it is no loss of generality to assume that D is a vector space over the rationals, and so that $\bar{D}$ is a closed linear subspace of B. By the Hahn-Banach theorem there exist functionals $x_\alpha^* \in B^*$, $\alpha < \Gamma$, such that

$$\|x_\alpha\| = x_\alpha^*(x_\alpha) \quad \text{and} \quad \|x_\alpha^*\| = 1, \quad \forall \alpha < \Gamma. \tag{2.2.1}$$

We claim that

$$\|x\| = \sup_{\alpha < \Gamma} x_\alpha^*(x), \quad \forall x \in \bar{D}. \tag{2.2.2}$$

The inequality $\geqq$ is obvious. So let $x \in \bar{D}$ and let $\varepsilon > 0$ be given. Then there exists $\alpha < \Gamma$ with $\|x - x_\alpha\| < \varepsilon$, and so

$$\|x\| \leqq \varepsilon + \|x_\alpha\| = \varepsilon + x_\alpha^*(x_\alpha) = \varepsilon + x_\alpha^*(x) + x_\alpha^*(x_\alpha - x) \leqq 2\varepsilon + x_\alpha^*(x).$$

Thus the converse inequality in (2.2.2) holds. From (2.2.2) we have that

$$\|f(s) - x_\beta\| = \sup_{\alpha < \Gamma} x_\alpha^*(f(s) - x_\beta), \quad \forall \beta < \Gamma, \quad \forall s \in f(S \setminus N)$$

since $\bar{D}$ is a vector space containing $f(s)$ and x_β. By Corollary 3.2 in [4], the function

$$s \mapsto \|f(s) - x_\beta\|$$

is μ-measurable for all $\beta < \Gamma$ and since $f(S \setminus N) \subseteqq \bar{D}$, we have

$$S \setminus N \subseteqq \bigcup_{\beta < \Gamma} \{s \mid \|f(s) - x_\beta\| < \varepsilon\}, \quad \forall \varepsilon > 0.$$

By [4, Theorem 3.1], there exist a countable set $\Lambda \subseteqq \{\alpha \mid \alpha < \Gamma\}$ and a null set $N_0 \supseteqq N$ such that

$$S \setminus N_0 \subseteqq \bigcup_{\beta \in \Lambda} \{s \mid \|f(s) - x_\beta\| < 2^{-n}\}, \quad \forall n \geqq 1.$$

Hence $f(S \setminus N_0) \subseteqq \overline{\{x_\beta \mid \beta \in \Lambda\}}$, and so f is essentially separably valued. Thus f is Bochner μ-measurable.

Proposition 2.3. *Let $(S, \mathscr{S}, \mu)$ be a probability space, where μ is a ξ-additive measure with $\xi > \aleph_0$; let B be a Banach space, and let $f\colon (S, \mathscr{S}, \mu) \to B$ be a* *function such that* $\operatorname{dens} f(S \setminus N) > \xi$ *for some* μ*-null set* N. *Then* $f \in \mathrm{LLN}(\mu, B)$ *if and only if* $f \in L^1_*(\mu, B) \cap L^1_w(\mu, B)$.

Proof. If $f \in \mathrm{LLN}(\mu, B)$, then by [5, Theorem 2.4], we have that $f \in L^1_*(\mu, B) \cap L^1_w(\mu, B)$. If $f \in L^1_w(\mu, B)$ and $\operatorname{dens} f(S \setminus N) > \xi$ for some μ-null set N, then by Theorem 2.2, f is Bochner μ-measurable and $f \in L^1_*(\mu, B)$ ensures that f is Bochner μ-integrable; but then $f \in \mathrm{LLN}(\mu, B)$ by the well-known result due to A. Beck ([1, p. 26]).

Corollary 2.4. *Let* $(S, \mathscr{S}, \mu)$ *be a probability space, where* μ *is a* ξ*-additive measure with* $\xi > \aleph_0$, *and let* T *be a set such that* $\operatorname{card} T < \xi$. *Then*

$$\mathrm{LLN}(\mu, l_p(T)) = L^1_w(\mu, l_p(T)) \cap L^1_*(\mu, l_p(T))$$

for $1 \leqq p < \infty$.

Proof. This is an immediate consequence of Proposition 2.3 because

$$\operatorname{dens} l_p(T) \leqq \operatorname{card} T \cdot \aleph_0 < \xi\aleph_0 = \xi,$$

and therefore $\operatorname{dens} f(S) < \xi$.

3. Functions which do not satisfy LLN.

In [3] E. Giné and J. Zinn give a necessary and sufficient condition for $f \in \mathrm{LLN}(\mu, l_\infty(T))$ provided that f satisfies some measurability condition (condition $\mathrm{NSM}(P)$ in [3]). We shall describe this necessary and sufficient condition. Let $(S, \mathscr{S}, \mu)$ be a probability space, T some set and $f: S \times T \to \mathsf{R}$ a stochastic process. Consider the random metrics defined by

$$d_{n,p}(t, t') = \left(\frac{1}{n} \sum_{i=1}^{n} |f(s_i, t) - f(s_i, t')|^p\right)^{1/p \vee 1} \quad \text{for } 0 < p < \infty, \tag{3.1}$$

$$d_{n,\infty}(t, t') = \max_{1 \leq i \leq n} |f(s_i, t) - f(s_i, t')| \tag{3.2}$$

for all $t, t' \in T$, their covering numbers $N_{np}(r, f)$ (i.e. the smallest integer N such that T may be covered by $N, d_{n,p}$-balls of radius r), and their metric entropies $H_{n,p}(r, f)$ defined by

$$H_{n,p}(r, f) = \ln N_{n,p}(r, f). \tag{3.3}$$

Then under a certain measurability condition it is shown in [3] that

$f \in \mathrm{LLN}(\mu, l_\infty(T))$ if and only if

$$s \mapsto F(s) = \sup_{t \in T} |f(s,t)| \text{ belongs to } L^1(\mu) \tag{3.4}$$

and

$$\lim_{n \to \infty} E^* \frac{H_{n,p}(r, f 1_{[F \leq M]})}{n} = 0, \quad \forall r > 0,\ \forall M > 0,\ \forall p \in (0, \infty]. \tag{3.5}$$

We shall show that there exists a function f satisfying (3.4) and (3.5), but $f \notin \mathrm{LLN}(\mu, l_\infty(T))$, hence the measurability condition in [3] is to be necessary for the entropy condition to imply $\mathrm{LLN}(\mu, l_\infty(T))$. Moreover, this function is Gelfand integrable which shows that in general $\mathrm{LLN}(\mu, B)$ is a proper subspace of $L^1_G(\mu, B) \cap L^1_*(\mu, B)$.

THEOREM 3.1. *Let $(S, \mathscr{S}, \mu)$ be a probability space, where S is a Polish space with* card $S = c$, *$\mathscr{S}$ is the Borel σ-algebra and μ is a diffuse probability measure. Let $\mathscr{F} = \{F_t | t \in T\}$ be the class of subsets of S that is established by Lemma* 1.1, *and let $f : S \times T \to \mathrm{R}$ be the function defined by*

$$f(s,t) = 1_{F_t}(s). \tag{3.1.1}$$

Then

$$f \in L^1_*(\mu, B) \cap L^1_G(\mu, B), \tag{3.1.2}$$

$$f \notin \mathrm{LLN}(\mu, B), \tag{3.1.3}$$

(3.1.4) *f satisfies the entropy condition* (3.4) *and* (3.5),

$$N_{n,p}(r, f) \leqq n+1, \quad \forall n \geqq 1,\ \forall r \geqq 0,\ \forall 0 < p \leqq \infty, \tag{3.1.5}$$

where B is any of the following Banach spaces

$$l_p(T) \ \textit{for}\ 1 < p \leqq \infty \ \textit{or}\ c_0(T).$$

PROOF. Since

$$S = \bigcup_{t \in T} F_t,$$

we have

$$\sum_{t \in T} |f(s,t)|^p = \sum_{t \in T} 1_{F_t}(s) = 1, \quad \forall s \in S,\ \forall 1 \leqq p < \infty$$

which implies that functions

$$f(s): t \mapsto 1_{F_t}(s)$$

belong to $l_p(T)$, $\|f(s)\|_p = 1$ for all $s \in S$ and all p, $1 \leqq p < \infty$, and that

$$\overline{\int_S}^{*} \|f(s)\|_p \mu(ds) = \int_S \|f(s)\|_p \mu(ds) = 1, \quad \forall 1 \leqq p < \infty.$$

Hence $f \in L^1_*(\mu, l_p(T))$ for all $1 \leqq p < \infty$. For every $s \in S$ there exists exactly one t such that $1_{F_t}(s) = 1$, showing that $f(s) \in c_0(T)$ for every $s \in S$ and therefore that $f \in L^1_*(\mu, c_0(T)) \subseteqq L^1_*(\mu, l_\infty(T))$.

Now we shall show that functions $s \mapsto f(s)$ belong to $L^1_G(\mu, l_p(T))$ for $1 < p \leqq \infty$ and to $L^1_G(\mu, c_0(T))$. Let $1 < p < \infty$; let q be such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and let $x = (x_t) \in l_p^*(T) = l_q(T)$. Then

$$\sum_{t \in T} |x_t|^q < \infty \tag{3.1.6}$$

implying that the set $T_0 = \{t | x_t \neq 0\}$ is countable. If

$$M = \{s \in S | x(f(s)) = \sum_{t \in T} x_t 1_{F_t}(s) \neq 0\},$$

then

$$M \subseteqq \bigcup_{t \in T_0} F_t.$$

Since F_t and T_0 are countable, then so is M. The measure μ is diffuse, and therefore $\mu(M) = 0$, which implies that $x(f(\cdot))$ is μ-measurable and

$$\int_S x(f(s))\mu(ds) = 0$$

for every $x \in l_p^*(T)$ and every $1 < p < \infty$. Thus

$$(3.1.7) \qquad \int_S f(s)\mu(ds) = 0 \quad \text{and} \quad f \in L_G^1(\mu, l_p(T)), \quad \forall p,\ 1 < p < \infty.$$

The same proof goes for $c_0(T)$ because $c_0(T)^* = l_1(T)$. The space $c_0(T)$ can be embedded naturally into $l^\infty(T)$. For every $x \in l^\infty(T)^*$ we define

$$x' = x|_{c_0(T)}.$$

Then $x' \in l_1(T)$ and

$$x'(f(s)) = x(f(s)), \quad \forall s \in S.$$

Now working with x' we conclude that

$$(3.1.8) \qquad \int_S x'(f(s))\mu(ds) = 0$$

and from (3.1.8) and (3.1.7) we conclude that

$$(3.1.9) \qquad \int_S f(s)\mu(ds) = 0 \quad \text{and} \quad f \in L_G^1(\mu, l_\infty(T)) \cap L_G^1(\mu, c_0(T)).$$

So we have proved (3.1.2). To prove (3.1.3), let

$$L_p^\infty = \left\{ (s_i) \in S^\infty \,\middle|\, \left\| \frac{1}{n} \sum_{i=1}^n f(s_i) \right\|_p = 1,\ \forall n \in \mathbf{N} \right\}, \quad 1 < p \leqq \infty.$$

Then by (1.1.2) we have for all $1 < p \leqq \infty$

$$L_p = \left\{ (s_i) \in S^\infty \,|\, \exists t \in T : \frac{1}{n} \sum_{i=1}^n 1_{F_t}(s_i) = 1,\ \forall n \in \mathbf{N} \right\}$$
$$= \{ (s_i) \in S^\infty \,|\, \exists t \in T : \{ s_i | i \in \mathbf{N} \} \subseteqq F_t \},$$

and it now easily follows that

$$L_p = \bigcup_{t \in T} F_t^\infty, \quad 1 < p \leqq \infty.$$

By (1.1.3) we have $B \cap L_p \neq \emptyset$ for every $B \in \mathscr{S}^\infty$ with $\mu^\infty(B) > 0$, which implies

$$(\mu^\infty)^*(L_p) = 1,$$

and so (3.1.3) follows.

To prove (3.1.4), it is enough to verify (3.5) for $M = 1$. Let $r > 0$, $n \geqq 1$, $0 < p \leqq \infty$ and $(s_j) \in S^\infty$ be given. Choose $t_j \in T$, $j = 1, \ldots, n$, such that $s_j \in F_{t_j}$ (see (1.1.4)), and let

$$U_n = T \setminus \{t_1, \ldots, t_n\}.$$

If $u, v \in U_n$, then

$$1_{F_u}(s_j) = 1_{F_v}(s_j) = 0, \quad \forall j = 1, \ldots, n.$$

So we have that U_n is a $d_{n,p}$-ball of radius 0. Clearly $\{t_j\}$ for $j = 1, \ldots, n$ are $d_{n,p}$-balls of radius 0. Hence

$$N_{n,p}(r,f) \leqq N_{n,p}(0,f) \leqq n+1, \quad \forall n \geqq 1,\ \forall r \geqq 0,\ \forall 0 < p \leqq \infty$$

since $T = U_n \bigcup \bigcup_{i=1}^n \{t_i\}$. In particular we have

$$\lim_{n\to\infty} E^* \frac{H_{n,p}(r,f)}{n} \leqq \lim_{n\to\infty} \frac{\log(n+1)}{n} = 0.$$

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