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ON EXTREMAL h-BASES A_4

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Summary.

Let $h \in \mathbb{N}$. For each set $A_4 = \{1 = a_1 < a_2 < a_3 < a_4\}$ with $a_i \in \mathbb{N}$ the h-range $n(h, A_4)$ is defined as the largest integer N, such that all positive integers $n \leqq N$ have a representation as a sum of at most h addends, all elements from the set A_4. In this paper we give a set A_4 for which

$$n(h, A_4) = \sigma_4(h/4)^4 + O(h^3) \qquad \text{where } \sigma_4 > 2.008.$$

1. Introduction.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Small letters represent non-negative integers, when not otherwise said. $[x]$ denotes the integral part of a real number x. We define an interval by

$$[a, b] = \{m \mid m, a, b \in \mathbb{Z},\ a \leqq m \leqq b\}.$$

The sum $A+B$ of two non-empty integer sets A, B is defined by

$$A+B = \{a+b \mid a \in A,\ b \in B\}.$$

Let $h \in \mathbb{N}$. Then we write hA for the h-fold sum $A+A+\ldots+A$.

We shall be concerned with finite integer sets

$$A = A_k = \{a_1, a_2, \ldots, a_k\}, \quad 1 = a_1 < a_2 < \ldots < a_k.$$

We have a *representation* of the integer n with respect to the *basis* A_k if

$$n = \sum_{l=1}^{k} x_l a_l, \quad x_l \in \mathbb{N}_0. \tag{1.1}$$

We say that A_k is a *h-basis* for a positive integer n if $[0, n] \subset hA'$, where

$$A' = A'_k = A_k \cup \{0\}.$$

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We denote the representation (1.1) a *h-representation* of n, even if less than h addends are used. The *h-range* $n(h, A_k)$ is

$$n(h, A_k) = n \quad \text{iff} \quad [0, n] \subset hA'_k \quad \text{and} \quad n+1 \notin hA'_k.$$

In the global case (Selmer [11]), h and k are given and the problem consists in determining *the extremal h-range*

$$n(h, k) = \max_{A_k} n(h, A_k),$$

and also the corresponding *extremal bases*, i.e. the bases A_k for which $n(h, A_k) = n(h, k)$.

We consider only bases A_k which are *h-admissible*, that is

$$a_k \leqq n(h, A_k).$$

Let us denote the smallest such h by h_0, hence

$$h_0 = h_0(A_k) = \min\{h \in \mathbb{N} \mid a_k \leqq n(h, A_k)\}.$$

For $k = 3$ (see Selmer [11]),

$$h_0 = h_0(A_3) = a_2 + [a_3/a_2] - 2. \tag{1.2}$$

2. Some results about h-ranges.

Apart from some tabulated values of the extremal h-range $n(h, k)$ for small h and k (see Mossige [7], [8] and Lunnon [6]), the exact value of $n(h, k)$ is known only for $k = 1$ (trivial), $k = 2$, $k = 3$ and for $h = 1$ (trivial).

Let k be given. Like Hofmeister [3] we write for $h \to \infty$,

$$n(h, k) \geqq c_k(h/k)^k + O(h^{k-1}).$$

For $k = 4$, Hofmeister and Schell [2] showed in 1970 that

$$n(h, 4) \geqq \frac{1989}{1024}(h/4)^4 + O(h^3).$$

Hofmeister and Schell showed in 1972 (see Hofmeister [5]) that

$$n(h, 4) \geqq 2(h/4)^4 + O(h^3). \tag{2.1}$$

In a lecture in Mainz in 1972, Hugo Schell (see also Hofmeister [5, p. 51]) presented the conjecture that:

for given $k \geqq 1$ and sufficiently large h,
the extremal h-range satisfies

$$n(h,k) = \frac{2^{k-1}}{k}(h/k)^k + O(h^{k-1}).$$

The conjecture is true for $k = 1, 2$ and 3, see Hofmeister [3].

Since 1971, it seems to have been generally believed that (2.1) holds with equality. Here we prove (Mossige [9])

$$n(h,4) \geqq c_4(h/4)^4 + O(h^3), \quad \text{where } c_4 > 2.008. \tag{2.2}$$

Hence, for $k = 4$, Schell's conjecture is not true.

We shall need the following result of Mrose [10]:

LEMMA 2.1. *Let* $k, m \in \mathbb{N}$. *If*

$$n(h, A_k) \geqq c_k(h/k)^k + O(h^{k-1}) \quad \textit{for all } h \in \mathbb{N}$$

with $h \equiv 0 \pmod{m}$ $(h \to \infty)$, *then we have for all* $h' \in \mathbb{N}$ $(h' \to \infty)$

$$n(h', A_k) \geqq c_k(h'/k)^k + O(h'^{k-1}).$$

3. The main steps in the search for bases A_4 with large h-ranges.

Let $k = 4$, $h = 12j$, $j = \alpha t$, where $\alpha, t \in \mathbb{N}$. For α sufficiently large we consider the h-range of the integer basis $C = \{a_1, a_2, a_3, a_4\}$, given in the regular form (see (4.2)) as

$$\begin{cases} a_1 = 1 \\ a_2 = (9j + b_1t + d_1)a_1 \\ a_3 = (4j + b_2t + d_2)a_1 + (3j + b_3t + d_3)a_2 \\ a_4 = (7j + b_4t + d_4)a_1 + (2j + b_5t + d_5)a_2 + (2j + b_6t + d_6)a_3, \end{cases} \tag{3.1}$$

where the ordered sets $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ and $P = (d_1, d_2, d_3, d_4, d_5, d_6)$, $B, P \subset \mathbb{Z}$. We denote the basis $C = C(B, P)$.

Our extensive calculations for $k = 4$ were performed on the Univac 1100/82 computer at the University of Bergen. We have earlier (Mossige [7]) determined the extremal h-range $n(h, 4)$ for all $h \leqq 28$, by scanning the complete universe of all admissible bases A_4 for each h. For larger h, the search had to be restricted to increasingly narrower sectors of the complete universes. The computed bases with the largest h-ranges for $29 \leqq h \leqq 75$ are given in [9]. It proved useful to concentrate the search to h_0-bases, but extensive calculations were also performed with $h > h_0$.

From this point on, in 1978, the further search was concentrated on bases with the *structure* given in (3.1). This is done without explicit computation of the h-range and hence, the amount of calculations is independent of h. The structure is the same as that Hofmeister and Schell used in [5], though they did not parametrize the basis in the same way.

4. The regular form and the $D(n; c_2, c_3, c_4)$ representation.

The representation

$$n = \sum_{l=1}^{k} e_l a_l, \quad e_l \geqq 0, \tag{4.1}$$

is *regular* (and unique) by the basis A_k if

$$\sum_{l=1}^{m} e_l a_l < a_{m+1}, \quad m = 1, 2, \ldots, k-1.$$

From now on, let $k = 4$. The basis A_4 is given in *regular form* when

$$\begin{cases} a_2 = y_1 a_1 \\ a_3 = y_2 a_1 + y_3 a_2 \\ a_4 = y_4 a_1 + y_5 a_2 + y_6 a_3 \end{cases} \tag{4.2}$$

are regular representations by the partial bases $A_l = \{a_1, a_2, \ldots, a_l\}$, $l = 2, 3$. From (4.1) with $k = 4$, for any $c_l \in \mathbb{Z}$,

$$\begin{cases} n = e_1 a_1 + e_2 a_2 + e_3 a_3 + e_4 a_4 + c_2(y_1 a_1 - a_2) + \\ \qquad + c_3(y_2 a_1 + y_3 a_2 - a_3) + c_4(y_4 a_1 + y_5 a_2 + y_6 a_3 - a_4) \\ \quad = (e_1 + c_4 y_4 + c_3 y_2 + c_2 y_1) a_1 + (e_2 + c_4 y_5 + c_3 y_3 - c_2) a_2 + \\ \qquad + (e_3 + c_4 y_6 - c_3) a_3 + (e_4 - c_4) a_4 \\ \quad = x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 \quad \text{(say).} \end{cases} \tag{4.3}$$

If all $x_l \geqq 0$, using Hofmeister's notation in [5], we call this the $D(n; c_2, c_3, c_4)$ representation of n (with respect to A_4 in the regular form (4.2)).

Clearly $D(n;0,0,0)$ is the regular representation of n. We call

$$(4.4)\qquad \begin{aligned} G(c_2,c_3,c_4) &= \sum e_l - \sum x_l \\ &= -c_4y_6-c_4y_5-c_4y_4-c_3y_3-c_3y_2-c_2y_1+c_4+c_3+c_2 \end{aligned}$$

the *gain* in the number of addends in the $D(n;c_2,c_3,c_4)$ representation compared to the regular one.

To determine the h-range of the basis A_4, we use that for each integer $n \in hA'_4$, we can find a h-representation $D(n;c_2,c_3,c_4)$ with $\sum x_l \leqq h$. (It is clear that given $x_1,\ldots,x_4$, then c_2,c_3,c_4 are uniquely determined by (4.3).)

5. The procedure to determine the h-range.

LEMMA 5.1. *Let the basis A_4 in (4.2) with $y_3 \geqq y_5+2$ be given. The representation $\sum e_la_l$ of the integer n is regular iff*

$$(5.1)\qquad \begin{cases} e_1 \leqq y_1-1 \\ e_2 \leqq y_3 \\ e_3 \leqq y_6 \\ e_2+e_3 \leqq \begin{cases} y_3+y_6-1 & \text{for } e_1 \leqq y_2-1 \\ y_3+y_6-2 & \text{for } e_1 \geqq y_2 \end{cases} \\ \text{if } e_2 = y_3 \quad \text{then } e_1 \leqq y_2-1, \\ \text{if } e_3 = y_6 \text{ and } e_2 = y_5 \quad \text{then } e_1 \leqq y_4-1 \text{ and else} \\ \text{if } e_3 = y_6 \quad \text{then } e_2 < y_5. \end{cases}$$

The proof is easy and can be found in [9].

LEMMA 5.2. *Let the basis C in (3.1) be given and let α be sufficiently large. Let the regular representations of $n \in [0, ha_4]$ satisfy (5.1). Then the representations*

$$(5.2)\qquad \begin{cases} D(n;-1,1,0), & D(n;-1,-1,1), \quad D(n;-2,1,1) \\ D(n;-1,-2,2), & D(n;-2,-1,2), \quad D(n;-2,-3,3) \end{cases}$$

and $D(n;-3,-4,5)$, given by (4.3), are the only representations that can give a positive gain (4.4).

PROOF. From (4.2), (4.3), and (5.1) we get

$$\begin{aligned} -c_4y_6+c_3 &\leqq e_3 \leqq y_6 \\ -c_4y_5-c_3y_3+c_2 &\leqq e_2 \leqq y_3 \\ -c_4y_4-c_3y_2-c_2y_1 &\leqq e_1 \leqq y_1-1. \end{aligned}$$

To use the representation (4.3), the gain (4.4) must be non-negative:

$$c_4y_6+c_4y_5+c_4y_4+c_3y_3+c_3y_2+c_2y_1-c_4-c_3-c_2 \leqq 0.$$

Let α be sufficiently large. When we substitute by (3.1) in the inequalities above, we get

(i) $c_4 \geqq -1$

(ii) $2c_4+3c_3 \geqq -3$

(iii) $7c_4+4c_3+9c_2 \geqq -9$

(iv) $11c_4+7c_3+9c_2 \leqq 0.$

From (ii), (iii), and (iv),

$$4c_4+3c_3 \leqq 9, \quad c_4 \leqq 6.$$

From this, it follows that we have only a finite number of different representations (4.3). For chosen $c_4 \in [-1, 6]$ we have

$$-3-2c_4 \leqq 3c_3 \leqq 9-4c_4,$$

and for chosen c_3 we have

$$-9-7c_4-4c_3 \leqq 9c_2 \leqq -11c_4-7c_3.$$

This gives us the following possibilities in addition to the seven given in Lemma 5.2

c_2	0	1	0	−1	−2	0	−1
c_3	0	0	1	2	4	−1	0
c_4	−1	−1	−1	−1	−1	0	0

From Lemma 5.1 it is easily seen that none of these representations $D(n; c_2, c_3, c_4)$ can be used, and Lemma 5.2 follows.

Our basis C in (3.1) satisfies the condition $y_3 \geqq y_5+2$ in Lemma 5.1 for α sufficiently large. Let $n \in hC'$ be given by (4.1) and satisfy (5.1). If possible, we apply the $D(n; c_2, c_3, c_4)$ representation in (5.2) with non-negative coefficients that gives the *largest* positive gain. Then

(5.3) $$e_1+e_2+e_3+e_4-G(c_2, c_3, c_4) \leqq h,$$

or we have for the regular representation itself that

(5.4) $$e_1+e_2+e_3+e_4 \leqq h.$$

It turns out that $D(n; -3, -4, 5)$ in our case never gives the largest gain.

Using (5.1) and the fact that the coefficients in (4.3) for the representations (5.2) must be non-negative, we split the integers n first into 3 disjoint sets

1. $0 \leqq e_1 \leqq y_2 - 1, \quad e_2 = y_3, \quad 0 \leqq e_3 \leqq y_6 - 1$
2. $0 \leqq e_1 \leqq y_1 - 1, \quad 0 \leqq e_2 \leqq y_3 - 1, \quad 0 \leqq e_3 \leqq y_6 - 1$
3. $0 \leqq e_1 \leqq y_1 - 1, \quad 0 \leqq e_2 \leqq y_5 - 1, \quad e_3 = y_6$ or
   $0 \leqq e_1 \leqq y_4 - 1, \quad e_2 = y_5, \quad e_3 = y_6.$

Splitting these sets further, we end up with 40 linear inequalities of the type (5.3) and (5.4) to determine the smallest value of e_4 with the corresponding integer $n \in hC'$ such that $n+1 \notin hC'$ (see [9]). It turns out that this e_4 has the form $e_4 = 3\alpha t + xt + y$, where $x, y \in \mathbb{Z}$. It follows that for α sufficiently large, we get

$$n(h, C) = \sigma_4 (h/4)^4 + O(h^3), \tag{5.5}$$

where the coefficient $\sigma_4 = \sigma_4(C)$.

6. The optimal C basis.

THEOREM 6.1. *Let $h = 12\alpha t$, $\alpha, t \in \mathbb{N}$. Let the basis C have $B = (15b, 14b, -15b, 23b, -2b, -20b)$, $P = (0, 0, 2, 0, 0, 0)$ where $b \in \mathbb{N}$. Let b and α be given, where $\alpha \geqq 25b$. Then the basis $C = C(B, P)$ with*

$$\begin{cases} a_2 = (9\alpha + 15b)ta_1 \\ a_3 = (4\alpha + 14b)ta_1 + ((3\alpha - 15b)t + 2)a_2 \\ a_4 = (7\alpha + 23b)ta_1 + (2\alpha - 2b)ta_2 + (2\alpha - 20b)ta_3 \end{cases} \tag{6.1}$$

is a h_0-basis with h-range

$$n(h, C) = (3\alpha + 45b)ta_4 + (\alpha - 13b)ta_2 + ((8\alpha - 2b)t - 2)a_1. \tag{6.2}$$

PROOF. From Lemma 5.2 and (4.4) we get the gains

(1) $G(-1, 1, 0) = (2\alpha + 16b)t - 2$
(2) $G(-1, -1, 1) = (5\alpha + 13b)t + 1$
(3) $G(-2, 1, 1) = 30bt - 2$
(4) $G(-1, -2, 2) = (\alpha + 11b)t + 3$
(5) $G(-2, -1, 2) = (3\alpha + 27b)t + 1$
(6) $G(-2, -3, 3) = (6\alpha + 24b)t + 4.$

For each integer $n \in hC'$, the regular representation (4.1), satisfying (5.1) is a h-representation itself, or we use the $D(n; c_2, c_3, c_4)$ representation, corresponding to the gains (1) to (6), that gives the largest gain. In each of the 40 cases below we mark to the left which representation that is used if different from the $D(n; 0, 0, 0)$ representation. In each line we only write the changes from the preceding line.

$e_1 \leqq (3\alpha - 3b)t - 1$, $e_2 = (3\alpha - 15b)t + 2$, $e_3 \leqq (2\alpha - 20b)t - 1$,
$e_4 \leqq (4\alpha + 38b)t$
$(3\alpha - 3b)t \leqq e_1 \leqq (4\alpha + 14b)t - 1$, $e_4 \leqq 1$
(4) $2 \leqq e_4 \leqq (4\alpha + 32b)t + 3$
$e_1 \leqq (5\alpha + b)t - 1$, $e_2 \leqq (\alpha - 13b)t$, $e_4 \leqq (4\alpha + 32b)t + 2$
$(5\alpha + b)t \leqq e_1 \leqq (7\alpha - 7b)t - 1$, $e_3 = 0$, $e_4 \leqq (4\alpha + 20b)t + 1$
$(7\alpha - 7b)t \leqq e_1 \leqq (8\alpha - 2b)t - 1$, $e_4 = 0$
(3) $1 \leqq e_4 \leqq (3\alpha + 45b)t - 1$
(1) $(5\alpha + b)t \leqq e_1 \leqq (8\alpha - 2b)t - 1$, $1 \leqq e_3 \leqq (2\alpha - 20b)t - 1$, $e_4 \leqq (3\alpha + 51b)t$
$(8\alpha - 2b)t \leqq e_1 \leqq (9\alpha + 15b)t - 1$, $e_3 = 0$, $e_4 = 0$
(3) $e_4 = 1$
(1) $1 \leqq e_3 \leqq (2\alpha - 20b)t - 1$, $e_4 \leqq 1$
(5) $e_3 \leqq (2\alpha - 20b)t - 1$, $2 \leqq e_4 \leqq (3\alpha + 45b)t + 3$
$e_1 \leqq (3\alpha - 3b)t - 1$, $(\alpha - 13b)t + 1 \leqq e_2 \leqq (3\alpha - 15b)t + 1$,
$e_4 \leqq (4\alpha + 38b)t + 1$
$(3\alpha - 3b)t \leqq e_1 \leqq (5\alpha + b)t - 1$, $(\alpha - 13b)t + 1 \leqq e_2 \leqq (2\alpha - 26b)t + 2$,
$e_4 \leqq (3\alpha + 45b)t$
$(2\alpha - 26b)t + 3 \leqq e_2 \leqq (3\alpha - 15b)t + 1$, $e_4 \leqq 1$
(4) $2 \leqq e_4 \leqq (3\alpha + 45b)t + 4$
$(5\alpha + b)t \leqq e_1 \leqq (6\alpha + 6b)t - 1$, $(\alpha - 13b)t + 1 \leqq e_2 \leqq (2\alpha - 26b)t + 2$,
$e_3 = 0$, $e_4 \leqq (4\alpha + 20b)t - 1$
$(2\alpha - 26b)t + 3 \leqq e_2 \leqq (3\alpha - 15b)t + 1$, $e_4 \leqq 1$
(4) $2 \leqq e_4 \leqq (4\alpha + 20b)t + 3$
(1) $(\alpha - 13b)t + 1 \leqq e_2 \leqq (3\alpha - 15b)t + 1$, $1 \leqq e_3 \leqq (2\alpha - 20b)t - 1$,
$e_4 \leqq (3\alpha + 45b)t - 1$
$(6\alpha + 6b)t \leqq e_1 \leqq (9\alpha + 15b)t - 1$, $e_3 = 0$, $e_4 = 0$
(1) $1 \leqq e_3 \leqq (2\alpha - 20b)t - 1$, $e_4 = 0$
(2) $(6\alpha + 6b)t \leqq e_1 \leqq (9\alpha + 3b)t - 1$, $e_3 \leqq (2\alpha - 20b)t - 1$,
$1 \leqq e_4 \leqq (3\alpha + 45b)t + 2$
(2) $(9\alpha + 3b)t \leqq e_1 \leqq (9\alpha + 15b)t - 1$, $1 \leqq e_4 \leqq 2$
(2) $(\alpha - 13b)t + 1 \leqq e_2 \leqq (3\alpha - 39b)t + 3$, $3 \leqq e_4 \leqq (3\alpha + 57b)t$
(6) $(3\alpha - 39b)t + 4 \leqq e_2 \leqq (3\alpha - 15b)t + 1$, $3 \leqq e_4 \leqq (4\alpha + 44b)t + 5$
$e_1 \leqq (5\alpha + b)t - 1$, $e_2 \leqq (\alpha - 13b)t$, $e_3 = (2\alpha - 44b)t$, $e_4 \leqq (4\alpha + 32b)t + 1$
(1) $(5\alpha + b)t \leqq e_1 \leqq (8\alpha - 2b)t - 1$, $e_4 \leqq (3\alpha + 51b)t - 1$

(1) $(8\alpha-2b)t \leqq e_1 \leqq (9\alpha+15b)t-1,\ e_4 \leqq 1$
(5) $2 \leqq e_4 \leqq (3\alpha+45b)t+2$
$e_1 \leqq (5\alpha+b)t-1,\ (\alpha-13b)t+1 \leqq e_2 \leqq (2\alpha-26b)t+2,\ e_4 \leqq (3\alpha+45b)t-1$
$e_1 \leqq (3\alpha-3b)t-1,\ (2\alpha-26b)t+3 \leqq e_2 \leqq (2\alpha-2b)t,\ e_4 \leqq (5\alpha+25b)t-2$
$(3\alpha-3b)t \leqq e_1 \leqq (5\alpha+b)t-1,\ e_4 \leqq 1$
(4) $2 \leqq e_4 \leqq (4\alpha+32b)t+4$
(1) $(5\alpha+b)t \leqq e_1 \leqq (6\alpha+6b)t-1,\ (\alpha-13b)t+1 \leqq e_2 \leqq (2\alpha-2b)t-1,$
$e_4 \leqq (4\alpha+32b)t$
(1) $(6\alpha+6b)t \leqq e_1 \leqq (9\alpha+15b)t-1,\ e_4 = 0$
(2) $1 \leqq e_4 \leqq (4\alpha+20b)t+3$
(1) $(5\alpha+b)t \leqq e_1 \leqq (6\alpha+6b)t-1,\ e_2 = (2\alpha-2b)t,\ e_4 \leqq (4\alpha+32b)t-1$
(1) $(6\alpha+6b)t \leqq e_1 \leqq (7\alpha+23b)t-1, e_4 = 0$
(2) $1 \leqq e_4 \leqq (6\alpha+12b)t+2$

We easily check that for $\alpha \geqq 25b$, line 7 in the above scheme gives the smallest integer

$$N+1 = (3\alpha+45b)ta_4+(\alpha-13b)ta_2+((8\alpha-2b)t-1)a_1$$

with no h-representation and such that all integers smaller than $N+1$ have a h-representation. Hence, from Lemma 5.2, $N+1$ has no h-representation and $n(h, C) = N$.

From (1.2) we immediately have $h_0(C \setminus \{a_4\}) = h$ hence, $h = h_0(C)$.

COROLLARY 6.1. *Let the basis* $C = C(B, P)$ *in* (6.1) *be given, with*

$$n(h, C) = \sigma_4(h/4)^4+O(h^3).$$

For given $\varepsilon > 0$, *we can choose the integers* α *and* b *such that*

$$\sigma_4 > s_4-\varepsilon,$$

where

$$s_4 = 2+3^{-4}2^{-6}g(\gamma) = 2.0080397 \tag{6.3}$$

$$g(\gamma) = -864\gamma-4536\gamma^2-594\gamma^3+81\gamma^4 \tag{6.4}$$

$$\gamma = \frac{11}{6}+\frac{1}{3}\sqrt{457}\cos\frac{\varphi+4\pi}{3},\quad \cos\varphi = \frac{7163}{\sqrt{457^3}},\quad 0 < \varphi < \pi/2 \tag{6.5}$$

$$\gamma = -0.09712372.$$

PROOF. From (6.2) and (6.1) we get for $t \to \infty$

$$n(h,C) = (3\alpha+45b)t(2\alpha-20b)t(3\alpha-15b)t(9\alpha+15b)t+O(t^3)$$
$$= t^4(162\alpha^4+270\alpha^3b-28350\alpha^2b^2+74250\alpha b^3+202500b^4)+O(t^3).$$

Since $h = 12\alpha t$, we get with $r = -20b/\alpha$

$$n(h,C) = (2+3^{-4}2^{-6}g(r))(h/4)^4+O(h^3).$$

To maximize $g(r)$, put $g'(\gamma) = 0$, where γ is given by (6.5).

Choosing α and b suitably, we can always make $r = -20b/\alpha$ as close to γ as we want, and hence $2+3^{-4}2^{-6}g(r)$ as close to s_4 as we want. In fact, already the choice $b = 1$, $\alpha = 206$ gives σ_4 with all the decimals of s_4 in (6.3).

The condition $\alpha \geqq 25b$ is clearly satisfied. The two other roots of $g'(r) = 0$ correspond to (local) minima of $g(r)$. The remaining values of r, where $g(r) \geqq g(\gamma)$ are either positive or so large negative that we cannot get $\alpha \geqq 25b$.

In combination with Lemma 2.1 with modulus 12α, we have

THEOREM 6.2. *For any given* $\varepsilon > 0$

$$n(h,4) \geqq c_4(h/4)^4+O(h^3),$$

where $c_4 > s_4-\varepsilon$. *Here* s_4 *is given in Corollary* 6.1.

7. The search for optimal bases.

The basis (6.1) of Theorem 6.1 was the result of a long and complicated search, described in detail in [9]. We shall here only indicate the method.

As mentioned in section 5, the conditions (5.1) lead to a set of 40 (mutually disjoint) cases for the basis (4.2). With the particular choice (6.1), these cases are given in section 6. With the general basis (3.1), we can similarly write down the 40 cases, with 40 corresponding inequalities resulting from the condition $\sum x_i \leqq h$ (when $n = \sum x_i a_i$). It turns out that if a choice of 9 of these inequalities are satisfied, so are the remaining ones (for α sufficiently large).

The set $P = (d_1, d_2, d_3, d_4, d_5, d_6)$ does not influence the coefficient σ_4 of (5.5). The choice of P in Theorem 6.1 turned out to be convenient (but other choices of P might give larger h-ranges). Over all integral sets $B = (b_1, b_2, b_3, b_4, b_5, b_6)$, we must then try to maximize σ_4 (a non-linear function in the b_i) under the constraints of the above-mentioned 9 inequalities (linear in the b_i). This is a formidable task, but it was greatly simplified by the early observation that 6 of the 9 inequalities could apparently be replaced by strict equalities. After a long computer search, the basis (6.1) finally emerged.

We then got a welcome confirmation of this result, by treating the optimization (with all 9 inequalities) as a problem in *real* variables. There is a NAG algorithm E04VDF for this (Numerical Algorithms Group (NAG), 7 Bandbury Road, Oxford, U.K.). The algorithm only gives a *local* maximum for σ_4, but it came out with the same result (6.1). We therefore feel confident that we have found the global maximum, at least with the *structure* of the basis (3.1) as our starting point.

With $P = (0,0,2,0,0,0)$, it follows from (3.1) and (1.2) that

$$h_0(C \setminus \{a_4\}) = h + (b_1 + b_3)t,$$

showing that we must have $b_1 + b_3 \leqq 0$. In the optimization, we usually get

$$h_0 = h_0(C) = h + (b_1 + b_3)t.$$

Choosing $b_1 + b_3 < 0$, we then get bases where $h - h_0$ is proportional to $h = 12\alpha t$. Also in this case, we can find bases C with the coefficient σ_4 very close to the optimal one in Corollary 6.1. As a simple example of $\sigma_4 > 2$, we mention

$$B = (3, 5, -9, 8, -2, -11), \quad \alpha = 141, \quad \sigma_4 = 2.0058.$$

This should be compared with the extremal bases for $k = 3$ (Hofmeister [1], [4]), which all have $h - h_0 \sim h/9$ (asymptotically, as $h \to \infty$).

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