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CONFORMAL MARTINGALES AND ANALYTIC FUNCTIONS

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Abstract.

We give a number of equivalent characterizations of conformal martingales in C^n, including a representation formula in terms of complex Ito integrals.

We then prove that if $m > 1$ there exist no diffusions X_t, Y_t on C^n, C^m respectively such that all analytic $\phi: C^n \to C^m$ map X_t into a time change of Y_t. However, given any "reasonable" Y_t on C^m and any such ϕ we may construct a diffusion X_t (which depends on ϕ) such that ϕ maps X_t into a time change of Y_t.

Finally we prove that the complex Ito and complex Stratonovich integrals in C^n coincide if the integrand is holomorphic.

Introduction.

Our main concern in this paper is to study conformal martingales in a complex notation. Our basic tool, the complex Ito formula, is by now well-known. It has been used by several people e.g. Varopoulos [9]. Following Fukushima and Okada [2], we define a conformal martingale $Z_t = (Z_1, Z_2, \ldots, Z_n) \in C^n$ by the requirement that all Z_i and all Z_iZ_j should be martingales. In an abstract way, we look at martingales as analogues of harmonic functions and conformal martingales as analogues of holomorphic functions. From our viewpoint this definition seems to be natural.

For information on conformal martingales in the plane see Getoor and Sharpe [3]. For information on stochastic integrals see Øksendal [5].

In *Theorem 1* we give several equivalent formulations of the concept conformal martingale. Some are well-known, but we believe that the matrix conditions (v), (vi) and (vii) are new.

Theorem 2 states that every conformal martingale $dX_t = \sigma dB_t$ in R^{2n} can be written as $dZ_t = UdW_t$ with U a complex $n \times m$ matrix and W_t a complex version of $2m$-dimensional Brownian motion.

Theorem 5 explains why there can be no pair of processes X_t in C^n, Y_t

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in $\mathbf{C}^m$ $m > 1$ such that $\phi(X_t)$ coincide in law with Y_t whenever $\phi: \mathbf{C}^n \to \mathbf{C}^m$ is analytic. (When $m = 1$ there are many such processes.)

Theorem 7 proves a weak Levy theorem in $\mathbf{C}^m$, $m > 1$; Given a reasonable $\phi: \mathbf{C}^n \to \mathbf{C}^m$ and a "nice" diffusion H_t in $\mathbf{C}^m$, there exists a diffusion Z_t and a time-change α_t such that $\phi(Z_{\alpha_t})$ coincide in law with H_t.

Theorem 9 states that if $dZ_t = U(Z_t)dW_t$ is a solution in the Stratonovich sense and U has holomorfic coefficients, then Z_t is also a solution in the Ito sense. This has immediate consequences in relation with Theorem 7, i.e. if H_t is given by a holomorphic U as above, Z_t can be constructed to have holomorphic entries in its matrix.

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1. Some notation and background in real variable theory.

A stochastic process X_t in $\mathbf{R}^n$ is called a *stochastic integral* iff

$$X_t(\omega) = X_0(\omega) + \int_0^t \sigma(s, \omega) dB_s(\omega) + \int_0^t b(s, \omega) ds$$

where B_s denotes m-dimensional Brownian notion and $\sigma(s, \omega), b(s, \omega)$ are processed adapted to the Brownian filtration. Integration with respect to dB_s-Brownian motion, cannot be defined as an ordinary integration. This is because the paths of Brownian motion are of unbounded variation with probability one. Nevertheless, the integrals are well defined mathematical objects. One defines them as limits of Riemann-sums with specific approximation points. Each system of approximation may give different integrals. The left-point approximations are called Ito-integrals. The mid-point approximations are called Stratonovich-integrals. Ito-integrals are usually preferred, since the results turn out to be martingales. Since all reasonable martingales can be represented as Ito-integrals, we can take this as a definition of a martingale:

A stochastic integral is called a martingale iff $b \equiv 0$.

Unless nothing else is said, we will work with Ito-integrals.

The same notation applies to any dimension, so σ may be an $n \times m$ matrix and b a vector. Supressing arguments, we will write a stochastic integral as

$$dX_t = \sigma dB_t + b dt$$

the interpretation being the same as the integral expression.

2. The Ito formula.

Stochastic integrals behave nicely under C^2-mappings, and the result is given by an explicit expression called Ito's formula, i.e. if $\phi : \mathsf{R}^n \to \mathsf{R}^m$ is C^2, then

$$d\phi(X_t) = \sum_i^n \frac{\partial \phi}{\partial x_i}(X_t) dX_i + \tfrac{1}{2} \sum_{i,j}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j}(X_t) dX_i dX_j.$$

To get this on the standard form, one treats the products $dX_i dX_j$ according to the usual rules of calculus using the formal relations

$$dt \cdot dt = dt \cdot dB_i = 0 \quad \text{and} \quad dB_i dB_j = \delta_{ij} dt.$$

This is proved by doing a second order Taylor-expansion of ϕ, and estimate the expressions.

If we instead do a complex second order Taylor-expansion, clearly this will be just as good in each real coordinate. The terms which should disappear must still disappear and it is only necessary to note that the products $dX_i dX_j$ are treated in the usual way, to get a version suitable to complex analysis:

Let $\phi : \mathsf{C}^n \to \mathsf{C}^m$ be C^2, then

$$d\phi(Z_t) = \sum_i^n \frac{\partial \phi}{\partial z_i} dZ_i + \sum_i^n \frac{\partial \phi}{\partial \bar{z}_i} d\bar{Z}_i + \tfrac{1}{2} \sum_{i,j}^n \frac{\partial^2 \phi}{\partial z_i \partial z_j} dZ_i dZ_j +$$

$$+ \tfrac{1}{2} \sum_{i,j}^n \frac{\partial^2 \phi}{\partial \bar{z}_i \partial \bar{z}_j} d\bar{Z}_i d\bar{Z}_j + \sum_{i,j}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dZ_i d\bar{Z}_j.$$

If ϕ happens to be holomorphic, things simplify, so

$$d\phi(Z_t) = \sum_i^n \frac{\partial \phi}{\partial z_i} dZ_i + \tfrac{1}{2} \sum_{i,j}^n \frac{\partial^2 \phi}{\partial z_i \partial z_j} dZ_i dZ_j.$$

We have done the usual identification between real and complex vectors:

$$(x_1, y_1, x_2, y_2, \ldots) \doteq (x_1 + iy_1, x_2 + iy_2, \ldots).$$

Throughout the text, we let X_t, Y_t denote real versions and Z_t, H_t denote complex versions of the processes. B_t is Brownian motion in real notation and W_t is Brownian motion in the complex case, i.e.

$$dW_t = \begin{bmatrix} dB_1 + idB_2 \\ dB_3 + idB_4 \\ \vdots \end{bmatrix} \overset{i}{=} \begin{bmatrix} dB_1 \\ dB_2 \\ \vdots \end{bmatrix} = dB_t.$$

With these conventions, it is easy to see that we have the formal rules

$$dt \cdot dt = dt \cdot dW_i = dt \cdot d\bar{W}_i = dW_i dW_j = d\bar{W}_i d\bar{W}_j = 0$$

$$dW_i d\bar{W}_j = 2\delta_{ij} dt.$$

3. Representation of stochastic integrals in R^n.

A real linear mapping $L: \mathrm{R}^{2n} \to \mathrm{R}^{2m}$ is called C-*linear* iff $L(\lambda z) = \lambda L(z)$ or all $z \in \mathrm{C}^n$, $\lambda \in \mathrm{C}$. The mapping L is called C-*antilinear* iff $L(\lambda z) = \bar{\lambda} L(z)$ for all $z \in \mathrm{C}^n$, $\lambda \in \mathrm{C}$. For example, a 2×2 matrix L is C-linear iff it has the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and C-antilinear iff it has the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

If $dX_t = \sigma dB_t + bdt$ is a stochastic integral in C^n we may assume that σ is a $2n \times 2m$ matrix and split σ in a C-linear and a C-antilinear part. Writing this out in complex language, it is clear that every stochastic integral in C^n can be represented in the form

$$dX_t = UdW_t + Vd\bar{W}_t + bdt$$

where U and V are complex $n \times m$ matrices.

4. Conformal martingales.

We will use the product definition of a conformal martingale:

DEFINITION. A process Z_t in C^n is a conformal martingale iff all Z_i and all $Z_i Z_j$ are martingales.

We will now prove a series of equivalent statements, eventually leading to a general representation theorem.

THEOREM 1. *Let*

$$dX_t = \sigma dB_t, \quad dZ_t = UdW_t + Vd\bar{W}_t$$

be the process $Z_t \in \mathbf{C}^n$ in real and complex notation.

Then the following statements are all equivalent:

(i) Z_t *is a conformal martingale,*
(ii) $dZ_i dZ_j = 0$ *all* i, j,
(iii) $\phi(Z_t)$ *is a martingale for all holomorphic* ϕ,
(iv) $\phi(Z_t)$ *is a conformal martingale for all holomorphic* ϕ,
(v) $UV^t + VU^t = 0$,
(vi) $\sigma\sigma^t$ *is* $\mathbf{C}$*-linear,*
(vii) $\sigma\sigma^t$ *has a* $\mathbf{C}$*-linear square root.*

If in addition, Z_t is a diffusion, all the above are equivalent to:

(viii) *The infinitesimal generator L of Z_t is given by*

$$Lf = \sum_{i,j}^{n} 2(UU^* + VV^*)_{ij} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}.$$

PROOF. (i) $\Leftrightarrow$ (ii). Apply the complex Ito formula to $\phi(z_1, z_2) = z_1 z_2$

$$d(Z_i Z_j) = Z_i dZ_j + Z_j dZ_i + dZ_i dZ_j.$$

Since dZ_t has no dt-terms the equivalence follows from the above expression.

(ii) $\Leftrightarrow$ (iii) is obvious from the complex Ito formula.

(iii) $\Leftrightarrow$ (iv), since composition of holomorphic mappings are holomorphic.

(ii) $\Leftrightarrow$ (v). One must compute

$$dZ_i = \sum_{k=1}^{m} U_{ik} dW_k + \sum_{k=1}^{m} V_{ik} d\bar{W}_k,$$

so

$$dZ_i dZ_j = \left(\sum_k^m U_{ik} dW_k + \sum_k^m V_{ik} d\bar{W}_k\right)\left(\sum_l^m U_{jl} dW_l + \sum_l^m V_{jl} d\bar{W}_l\right).$$

Using the formal rules for multiplication, we get

$$dZ_i dZ_j = \sum_k^m (U_{ik} V_{jk} + V_{ik} U_{jk}) \cdot 2dt \Rightarrow dZ_i dZ_j = (UV^t + VU^t)_{ij} \cdot 2dt.$$

(v) $\Leftrightarrow$ (vi). We need to mediate between the real and the complex notation, and interprete all matrices as linear transformations on R^{2m}.

We let $\Delta : \mathsf{R}^{2m} \to \mathsf{R}^{2n}$ be a real-linear transformation, and let C and D denote the complex-conjugation operators on R^{2m} and R^{2n} respectively, i.e.

$$C(x_1, y_1, x_2, y_2, \ldots) = (x_1, -y_1, x_2, -y_2, \ldots).$$

If Δ happens to be C-linear, it can be represented by a real $2n \times 2m$ matrix A, or by a complex $n \times m$ matrix M. Then Δ^{adj} (= Hilbert-space adjoint) can be represented by A^t or M^*. But what is the linear transformation corresponding to M^t? To understand this, we have to be able to express complex conjugation of matrix-elements in M in terms of linear transformations.

We have

$$\bar{M}z = \overline{M\bar{z}} = (D\Delta C)z,$$

so $\bar{M}$ represents the linear transformation $D\Delta C$. That is, we have the following scheme:

If Δ is C-linear

$$\begin{array}{lll} A & \leftarrow \Delta & \rightarrow M \\ A^t & \leftarrow \Delta^{\text{adj}} & \rightarrow M^* \\ DAC & \leftarrow D\Delta C & \rightarrow \bar{M} \\ CA^tD & \leftarrow C\Delta^{\text{adj}}D & \rightarrow M^t. \end{array}$$

Note that C and D are self-adjoints, and that $C^2 = I$, $D^2 = I$.

In the rest of this proof equalities are to be interpreted as equalities between linear transformations, and the operations t and $*$ refer to the matrix in question. Note that the relation between σ, U and V is

$$\begin{aligned} \sigma &= U + VC \\ \sigma\sigma^t &= \sigma\sigma^{\text{adj}} = (U+VC)(U+VC)^{\text{adj}} \\ &= UU^{\text{adj}} + VCC^{\text{adj}}V^{\text{adj}} + UC^{\text{adj}}V^{\text{adj}} + VCU^{\text{adj}} \\ &= UU^{\text{adj}} + VV^{\text{adj}} + UCV^{\text{adj}}DD + VCU^{\text{adj}}DD \\ &= UU^* + VV^* + (UV^t + VU^t)D, \end{aligned}$$

so $(UV^t + VU^t)D$ is the C-antilinear part of $\sigma\sigma^t$.

(vi) $\Leftrightarrow$ (vii) is trivial.

(v) $\Leftrightarrow$ (viii) will follow immediately from the following proposition:

PROPOSITION 1. *Let*

$$dZ_t = U(Z_t)dW_t + V(Z_t)d\bar{W}_t + b(Z_t)dt$$

be a diffusion. Then Z_t has an infinitesimal generator

$$Lf = \sum_{i,j} a_{ij} \frac{\partial^2 f}{\partial z_i dz_j} + \sum_{i,j} \overline{a_{ij}} \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} + 2\sum_{i,j} c_{ij} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} + \sum_i b_i \frac{\partial f}{\partial z_i} + \sum_i \bar{b}_i \frac{\partial f}{\partial \bar{z}_i}$$

where $(a_{ij}) = UV^t + VU^t$, $(c_{ij}) = UU^* + VV^*$, $(b_i) = b$.

PROOF. One could of course just translate from the real expression, but an imitation of the real proof is much easier. Let f in C^2 and apply the complex Ito formula

$$df(Z_t) = \sum_i \frac{\partial f}{\partial z_i} dZ_i + \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{Z}_i + \\ + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} dZ_i dZ_j + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} d\bar{Z}_i d\bar{Z}_j + \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} dZ_i d\bar{Z}_j.$$

From the proof of Theorem 1 (ii) $\Leftrightarrow$ (v) we know that

$$dZ_i dZ_j = (UV^t + VU^t)_{ij} \cdot 2dt.$$

We need the corresponding expression for $dZ_i d\bar{Z}_j$ that is

$$dZ_i d\bar{Z}_j = \left(\sum_k U_{ik} dW_k + \sum_k V_{ik} d\bar{W}_k\right)\left(\sum_l \bar{U}_{jl} d\bar{W}_l + \sum_l \bar{V}_{jl} dW_l\right)$$

$$= \sum_k (U_{ik}\bar{U}_{jk} + V_{ik}\bar{V}_{jk}) \cdot 2dt = 2(UU^* + VV^*)_{ij} dt.$$

Then if we collect the dW and the dt-terms,

$$df(Z_t) = \left\{\sum_i \frac{\partial f}{\partial z_i} b_i + \sum_i \frac{\partial f}{\partial \bar{z}_i} \bar{b}_i + \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} a_{ij} + \sum_{i,j} \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} \overline{a_{ij}} + \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} 2c_{ij}\right\} dt + \\ + \sum_{i,k} U_{ik}(Z_t) \frac{\partial f}{\partial z_i}(Z_t) dW_k + \sum_{i,k} V_{ik}(Z_t) \frac{\partial f}{\partial z_i}(Z_t) d\bar{W}_k + \\ + \sum_{i,k} \bar{U}_{ik}(Z_t) \frac{\partial f}{\partial \bar{z}_i}(Z_t) d\bar{W}_k + \sum_{i,k} \bar{V}_{ik}(Z_t) \frac{\partial f}{\partial \bar{z}_i}(Z_t) dW_k.$$

Taking expectations, all dW-terms disappear by the martingale property of Ito-integrals, and

$$E^z f(Z_t) = f(z) + \int_0^t E^z \Bigg[\sum_i \frac{\partial f}{\partial z_i} b_i + \sum_i \frac{\partial f}{\partial \bar{z}_i} \bar{b}_i + \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} a_{ij} + \\ + \sum_{i,j} \frac{\partial^2 f}{\partial \bar{z}_i \partial \bar{z}_j} \bar{a}_{ij} + \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} 2c_{ij} \Bigg] ds.$$

This gives

$$Af(z) = \lim_{t \downarrow 0} \frac{E^z(f(Z_t)) - f(z)}{t} = Lf(z).$$

5. A representation theorem for conformal martingales.

We see that the condition (v), $UV^t + VU^t = 0$ in Theorem 1 is trivially satisfied if $V = 0$. We will prove that all conformal martingales can be represented in this way. To do this we need the fact that any $n \times n$ matrix has a polar representation. This is true because we are in a finite dimensional Hilbert space. For the sake of complementeness we give a proof.

LEMMA 1. *If σ is a $n \times n$ matrix and $\sqrt{\sigma\sigma^t}$ is a square root of $\sigma\sigma^t$, there exist an orthogonal matrix σ_0 such that $\sigma = (\sqrt{\sigma\sigma^t})\sigma_0$.*

PROOF. Define a linear mapping

$$L: \text{Range}(\sqrt{\sigma\sigma^t}) \to \text{Range}(\sigma^t)$$

by $L\sqrt{\sigma\sigma^t}(x) = \sigma^t(x)$. Put $H = \text{Range}(\sqrt{\sigma\sigma^t})$, $G = \text{Range}(\sigma^t)$. Since

$$\|\sqrt{\sigma\sigma^t}(x)\| = (\sqrt{\sigma\sigma^t}x, \sqrt{\sigma\sigma^t}x)^{1/2} = (\sigma\sigma^t x, x)^{1/2} = (\sigma^t x, \sigma^t x)^{1/2} = \|\sigma^t x\|.$$

L is an isometry. $\sqrt{\sigma\sigma^t}$ and σ^t have the same kernel so $\dim H = \dim G$. Then $\dim H^\perp = \dim G^\perp$ and there exist a linear isometry $\Delta : H^\perp \to G^\perp$. Define a linear mapping $\tilde{\sigma}_0 : \mathsf{R}^n \to \mathsf{R}^n$ by

$$\tilde{\sigma}_0(x) = L(x_H) + \Delta(x_{H^\perp}).$$

Then

$$(\tilde{\sigma}_0(x), \tilde{\sigma}_0(x)) = (L(x_H), L(x_H)) + (\Delta x_{H^\perp}, \Delta x_{H^\perp}) \\ = (x_H, x_H) + (x_{H^\perp}, x_{H^\perp}) = (x, x),$$

so $\tilde{\sigma}_0$ is orthogonal and coincide with L on H. That is

$$\tilde{\sigma}_0\sqrt{\sigma\sigma^t} = \sigma^t \Rightarrow \sigma = (\sqrt{\sigma\sigma^t})\tilde{\sigma}_0^t.$$

THEOREM 2. *Let $dX_t = \sigma dB_t$ be a conformal martingale in* R^{2n}. *Then there exist a complex $n \times m$ matrix $\tilde{U}$ and a complex Brownian motion $\tilde{W}_t$ such that in complex notation, dX_t can be written as $dZ_t = \tilde{U}d\tilde{W}_t$.*

PROOF. Assume first that σ is a $2n \times 2n$ matrix. By Theorem 1 (vii) and the lemma, we know that $dX_t = (\sqrt{\sigma\sigma^t})\sigma_0 dB_t$. Put $\sigma_{\mathsf{C}} = \sqrt{\sigma\sigma^t}$ and $dY_t = \sigma_0 dB_t$; $dX_t = \sigma_{\mathsf{C}} dY_t$.

By a theorem of McKean [4], a stochastic integral $dY_t = \sigma_0 dB_t$ is a Brownian motion if and only if $\sigma_0\sigma_0^t = I$. This being satisfied, Y_t is Brownian motion, and since σ_{C} is C-linear, the conclusion follows by writing everything out in complex notation.

If σ is not a $2n \times 2n$ matrix, we can add columns or rows of zeros to arrange this. Addition of such columns do not change the process, so the result follows as before. Addition of rows do not change the conformity. We then do the construction in a larger space. Since C-linearity is not changed when we remove rows of zeros to get back to the old dimension, the result is still true.

6. The Lévy theorem.

The famous Lévy theorem says the following:

If $\phi: \mathsf{C} \to \mathsf{C}$ is holomorphic and W_t is Brownian motion in C, $\phi(W_t)$ is again Brownian motion except from a change of time scale.

Let us see why this is true. To do this we need a strenghthening of the McKean theorem. First we discuss the concept of time change.

Let $c(t, \omega) \geqq 0$ be a process adapted to the Brownian filtration. Let $\beta_t(\omega) = \int_0^t c(s, \omega)ds$ and put $\alpha_t = \inf\{s; \beta_s > t\}$.

Then the following theorem applies, see Øksendal [6].

THEOREM 3 (Øksendal). *Let*

$$dX_t = \sigma(t, \omega)dB_t + b(t, \omega)dt \quad \textit{be a stochastic integral,}$$

$$dY_t = \tilde{\sigma}(Y_t)dB_t + \tilde{b}(Y_t)dt \quad \textit{be a diffusion.}$$

If $c(t, \omega)$ and α_t are as above, the following statements are equivalent

(i) $\sigma\sigma^t(t, \omega) = c(t, \omega)\tilde{\sigma}\tilde{\sigma}^t(X_t)$, $E_{\alpha_t}[b|X] = \tilde{b}(X)E_{\alpha_t}[c|X]$.

(ii) X_{α_t} coincide in law with Y_t.

Since we are only going to use the first part of (i), we leave the second part undefined, noting only that it is trivially satisfied if b and $\tilde{b}$ are both zero. The interested reader can consult Øksendal's paper.

REMARK. This allows α_t to have jumps. If we only want to consider continuous time changes, β_t must be strictly increasing. α_t is called a time change, and we will only consider time changes of this type.

As a special case we get:

A stochastic integral coincide in law with B_t up to a change of time scale if $b = 0$ and $\sigma\sigma^t = c(t, \omega)I$.

From this we easily get the following;

THEOREM 4. (Lévy theorem, strong global form). *Let $\phi: \mathbf{C}^n \to \mathbf{C}$ be holomorphic and let $dZ_t = U(t, \omega)dW_t$ be a conformal martingale such that $t \to \int_0^t |J_{\mathbf{C}}\phi(Z_s)U(s, \omega)|^2 ds$ is strictly increasing. Then $\phi(Z_t)$ is a continuous time change of Brownian motion.*

PROOF. $J_{\mathbf{C}}\phi$ denotes the complex Jacobian matrix. By the complex Ito formula

$$d\phi(Z_t) = J_{\mathbf{C}}\phi(Z_t)dZ_t = J_{\mathbf{C}}\phi(Z_t)U(t, \omega)dW_t = \tilde{U}(t, \omega)dW_t.$$

Since in this case $\tilde{U}\tilde{U}^*$ is a 1×1 matrix

$$\tilde{U}\tilde{U}^*(t, \omega) = |J_{\mathbf{C}}\phi(Z_t)U(t, \omega)|^2 \cdot I,$$

and the result follows trivially from the complex version of Øksendal's theorem.

In higher dimensions the Lévy theorem fails completely. Not only is it false, there exists no reasonably large class of functions and no pair of processes X_t, Y_t for which the conclusion of the Lévy theorem is true. The following theorem explains this.

THEOREM 5 (Negative Lévy theorem). *Let X_t, Y_t be processes in $\mathbf{C}^2$ with $\sigma(t, \omega) \neq 0$ and*

$$dX_t = \sigma(t, \omega)dB_t + b(t, \omega)dt$$

$$dY_t = D(Y_t)dB_t + F(Y_t)dt.$$

Let $\phi_j: \mathbf{C}^2 \to \mathbf{C}^2$, $j = 1, 2, 3, 4$ be given by

$$\phi_1(z, w) = (z, w),$$

$$\phi_2(z, w) = (w, z),$$

$$\phi_3(z, w) = (2z, w),$$

$$\phi_4(z, w) = (z + w, w).$$

Then there exists a j such that $\phi_j(X_{\alpha_t})$ does not coincide in law with Y_t for any time-change α_t.

PROOF. Assume the converse to get a contradiction i.e. for each j there exists α_t such that $\phi_j(X_{\alpha_t})$ coincide with Y_t. By Ito's formula we have

$$d\phi_j(X_t) = J_{\mathrm{R}}\phi_j(X_t)\sigma(t,\omega)dB_t + dt\text{-terms}.$$

By Øksendal's theorem, it is necessary that there exist functions $c_j(t,\omega)$ such that

$$(*) \qquad J_{\mathrm{R}}\phi_j(X_t)\sigma(t,\omega)\sigma(t,\omega)^t J_{\mathrm{R}}\phi_j(X_t)^t = c_j(t,\omega)D(X_t)D(X_t)^t.$$

Put $P = \sigma(t,\omega)\sigma(t,\omega)^t$, $Q = D(X_t)D(X_t)^t$.

P and Q are 4×4 matrices consisting of 4 2×2 blocks.

$$P = \begin{bmatrix} F & G \\ H & I \end{bmatrix}, \quad Q = \begin{bmatrix} J & K \\ L & M \end{bmatrix}.$$

Assume $P \neq 0$. Now we have:

$$J_{\mathrm{R}}\phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J_{\mathrm{R}}\phi_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$J_{\mathrm{R}}\phi_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J_{\mathrm{R}}\phi_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we insert the three first expressions in $(*)$ and multiply out, we will get:

$$\begin{bmatrix} F & G \\ H & I \end{bmatrix} = c_1(t,\omega)\begin{bmatrix} J & K \\ L & M \end{bmatrix}$$

$$\begin{bmatrix} I & H \\ G & F \end{bmatrix} = c_2(t,\omega)\begin{bmatrix} J & K \\ L & M \end{bmatrix}$$

$$\begin{bmatrix} 4F & 2G \\ 2H & I \end{bmatrix} = c_3(t,\omega)\begin{bmatrix} J & K \\ L & M \end{bmatrix}$$

The only nonzero P satisfying these three relations are

$$P = \begin{bmatrix} O & G \\ G & O \end{bmatrix}$$

If we use this last expression in $(*)$ with $j = 4$, then

$$\begin{bmatrix} 2G & G \\ G & O \end{bmatrix} = c_4(t,\omega)\begin{bmatrix} J & K \\ L & M \end{bmatrix}.$$

This cannot be a multiple of P unless $P \equiv 0$.

We see that the Lévy theorem fails in higher dimensions because we can manipulate 2×2 blocks. It is, however, possible to prove a weak version in which the process X_t depends on the ϕ given. We first need some tools.

7. A Lévy theorem in higher dimensions.

THEOREM 6 (Csink and Øksendal [1]). *Let X_t, Y_t be diffusions on open sets $\Omega_1 \subset \mathrm{R}^n$, $\Omega_2 \subset \mathrm{R}^m$ respectively. Let A and $\hat{A}$ be the generators, and assume $\phi : \Omega_1 \to \Omega_2$ is C^2. If there exists a continuous function $\lambda : \Omega_1 \to [0, \infty)$ such that*

$$A[f \circ \phi](x) = \lambda(x)\hat{A}[f](\phi(x))$$

then there exists a time change α_t such that $\phi(X_{\alpha_t}) \sim Y_t$.

More exactly: with $\beta_t = \int_0^t \lambda(X_s)ds, \alpha_t = \inf\{s ; \beta_s > t\}$ the limit $\lim_{t \to \tau_{\Omega_1}} \phi(X_t)$ exists a.s., and the process

$$M_t(\omega, \hat{\omega}) = \begin{cases} \phi(X_{\alpha_t}), & t < \beta_{\tau_{\Omega_1}} \\ Y_{t-\beta_{\tau_{\Omega_1}}}, & t \geqq \beta_{\tau_{\Omega_1}} \end{cases}$$

with a natural probability law such that $Y_{t-\beta_{\tau_{\Omega_1}}}$ starts at $\lim_{t \to \tau_{\Omega_1}} \phi(X_t)$, coincides in law with Y_t.

REMARK. The original proof, see Csink and Øksendal [1], required $\lambda > 0$ except on an X_t-finely nowhere dense subset. Then the time change is continuous. Øksendal [6] extended this, allowing α_t to have jumps.

THEOREM 7. *Let $\phi : \Omega_1 \subseteqq \mathrm{C}^n \to \Omega_2 \subseteqq \mathrm{C}^m$ be holomorphic, and assume that $\max \operatorname{rank} J_{\mathrm{C}}\phi = m$. Let $dH_t = E(H_t)dW_t$ be a conformal diffusion in Ω_2. Then there exists a conformal diffusion $dZ_t = F(Z_t)dW_t$ in Ω_1 and a time change α_t such that $\phi(Z_{\alpha_t})$ coincide in law with H_t in the precise sense of the Csink-Øksendal theorem.*

Before we can prove Theorem 7 we need two simple lemmas.

LEMMA 2. *Let $\phi : \mathbf{C}^n \to \mathbf{C}^m$ be holomorphic, and let*

$$A[f](z) = \sum_{i,j}^{n} c_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z).$$

Then

$$A[f \circ \phi](z) = \sum_{k,l}^{m} [J_{\mathbf{C}}\phi(z) \circ [c_{ij}(z)] \circ J_{\mathbf{C}}\phi(z)^*]_{kl} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l}(\phi(z)),$$

where $J_{\mathbf{C}}\phi$ denotes the complex Jacobian matrix of ϕ, $(\partial\phi_i/\partial z_j)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

PROOF. By the complex version of the chain rule

$$\frac{\partial^2 f \circ \phi}{\partial z_i \partial \bar{z}_j}(z) = \sum_{k,l}^{m} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l}(\phi(z)) \frac{\partial \phi_k}{\partial z_i}(z) \overline{\frac{\partial \phi_l}{\partial z_j}}(z)$$

and the lemma follows easily from this.

In the next lemma we assume that C and D are complex matrices which are positive semidefinite and self-adjoint, i.e. they correspond to generators of processes.

LEMMA 3. *Let $\phi : \mathbf{C}^n \to \mathbf{C}^m$ and assume $m \leqq n$. Let $A_1(z), A_2(z), \ldots, A_s(z)$ denote all $m \times m$ submatrices of $J_{\mathbf{C}}\phi(z)$. Given $D(z)$, an $m \times m$ matrix as above, there exist an $n \times n$ matrix $C(z)$ such that*

$$J_{\mathbf{C}}\phi(z) \circ C(z) \circ J_{\mathbf{C}}\phi(z)^* = \sum_{i=1}^{s} |\det A_i(z)|^2 D(\phi(z)).$$

PROOF. Find matrices $B_1 B_2 \ldots B_s$ such that $A_i B_i = \det A_i I_{\mathbf{C}^m}$ (that is $B_i = \operatorname{Adj} A_i$). Put

$$E_i(z) = B_i(z) D(\phi(z)) B_i^*(z).$$

Then construct $n \times n$ matrices $C_1 C_2 \ldots C_s$ with E_i as submatrices and then only zeros such that

$$J_{\mathbf{C}}\phi \circ C_i \circ J_{\mathbf{C}}\phi^*(z) = A_i B_i(z) D(\phi(z)) B_i^* A_i^*(z) = |\det A_i(z)|^2 D(\phi(z)).$$

Then put $C = \sum_{i=1}^{s} C_i$.

PROOF OF THEOREM 7. Put $D(z) = EE^*(z)$ in Lemma 3, and let Z_t have the generator

$$\sum_{i,j}^{n} c_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z)$$

where $C(z)$ is the matrix given by the lemma. By Lemma 2 the conditions in the Csink-Øksendal theorem are satisfied, and the result follows.

REMARK. (i) If $n = m$ it is easy to see that the proces

$$dZ_t = \operatorname{Adj} J_{\mathbf{C}}\phi(Z_t)E(\phi(Z_t))dW_t$$

solves the problem. Then if E has holomorphic coefficients, F will also have this property.

(ii) In case ϕ is everywhere of maximal rank, the time change factor is strictly positive. Then the time changed process is a diffusion, and the theorem is true without time change.

(iii) Again, the time change need not be nice. The continuity of the time change, is related to the time the process spends on the zero set N of the time change rate λ. If E is holomorphic, it is easy to see that N is the intersection of zero sets of analytic functions. In some cases, it is possible to prove that the time-scaling is "nice" if one starts Z_t outside N. It is never possible to start the process at a point where $\operatorname{rank} J_{\mathbf{C}}\phi \leqq m-2$, since from the construction the diffusion Z_t will only have the constant solution. In some cases e.g. $\phi(z, w) = (z^2, w)$, it is possible to start the process everywhere.

In relation to the last remark we have the following:

THEOREM 8. *Let $dZ_t = F(Z_t)dW_t$ be a conformal diffusion in $D \subset \mathbf{C}^n$. Let $f: D \to \mathbf{C}$ be analytic, let $P \subset \mathbf{C}$ be a polar set* (i.e. *P has logarithmic capacity zero*) *and put*

$$K = f^{-1}(P).$$

Then if Z_t starts outside K, it never hits K, a.s.

PROOF. We know that, up to the exit time τ from D of Z_t, the process $f(Z_t)$ is a time change of Brownian motion W_t in $\mathbf{C}$. More precisely, if Z_s^z starts at z then

$$M_t = \begin{cases} f(Z_{\alpha_t}^z); & t < \beta_\tau \\ f^* + W_{t-\beta_\tau}^0; & t \geqq \beta_\tau \end{cases}$$

coincides in law with Brownian motion starting at $f(z)$. Here

$$\beta_t = \int_0^t \sum_j \left|\frac{\partial f}{\partial z_j}\right|^2 (Z_s)ds, \quad \alpha_t = \inf\{a; \beta_s > t\}$$

and

$$f^* = \lim_{t \to \tau} f(Z_t), \text{ which exists a.s. on } \{\omega; \beta_\tau < \infty\}.$$

Since P is polar we know that $M_t \notin P$ for all $t > 0$, a.s. So if $z \notin K$, then $M_t \notin P$ for all $t \geqq 0$, a.s.

Hence

$$f(Z_{\alpha_t}) \notin P \quad \text{for all } 0 \leqq t < \beta_\tau, \text{ a.s.}$$

To complete the proof we need to know that the two paths

$$\{f(Z_{\alpha_t}); 0 \leqq t \leqq \beta_\tau\} \quad \text{and} \quad \{f(Z_t); 0 \leqq t \leqq \tau\}$$

coincide a.s.

For this it suffices to prove the following:

If β_t is constant on (t_1, t_2), then $f(Z_t)$ is constant on (t_1, t_2) (a.s.).

The last statement follows from Ito's formula (see Proposition 1 above):

$$f(Z_t) = f(z) + \int_0^t \sum_j \frac{\partial f}{\partial z_j}(Z_t) dZ_j,$$

so if β_t is constant on (t_1, t_2), then $\sum_j |\partial f / \partial z_j(Z_t)|^2$ is zero there, hence $f(Z_t)$ is constant there. That completes the proof.

COROLLARY 1. *Let* $dZ_t = U(Z_t) dW_t$ *in* $D \subset \mathbf{C}^n$, *where* $U \in \mathbf{C}^{n \times n}$ *has analytic entries. Put*

$$N = \{z; \det U(z) = 0\}.$$

If Z_t *starts outside* N, *it never hits* N, a.s.

In Theorem 7 consider the case where $\phi: \mathbf{C}^n \to \mathbf{C}^n$ and E is everywhere invertible. Then

$$dZ_t = \operatorname{Adj} J_{\mathbf{C}} \phi(Z_t) E(\phi(Z_t)) dW_t$$

solves the problem. The time change rate in this case is

$$\lambda(t, \omega) = |\det J_{\mathbf{C}} \phi(Z_t(\omega))|^2.$$

This is nonzero whenever the matrix of Z_t is invertible. The last theorem then gives that Z_t never hits the zero set of the time change if it starts outside this set, i.e. the time change is continuous in this case.

8. The Stratonovich integral.

As we mentioned in the very beginning, we can define different stochastic integrals if we choose different approximation points. The two most popular choices lead to the Ito and the Stratonovich integral. It is a rather remarkable property that the Stratonovich solution coincides with the Ito solution in case we have a diffusion with holomorphic coefficients. We will now prove this. We start with a lemma.

LEMMA. *Let $dZ_t = b(Z_t)dt$. Then, in real notation, the characteristic operator L of Z_t is*

$$L = \sum_{i=1}^{n} \operatorname{Re}(b_i) \frac{\partial}{\partial x_{2i-1}} + \operatorname{Im}(b_i) \frac{\partial}{\partial x_{2i}}.$$

PROOF.

$$L = \sum_{i=1}^{n} b_i \frac{\partial}{\partial z_i} + \bar{b}_i \frac{\partial}{\partial \bar{z}_i} = \sum_{i=1}^{n} \operatorname{Re}(b_i)\left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i}\right) + \operatorname{Im}(b_i) i \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}\right)$$

$$= \sum_{i=1}^{n} \operatorname{Re}(b_i) \frac{\partial}{\partial x_i} + \operatorname{Im}(b_i) \frac{\partial}{\partial y_i} = \sum_{i=1}^{n} \operatorname{Re}(b_i) \frac{\partial}{\partial x_{2i-1}} + \operatorname{Im}(b_i) \frac{\partial}{\partial x_{2i}}.$$

PROPOSITION 2. *Let Z_t be the solution of $dZ_t = U(Z_t)dW_t$ in the Stratonovich sense. Then Z_t solves the corresponding Ito equation*:

$$dZ_t = U(Z_t)dW_t + b(Z_t)dt$$

where

$$b_i = \sum_{k,j}^{n,m} 2 \frac{\partial U_{ij}}{\partial \bar{z}_k} \bar{U}_{kj}.$$

PROOF. Let $dX_t = \sigma(X_t)dB_t$ be the corresponding real expression. Then it is well-known that X_t solves the Ito equation

$$(*) \qquad dX_t = \sigma(X_t)dB_t + \hat{b}(X_t)dt$$

where

$$\hat{b}_i = \sum_{k,j}^{2m,2n} \frac{\partial \sigma_{ij}}{\partial x_k} \sigma_{kj}.$$

To prove the proposition, it is enough to prove that Z_t and $(*)$ have the same infinitesimal generator, and it is enough only to consider the first order terms.

Let $U_{ij} = c_{ij} + id_{ij}$. Then the coefficients of σ are given by

$$\begin{aligned} \sigma_{2i-1\,2j-1} &= c_{ij} \\ \sigma_{2i-1\,2j} &= -d_{ij} \\ \sigma_{2i\,2j} &= d_{ij} \\ \sigma_{2i\,2j} &= c_{ij} \end{aligned} \quad \left(\text{using the identification } a+ib \sim \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$$

To use the lemma, we must find the real and imaginary part of b_i.

$$b_i = \sum_{k,j}^{n,m} 2\frac{\partial U_{ij}}{\partial \bar{z}_k}\bar{U}_{kj} = \sum_{k,j}^{n,m}\left(\frac{\partial U_{ij}}{\partial x_k} + i\frac{\partial U_{ij}}{\partial y_k}\right)\bar{U}_{kj}$$

$$= \sum_{k,j}^{n,m}\left(\frac{\partial c_{ij}}{\partial x_k} + i\frac{\partial d_{ij}}{\partial x_k} + i\frac{\partial c_{ij}}{\partial y_k} - \frac{\partial d_{ij}}{\partial y_k}\right)(c_{kj} - id_{kj})$$

$$= \sum_{k,j}^{n,m}\frac{\partial c_{ij}}{\partial x_n}c_{kj} - \frac{\partial d_{ij}}{\partial y_k}c_{kj} + \frac{\partial d_{ij}}{\partial x_k}d_{kj} + \frac{\partial c_{ij}}{\partial y_k}d_{kj} +$$

$$+ i\sum_{k,j}^{n,m} - \frac{\partial c_{ij}}{\partial x_k}d_{kj} + \frac{\partial d_{ij}}{\partial x_k}c_{kj} + \frac{\partial c_{ij}}{\partial y_k}c_{kj} + \frac{\partial d_{ij}}{\partial y_k}d_{kj}.$$

We now use the lemma and express everything in terms of σ.

$$L = \sum_{i,j,k}^{n,m,n}\left\{\frac{\partial\sigma_{2i-1\,2j-1}}{\partial x_{2k-1}}\sigma_{2k-1\,2j-1} + \frac{\partial\sigma_{2i-1\,2j}}{\partial x_{2k}}\sigma_{2k\,2j} + \right.$$

$$\left. + \frac{\partial\sigma_{2i-1\,2j}}{\partial x_{2k-1}}\sigma_{2k-1\,2j} + \frac{\partial\sigma_{2i-1\,2j-1}}{\partial x_{2k}}\sigma_{2k\,2j-1}\right\}\frac{\partial}{\partial x_{2i-1}} +$$

$$+ \left\{\frac{\partial\sigma_{2i\,2j}}{\partial x_{2k-1}}\sigma_{2k-1\,2j} + \frac{\partial\sigma_{2i\,2j-1}}{\partial x_{2k-1}}\sigma_{2k-1\,2j-1} + \right.$$

$$\left. + \frac{\partial\sigma_{2i\,2j}}{\partial x_{2k}}\sigma_{2k\,2j} + \frac{\partial\sigma_{2i\,2j-1}}{\partial x_{2k}}\sigma_{2k\,2j-1}\right\}\frac{\partial}{\partial x_{2i}}$$

$$= \sum_{i,j,k}^{2n,2m,n}\frac{\partial\sigma_{ij}}{\partial x_k}\sigma_{kj}\frac{\partial}{\partial x_i}.$$

Since the last expression is the generator of the real process, this proves the proposition.

As a corollary we get:

THEOREM 9. *If $dZ_t = U(Z_t)dW_t + b(Z_t)dt$ and U has holomorphic coefficients, then the Stratonovich solution coincide with the Ito solution.*

REMARK. The correction is the same if we have an extra b-term.

Theorem 9 may have consequences in geometry. On manifolds Stratonovich integrals are preferred, but they are not necessarily martingales. Now, given a minifold M, we can construct a martingale diffusion on M if we can find a holomorphic U such that $U(z)(\mathbf{C}^m) \subseteqq TM_z$ for all $z \in M$. Stratonovich solutions are also required in the Stroock and Varadhan [8] support theorems.

An interesting question is the following: Is it possible that the solutions are independent of the approximation points in this case? Is there a measure behind the result? We do not know, but we think it should be investigated further.

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