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TWO-SHEETED COVERINGS OF THE DISC

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1. Introduction.

Let X be a Riemann surface in which there is a holomorphic map φ to the open unit disc $\boldsymbol{D}$ with φ of constant valence 2. One says that X, or the pair (X, φ), is a two-sheeted covering of $\boldsymbol{D}$. Let N be the number of branch points of φ. Then $1 \leqq N \leqq \infty$. We will prove that if $N > 2$ and if φ_1, like φ, is a holomorphic map of X to $\boldsymbol{D}$ with φ_1 of valence 2, then $\varphi_1 = A(\varphi)$, where A is an automorphism of $\boldsymbol{D}$. This means in part that φ_1 and φ have the same branch points, and thus one may speak without ambiguity of the branch points of X. (This is not possible if $1 \leqq N \leqq 2$. Then X is, conformally, an annulus or the disc, and one finds that $\varphi_1 = A(\varphi(B))$ with $B \in \operatorname{Aut} X$.) It is a corollary of "$\varphi_1 = A(\varphi)$" (or for that matter of "$\varphi_1 = A(\varphi(B))$") that if (X_1, φ_1) and (X_2, φ_2) are two-sheeted coverings of $\boldsymbol{D}$, if

$$\Delta_k = \{\varphi_k(x) : x \text{ is a branch point of } \varphi_k\},$$

and if X_1 and X_2 are conformally equivalent, then $\Delta_2 = A(\Delta_1)$ with $A \in \operatorname{Aut} \boldsymbol{D}$. This should be new if N is infinite. (The converse holds, i.e., X_1 and X_2 are conformally equivalent if $\Delta_2 = A(\Delta_1)$ with $A \in \operatorname{Aut} \boldsymbol{D}$, but I would guess this is not new.)

The theorem that gives $\varphi_1 = A(\varphi)$ if $N > 2$ is in part 2. If $N = \infty$, the theorem gives more. Then it identifies the holomorphic maps, of X to $\boldsymbol{D}$, of constant, finite valence, not just those of valence 2. Then in part 3 we use the theorem to study the proper holomorphic maps of X to itself. E.g., we will prove that if $f \in \operatorname{Prop} X$ but $\notin \operatorname{Aut} X$, and if f fixes a point, then the point is a branch point of X.

2. The main theorem.

A mapping $\varphi : X \to Y$ of topological spaces X and Y is said to be proper if inverses of bounded sets are bounded, i.e., if $\{\varphi \in E\}$ is bounded in X

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whenever E is bounded in Y. A set is bounded if it is contained in a compact set.

Let X be a Riemann surface in which there is a holomorphic map φ to the open unit disc $\boldsymbol{D}$ with φ of constant, finite valence m. That is, $|\varphi| < 1$, while if $|\xi| < 1$, then the set where $\varphi = \xi$ consists of m points counting multiplicities. In other words, if $\partial(F, w)$ is the order of vanishing of F at w, then

$$\sum_{\varphi(x) = \xi} (1 + \partial(\varphi', x)) = m. \tag{2.1}$$

One says that X, or the pair (X, φ), is an m-sheeted covering of $\boldsymbol{D}$. Let N be the number of points, in X, of branch order $m-1$. Then $0 \leqq N \leqq \infty$. (The branch order of x is $\partial(\varphi', x)$.)

Let φ_1, like φ, be a proper holomorphic map of X to $\boldsymbol{D}$. Put $m_1 =$ the number of times φ_1 vanishes counting multiplicities. Then $m_1 < \infty$.

THEOREM 1. *If* $m_1 < N$, *then*

(i) m *divides* m_1;

(ii) $\varphi_1 = g(\varphi)$, *where* g *is a proper holomorphic map of* $\boldsymbol{D}$ *to itself of valence* m_1/m, *in other words,* g *is a finite Blaschke product that vanishes* m_1/m *times counting multiplicities.*

Thus if the number of points of branch order $m-1$, namely N, is infinite, then every proper holomorphic map of X to $\boldsymbol{D}$ is of valence zero mod m, and is obtained from φ by composing with a finite Blaschke product.

2.1. THE PROOF OF THEOREM 1. Our proof is elementary. Its main ingredient is an old lemma. If $f \in \mathcal{O}(\boldsymbol{D})$, let $N(f)$ be the number of times f vanishes in $\boldsymbol{D}$, counting multiplicities. Then the lemma is this:

LEMMA 1. *Let f be a finite Blaschke product. Let A be holomorphic in* $\boldsymbol{D}$ *and bounded by* 1 *there. If* $N(A-f) > N(f)$, *then* $A = f$.

PROOF. (i) Let $N(f) = 0$. Then f is a constant of modulus one, hence by the principle of maximum, $A = f$.

(ii) Let $N(f) > 0$. Because $N(A-f)$ is positive, there is, like in (i), a point ξ in $\boldsymbol{D}$ with $A(\xi) = f(\xi)$, but here $|f(\xi)| < 1$. Let $\zeta = f(\xi)$, and put

$$A_1 = \frac{\zeta - A}{1 - \bar{\zeta} A}, \quad f_1 = \frac{\zeta - f}{1 - \bar{\zeta} f}, \quad \text{and} \quad w = \frac{\xi - z}{1 - \bar{\xi} z}.$$

Then $A_1 = wA_2$, $f_1 = wf_2$, and

$$(1 - \bar{\zeta} A)(1 - \bar{\zeta} f)(A_2 - f_2) w = (|\zeta|^2 - 1)(A - f).$$

The identity gives

$$1+N(A_2-f_2) > 1+N(f_2)$$

because

$$N(A-f) > N(f) = N(f_1) = 1+N(f_2).$$

By the induction hypothesis, $A_2 = f_2$, which means $A = f$.

THE PROOF OF THEOREM 1. If $|\xi| < 1$, put

$$f(\xi) = \prod_{\varphi(x) = \xi} \varphi_1(x)^{1+\partial(\varphi', x)}$$

and

$$A(\xi) = \left(\frac{1}{m} \sum_{\varphi(x) = \xi} (1+\partial(\varphi', x))\varphi_1(x)\right)^m.$$

It is plain that f and A are holomorphic in $\boldsymbol{D}$, with $|f| < 1$ and $|A| < 1$ there. (Less briefly: To each point x there is an open disc V of center x and a (cyclic) group G, of $1+\partial(\varphi', x)$ automorphisms of V that fix x, such that the orbits of points are the fibers of $\varphi|V$. In other words, if $y \in V$, then the set of points $\sigma(y)$, $\sigma \in G$, is the set in V where $\varphi = \varphi(y)$. We may identify $\mathcal{O}(\varphi(V))$ with $\{g(\varphi)\colon g \in \mathcal{O}(\varphi(V))\}$; then $\mathcal{O}(V)$ is an overring of $\mathcal{O}(\varphi(V))$. The group G serves to tell who is in $\mathcal{O}(\varphi(V))$. The test is this: let $g \in \mathcal{O}(V)$; then $g \in \mathcal{O}(\varphi(V))$ iff $g(\sigma) = g$ for every σ in G. Put

$$\theta = \prod_{\sigma \in G} \varphi_1(\sigma);$$

then by the test, $\theta \in \mathcal{O}(\varphi(V))$, which means $\theta = g(\varphi)$ with g holomorphic in $\varphi(V)$. Let $|\xi| < 1$. If f_ξ is the germ of f at ξ, and g_x the germ of g at $\varphi(x)$, then by (2.1),

$$f_\xi = \prod_{\varphi(x) = \xi} g_x. \tag{2.2}$$

This proves that $f_\xi \in \mathcal{O}_\xi$, which means $f \in \mathcal{O}(\boldsymbol{D})$. Likewise, $A \in \mathcal{O}(\boldsymbol{D})$). And it is plain that f *is proper, because if* m sequences in the disc $\boldsymbol{D}$ converge to the boundary, then their termwise product does too. Let l be the number of times f vanishes in $\boldsymbol{D}$. Then it is plain, once more, that $l = m_1$, which is to say that f, like φ_1, vanishes m_1 times. (Less briefly: Let $|\xi| < 1$. Then by (2.2),

$$\partial(f,\xi) = \sum_{\varphi(x)=\xi} \partial(g_x,\xi), \tag{2.3}$$

while by the corollary to Lemma 7 (infra),

$$\begin{aligned}\partial(g,\varphi(x))(1+\partial(\varphi',x)) &= \partial(g(\varphi),x) = \partial(\theta,x)\\ &= \sum_{\sigma\in G}\partial(\varphi_1(\sigma),x) = \sum_{\sigma\in G}\partial(\varphi_1,\sigma(x))(1+\partial(\sigma',x))\\ &= \partial(\varphi_1,x)(1+\partial(\varphi',x)),\end{aligned}$$

that is, $\partial(g,\varphi(x)) = \partial(\varphi_1,x)$, hence by (2.3),

$$\partial(f,\xi) = \sum_{\varphi(x)=\xi}\partial(\varphi_1,x).$$

Then

$$l = \sum_{\xi\in \boldsymbol{D}}\partial(f,\xi) = \sum_{\xi\in \boldsymbol{D}}\sum_{\varphi(x)=\xi}\partial(\varphi_1,x) = \sum_{x\in X}\partial(\varphi_1,x) = m_1.)$$

Let $x\in X$. If the branch order of x is $m-1$, then

$$A(\varphi(x)) = \varphi_1(x)^m = f(\varphi(x)).$$

This means that

$$N(A-f) \geqq N \tag{2.4}$$

if the left side is the number of times $A-f$ vanishes in $\boldsymbol{D}$. The inequality (2.4), plus the hypothesis $N > m_1$, implies that $N(A-f) > N(f)$. Then by the lemma, $A = f$.

We may identify $\mathcal{O}(\boldsymbol{D})$ with $\{g(\varphi): g\in\mathcal{O}(\boldsymbol{D})\}$; then, in words used before, $\mathcal{O}(X)$ is an overring of $\mathcal{O}(\boldsymbol{D})$. It is to be proved that $\varphi_1\in\mathcal{O}(\boldsymbol{D})$. If $x\in X$, let $x_1,\ldots,x_m$ be the points where $\varphi = \varphi(x)$. Because there are m points in the list, it is understood that x is in the list $1+\partial(\varphi',x)$ times. Put $w_k = \varphi_1(x_k)$. Then we have proved that

$$\left(\frac{1}{m}\sum_{k=1}^{m} w_k\right)^m = \prod_{k=1}^{m} w_k. \tag{2.5}$$

If $m = 2$, the identity (2.5) implies that $w_1 = w_2$, which means $\varphi_1\in\mathcal{O}(\boldsymbol{D})$. If $m > 2$, the identity (2.5) does not imply that the w_k agree.

Let $-1 < t < 1$. Then $(t-\varphi_1)/(1-t\varphi_1)$, like φ_1, is a proper holomorphic map of X to $\boldsymbol{D}$; it vanishes m_1 times because φ_1, being holomorphic and proper, is of constant valence. Hence by "$A = f$",

$$\left(\frac{1}{m}\sum_{k=1}^{m}\frac{t-w_k}{1-tw_k}\right)^m = \prod_{k=1}^{m}\frac{t-w_k}{1-tw_k}. \tag{2.6}$$

Then (2.6) holds everywhere in t because both sides are rational. Let n be the number of w_k that equal w_1. Then the right side of (2.6) has a pole of order n at $1/w_1$, while the left side has a pole of order m there. Thus $n = m$, which means, once more, that $\varphi_1 \in \mathcal{O}(\boldsymbol{D})$.

We have proved that $\varphi_1 = g(\varphi)$ with g holomorphic in $\boldsymbol{D}$; $|g| < 1$ there because $\varphi(X) = \boldsymbol{D}$. Then g is proper.

(Proof. Let E, contained in $\boldsymbol{D}$, be bounded in $\boldsymbol{D}$. Put $F = \{g \in E\}$ and $G = \{\varphi \in F\}$. Then $G = \{\varphi_1 \in E\}$, hence G is bounded in X, hence $\varphi(G)$, i.e. F, is bounded in $\boldsymbol{D}$.)

Finally, if k is the number of times g vanishes in $\boldsymbol{D}$, then by (2.1), $m_1 = km$. (Less briefly: By (2.1),

$$km = \sum_{\xi\in \boldsymbol{D}} \partial(g,\xi) \sum_{\varphi(x)=\xi} (1+\partial(\varphi',x)) = \sum_{x\in X} \partial(g,\varphi(x))(1+\partial(\varphi',x)),$$

while by the corollary to Lemma 7 (infra),

$$\text{the last sum} = \sum_{x\in X} \partial(g(\varphi),x) = m_1.)$$

2.2. How good is the theorem, in other words, can one say more if the number of points of branch order $m-1$ is finite? I think it is fair to say no.

Let $m = 2$, let $N < \infty$, put

$$\Delta = \{\varphi(x) : x \text{ is a branch point of } \varphi\},$$

and let γ be "the" finite Blaschke product that vanishes to order one everywhere in Δ. The number of points in Δ, like the number in Δ_1 ((2.7) infra), is N. By part 3 (infra), $\gamma(\varphi) = \theta^2$ with $\theta \in \mathcal{O}(X)$. Then θ is proper, but it is not a $g(\varphi)$. By (3.6′), it vanishes N times.

Put

$$\Delta_1 = \{x \in X : x \text{ is a branch point of } \varphi\}. \tag{2.7}$$

Because $m = 2$, the number of points in Δ_1 is N. Let γ_1 be holomorphic in X, vanish to odd order infinitely often there, and vanish to odd order everywhere in Δ_1. (If we like, $\gamma_1 = B(\varphi)\theta$ with B a Blaschke product that vanishes to odd order infinitely often in $\boldsymbol{D}$, but vanishes nowhere in Δ.) Let (X_1, φ_1) be the Riemann surface of $\sqrt{\gamma_1}$, in other words, (X_1, φ_1) is a 2-sheeted covering of X with $\gamma_1(\varphi_1)$ a square in $\mathcal{O}(X_1)$. If $\varphi_2 = \varphi(\varphi_1)$, then the pair (X_1, φ_2) is a 4-sheeted covering of the disc $\boldsymbol{D}$ with N points of branch order 3, no points of branch order 2, and infinitely many points of branch order 1, while $\theta(\varphi_1)$ is a proper holomorphic map of X_1 to $\boldsymbol{D}$ that is not a $g(\varphi_2)$ because θ is not a $g(\varphi)$. Because θ vanishes N times, $\theta(\varphi_1)$ vanishes $2N$ times.

3. Proper holomorphic maps of X to itself.

We return to part 1. Accordingly, the pair (X, φ) is a two-sheeted covering of $\boldsymbol{D}$ and N is the number of branch points of φ. Let $\operatorname{Prop} X$ be the semigroup of proper holomorphic maps of X to itself. Then $\operatorname{Prop} X$ contains $\operatorname{Aut} X$. If $3 \leqq N < \infty$, then by Riemann-Hurwitz, $\operatorname{Prop} X = \operatorname{Aut} X$. (If $N = 2$, then by an ad hoc proof, $\operatorname{Prop} X = \operatorname{Aut} X$.) What if N is infinite? Then the inclusion can be proper, and it is this we study here. (If $N = 1$, $X \equiv \boldsymbol{D}$, hence the inclusion is proper.)

Let $N = \infty$. Then by putting $\varphi_1 = \varphi(f)$ if $f \in \operatorname{Prop} X$, we may use Theorem 1 to study $\operatorname{Prop} X$. E.g., we find that if $H^\infty(X)$ separates points in X, if $f \in \operatorname{Prop} X$, and if f fixes a point, then $f \in \operatorname{Aut} X$. (This is Theorem 4.)

Put

$$\Delta = \{\varphi(x) : x \text{ is a branch point of } \varphi\},$$

and let $T = \Delta'$ (the derived set of Δ). Then Δ is a discrete set in the disc $\boldsymbol{D}$, while T is a closed set in the circle $\partial\boldsymbol{D}$. Both are nonempty (Δ is infinite, but T may be finite).

Let $f \in \operatorname{Prop} X$; then by Theorem 1, $\varphi(f) = \hat{f}(\varphi)$ with $\hat{f} \in \operatorname{Prop} \boldsymbol{D}$. Which finite Blaschke products are $\hat{f}$'s? The test is this: Let g be a finite Blaschke product. Then the following imply one another:

(i) g is an $\hat{f}$.

(ii) Let $\xi \in \boldsymbol{D}$. Then $\xi \in \Delta$ iff $g(\xi) \in \Delta$ and g' vanishes to even order at ξ. (If $g'(\xi) \neq 0$, the order of vanishing is zero, which is even.)

We begin by proving the "$g = \hat{f}$ test". The proof is lengthy.

To φ corresponds a period 2 automorphism of X, called σ. The proof and precise statement is this: Let $x \in X$. Then the set where $\varphi = \varphi(x)$ consists of x plus one other point, say y. It is understood that $y = x$ if x is a branch point of φ. Put $\sigma(x) = y$. Then $\varphi(\sigma) = \varphi$ and $\sigma(\sigma) = \iota$. (Iota = the identity map.) To each point there is a disc in which σ is either the identity or minus the identity, which means σ is holomorphic. (The first alternative holds if the point is not a branch point, the second if it is.) Then $\sigma \in \operatorname{Aut} X$.

We have

$$\sigma(x) = x \quad \text{iff } x \text{ is a branch point of } \varphi. \tag{3.1}$$

LEMMA 2. *Let $f \in \operatorname{Prop} X$. Then $f(\sigma) = \sigma(f)$. In words, the proper maps of X to itself commute with the period 2 automorphism that corresponds to φ.*

PROOF. By Theorem 1,

$$\varphi(f(\sigma)) = \hat{f}(\varphi(\sigma)) = \hat{f}(\varphi) = \varphi(f),$$

hence either $f(\sigma) = \sigma(f)$, which is to be proved, or $f(\sigma) = f$.

Let $f(\sigma) = f$. Then $f = g(\varphi)$ with $g : \boldsymbol{D} \to X$. This gives $\hat{f} = \varphi(g)$ because $\hat{f}(\varphi) = \varphi(f)$, hence $N(\hat{f}')$ is infinite. $N(\hat{f}')$ is the number of times $\hat{f}'$ vanishes in $\boldsymbol{D}$, counting multiplicities.) But $N(\hat{f}') = k-1$ if k is the valence of f.

We identify, once more, $\mathcal{O}(\boldsymbol{D})$ with $\{g(\varphi) : g \in \mathcal{O}(\boldsymbol{D})\}$; then, in words used twice before, $\mathcal{O}(X)$ is an overring of $\mathcal{O}(\boldsymbol{D})$. The automorphism σ serves to tell who is in $\mathcal{O}(\boldsymbol{D})$. The test is this: let $f \in \mathcal{O}(X)$; then $f \in \mathcal{O}(\boldsymbol{D})$ iff $f(\sigma) = f$.

Let $\theta_1 \in \mathcal{O}(X)$ but $\notin \mathcal{O}(\boldsymbol{D})$, put $\theta_2 = \theta_1 - \theta_1(\sigma)$, and put $\gamma_1 = \theta_2^2$. Then γ_1 is holomorphic in $\boldsymbol{D}$. Because γ_1 does not vanish everywhere there, its order of vanishing, pointwise, is even or odd. The alternative is that the order of vanishing is everywhere infinite. Let $\gamma_2 \in \mathcal{O}(\boldsymbol{D})$ with $\partial(\gamma_2, z) = k$ if $\partial(\gamma_1, z) = 2k$ or $2k+1$. Then $\gamma_1 = \gamma_2^2 \gamma$ with γ holomorphic in $\boldsymbol{D}$ and all zeros of γ, if any, simple. Put

$$\theta = \theta_2 / \gamma_2; \tag{3.2}$$

then $\theta^2 = \gamma$, hence $\theta \in \mathcal{O}(X)$. By (3.2),

$$\theta(\sigma) = -\theta. \tag{3.3}$$

LEMMA 3. *The pair $(1, \theta)$ is a basis of the $\mathcal{O}(\boldsymbol{D})$-module $\mathcal{O}(X)$.*

PROOF. Let $g \in \mathcal{O}(X)$, and put

$$A = (g + g(\sigma))/2, \quad B = (g - g(\sigma))/2\theta.$$

Then $g = A + B\theta$. It is to be proved that B, which is meromorphic in $\boldsymbol{D}$, is, like A, holomorphic there. But $B^2\gamma$ is holomorphic, hence B is too because the zeros of γ are simple.

We have proved that the pair $(1, \theta)$ generates the $\mathcal{O}(\boldsymbol{D})$-module $\mathcal{O}(X)$. On the other hand if $A + B\theta = 0$, with A and B in $\mathcal{O}(\boldsymbol{D})$, then by (3.3), $A - B\theta = 0$, hence $A = B = 0$. This means the pair $(1, \theta)$ is independent over $\mathcal{O}(\boldsymbol{D})$.

COROLLARY 1. *θ separates the points of fibers of φ.*

PROOF. $\mathcal{O}(X)$ separates the points of X.

LEMMA 4. *The set where θ vanishes is the set of branch points of φ, in symbols,*

$$\{\theta = 0\} = \{x \in X : \sigma(x) = x\} \tag{3.4}$$

by (3.1). *Equivalently, Δ is the set where γ vanishes.*

PROOF. By (3.3), the left side of (3.4) contains the right side.

Let $\theta(x) = 0$. Then by (3.3) once more, $\theta(\sigma(x)) = 0$, hence by Corollary 1, $\sigma(x) = x$. This proves that the right side of (3.4) contains the left side. (Alternatively, let $x \in X$. Then because $\theta^2 = \gamma(\varphi)$,

$$2\partial(\theta, x) = \partial(\gamma, \varphi(x))(1 + \partial(\varphi', x)) \tag{3.5}$$

by the corollary to Lemma 7 (infra). Thus if $\theta(x)$ vanishes, then $\partial(\varphi', x)$ is odd, which means x is a branch point of φ.)

LEMMA 5. *Let g be a finite Blaschke product. If g is an $\hat{f}$, then $\gamma(g) = B^2\gamma$ with $B \in \mathcal{O}(\boldsymbol{D})$.*

PROOF. By Lemma 2, $\theta(f(\sigma)) = \theta(\sigma(f)) = -\theta(f)$, hence by Lemma 3, $\theta(f) = B\theta$. Then

$$\gamma(g(\varphi)) = (B^2\gamma)(\varphi)$$

because the left side $= \gamma(\varphi(f)) = \theta^2(f)$.

The converse is true too: g is an $\hat{f}$ if $\gamma(g) = B^2\gamma$. This is Lemma 6 (infra). Let $x \in X$. Then by (3.5),

$$\theta'(x) \neq 0 \quad \text{if } x \text{ is a branch point of } \varphi, \tag{3.6'}$$

while of course

$$\varphi'(x) \neq 0 \quad \text{if } x \text{ is not a branch point of } \varphi. \tag{3.6''}$$

LEMMA 6. *Let $g \in \operatorname{Prop} \boldsymbol{D}$, and let*

$$\gamma(g) = B^2\gamma \quad \text{with } B \in \mathcal{O}(\boldsymbol{D}). \tag{3.7}$$

Then $g(\varphi) = \varphi(f)$ with $f \in \operatorname{Prop} X$, that is, g is an $\hat{f}$.

PROOF. Let $x \in X$. Then, counting multiplicities, the set where $\varphi = g(\varphi(x))$ consists of two points, y_1 and y_2 say. By (3.7), $\theta(y_1) = B(x)\theta(x)$ or $\theta(y_1) = -B(x)\theta(x)$. If the first alternative holds, put $f(x) = y_1$, if the second holds, put $f(x) = y_2$. (If both hold, $y_1 = y_2$.) Then

$$\varphi(f) = g(\varphi) \quad \text{and} \quad \theta(f) = B\theta. \tag{3.8}$$

By (3.6) and (3.8), f is holomorphic because it is continuous, while by the first identity in (3.8), f is proper.

LEMMA 7. *Let f and g be formal power series with $f(0) = 0$ but $f \neq 0$. If ∂G is the order of vanishing of the formal power series G, then*

$$\partial(g(f)) = (\partial g)(1 + \partial f').$$

PROOF. Without loss of generality $g \neq 0$. If

$$g = g_l t^l + \ldots, \quad g_l \neq 0,$$

and

$$f = f_k s^k + \ldots, \quad f_k \neq 0, \quad k > 0,$$

then

$$g(f) = g_l(f_k s^k + \ldots)^l + \ldots = g_l f_k^l s^{kl} + \ldots$$

and

$$f' = k f_k s^{k-1} + \ldots,$$

hence

$$\partial(g(f)) = kl = (1 + k - 1)l = (1 + \partial f')\partial g.$$

COROLLARY. *Let X_1 and X_2 be Riemann surfaces, let f be holomorphic in X_1, with values in X_2, but not a constant, and let g be holomorphic in X_2. If $x \in X_1$, and if $\partial(F, w)$ is the order of vanishing of F at w, then*

$$\partial(g(f), x) = \partial(g, f(x))(1 + \partial(f', x)).$$

We now come to the proof of the "$g = \hat{f}$ test". Accordingly, g is a finite Blaschke product.

Let (i) hold. By Lemma 5,

$$\gamma(g) = B^2\gamma, \quad B \in \mathcal{O}(\boldsymbol{D}). \tag{3.9}$$

Then

$$\gamma'(g)g' = 2BB'\gamma + B^2\gamma'. \tag{3.10}$$

A. Let $\xi \in \Delta$. Then by (3.9) and Lemma 4, $g(\xi) \in \Delta$. Because $\partial(\gamma, \xi) = 1$,

$$\gamma(\xi + t) = \gamma_1 t + \ldots, \quad \gamma_1 \neq 0,$$

while

$$B(\xi + t) = B_p t^p + \ldots, \quad B_p \neq 0.$$

These give

$$2BB'\gamma + B^2\gamma' = (2p+1)B_p^2\gamma_1 t^{2p} + \ldots,$$

hence by (3.10),

$$2p = \partial(\gamma'(g)g', \xi) = \partial(\gamma'(g), \xi) + \partial(g', \xi) = \partial(g', \xi)$$

because $\gamma'(g(\xi)) \neq 0$. In words, g' vanishes to even order at ξ.

B. Let $g(\xi) \in \Delta$ and let $\partial(g', \xi) = 2l$. Then

$$\gamma'(g(\xi + t)) = A_0 + \ldots, \quad A_0 \neq 0,$$

and

$$g'(\xi + t) = G_{2l} t^{2l} + \ldots, \quad G_{2l} \neq 0,$$

while

$$B(\xi + t) = B_p t^p + \ldots, \quad B_p \neq 0,$$

and

$$\gamma(\xi + t) = \gamma_k t^k + \ldots, \quad \gamma_k \neq 0.$$

These plus (3.10) give

$$A_0 G_{2l} t^{2l} + \ldots = (2p + k)B_p^2 \gamma_k t^{2p+k-1} + \ldots .$$

If $k = 0$, that is, if $\xi \notin \Delta$, then by (3.9), $p > 0$, hence $2l = 2p-1$. This proves $\xi \in \Delta$.

We have proved that (i) gives (ii).

Let (ii) hold. Then $\gamma(g) = A\gamma$ with $A \in \mathcal{O}(\boldsymbol{D})$. It is to be proved that A is a square.

Let $A(\xi) = 0$. Then $g(\xi) \in \Delta$, hence

$$1+\partial(g', \xi) = \partial(\gamma, g(\xi))(1+\partial(g', \xi)) = \partial(\gamma(g), \xi) \tag{3.11}$$

by the corollary to Lemma 7.

A. Let $\xi \in \Delta$. Then $\partial(g', \xi) = 2l$. By (3.11),

$$1+2l = \partial(\gamma(g), \xi) = \partial(A, \xi)+\partial(\gamma, \xi) = \partial(A, \xi)+1$$

or $2l = \partial(A, \xi)$.

B. Let $\xi \notin \Delta$. Then $\partial(g', \xi) = 1+2l$. By (3.11) once more,

$$2+2l = \partial(\gamma(g), \xi) = \partial(A, \xi)+\partial(\gamma, \xi) = \partial(A, \xi).$$

We have proved that A vanishes to odd order nowhere in $\boldsymbol{D}$. This means A is a square, in other words, $\gamma(g) = B^2\gamma$ with $B \in \mathcal{O}(\boldsymbol{D})$. Thus by Lemma 6, (ii) gives (i).

Corollary 2. *Let g be a finite Blaschke product. If g is an $\hat{f}$, then T, the derived set of Δ, is completely g-invariant. In symbols,*

$$T = \{g \in T\}. \tag{3.12}$$

Proof. If k is the valence of g, then by the $g = \hat{f}$ test:

(3.13) there are at most $k-1$ points in $\{g \in \Delta\} \setminus \Delta$.

Let $g(\xi) \in T$ with $\xi \notin T$. Then there are infinitely many points in $\{g \in \Delta\} \setminus \Delta$, but this contradicts (3.13). This proves that the left side of (3.12) contains the right.

Let $\xi \in \Delta$. Then by the $g = \hat{f}$ test, $g(\xi) \in \Delta$. Thus T contains $g(T)$, in other words, the right side of (3.12) contains the left.

3.1. *Four theorems on* Prop X. Let $g \in \operatorname{Prop} \boldsymbol{D}$. This, i.e., g is a finite Blaschke product that is not a constant, is a standing hypothesis. Let $E \subset T$.

Then by $E \overset{g}{=} T$ we mean that g, if restricted to E, is a homeomorphism of E with T. This amounts to saying E is closed, g is univalent in E, and $g(E) = T$. Let k be the valence of g.

COROLLARY 3. *Let g be an $\hat{f}$, and let $T \neq \partial \boldsymbol{D}$. Then*

$$T = \bigcup_{l=1}^{k} T_l$$

with the T_l disjoint and with $T_l \overset{g}{=} T$. This means in part that T is the union of k disjoint copies of itself.

PROOF. Let $\xi \in \partial \boldsymbol{D} \setminus T$. Here we use $T \neq \partial \boldsymbol{D}$. The circle $\partial \boldsymbol{D}$ is the union of k nonoverlapping arcs whose endpoints, $z_1, \ldots, z_k$, satisfy $g(z) = \xi$. If A_l is the lth arc, let $T_l = A_l \cap T$. Then T_l is closed, g is univalent in T_l because, by Corollary 2, the endpoints of A_l are not in T, and, by Corollary 2 once more, $g(T_l) = T$. In brief, $T_l \overset{g}{=} T$, while the T_l, whose union is T, are disjoint because, once more, the endpoints of the A_l are not in T.

If l is a positive integer, let g_l be the lth iterate of g. In symbols,

$$g_1 = g,\ g_2 = g_1(g), \ldots, g_l = g_{l-1}(g).$$

If $l = 0$, $g = \iota$. Then g_l, like g, is a proper holomorphic map of $\boldsymbol{D}$, but its valence is k^l.

COROLLARY 4. *Let g be an $\hat{f}$, let $T \neq \partial \boldsymbol{D}$, and let l be a positive integer. Then T is the union of k^l disjoint copies of itself.*

PROOF. g_l is an $\hat{f}$.

Here is the first theorem on Prop X.

THEOREM 2. *Let $T \neq \partial \boldsymbol{D}$. If T is not the union of large numbers of disjoint copies of itself,* e.g., *if the number of components of T is finite, or if the number of isolated points of T is finite, then* Prop X = Aut X.

PROOF. Let $f \in$ Prop X. By Corollary 4, $\hat{f}$ is univalent, which means f is too.

Put $\theta'(t) = e^{it}g'(e^{it})/g(e^{it})$ if $-\infty < t < \infty$, and let $i\theta(0) = \log g(1)$. Then

$$g(e^{it}) = e^{i\theta(t)}. \tag{3.14}$$

If $\zeta_1, \ldots, \zeta_k$ are the zeros of g, i.e., if

$$g(z) = e^{i\tau} \prod_{m=1}^{k} \frac{\zeta_m - z}{1 - \bar{\zeta}_m z},$$

then

$$\theta'(t) = \sum_{m=1}^{k} (1-|\zeta_m|^2)/|1-\bar{\zeta}_m e^{it}|^2 \tag{3.15}$$

which means in part

$$\theta' > 0. \tag{3.16}$$

Put $\mu = \min \theta'$. Then $\mu > 0$. Let $|\xi| = 1$. Then there are k points, $t_1 < t_2 < \ldots < t_k$, in the interval $[0, 2\pi)$, that satisfy $g(e^{it}) = \xi$. Put $t_{k+1} = t_1 + 2\pi$. Because $\mu \leqq \theta'$,

$$(t_{l+1} - t_l)\mu < \theta(t_{l+1}) - \theta(t_l),$$

while by (3.14) and (3.16), the right side is 2π. Thus

$$t_{l+1} - t_l < 2\pi/\mu \quad \text{if } 1 \leqq l \leqq k. \tag{3.17}$$

LEMMA 8 (Fatou [1]). *Let $|\xi| = 1$. If g is not univalent and if g fixes the origin, then the union, over positive integers l, of the sets where $g_l = \xi$ is dense in $\partial \boldsymbol{D}$.*

PROOF. We have $g'_1 = g'$, $g'_2 = g'_1(g)g', \ldots, g'_l = g'_{l-1}(g)g'$. Thus in $\partial \boldsymbol{D}$, $|g'_l| \geqq \mu^l$ because $|g'(e^{it})| = \theta'(t)$. Then by (3.17), the circle $\partial \boldsymbol{D}$ is the union of k^l (nonoverlapping) arcs of length $< 2\pi/\mu^l$ whose endpoints, $z_1, \ldots, z_n$, satisfy $g_l(z) = \xi$, while by (3.15), $\mu > 1$ because $k \geqq 2$ and $g(0) = 0$. This proves that the union in the lemma is dense in the circle.

COROLLARY 5. *Let $g \in \operatorname{Prop} \boldsymbol{D}$ but $\notin \operatorname{Aut} \boldsymbol{D}$. Let $\xi \in \partial \boldsymbol{D}$. If g fixes a point in $\boldsymbol{D}$, then*

$$\bigcup_{l=1}^{\infty} \{g_l = \xi\}$$

is dense in $\partial \boldsymbol{D}$.

PROOF. The g in the corollary is conjugate to a g that fixes the origin, namely, $A(g(A))$ if A is the period 2 automorphism of $\boldsymbol{D}$ that takes the origin to the point fixed by g.

Here is our second theorem on Prop X.

THEOREM 3. *Let $T \neq \partial \boldsymbol{D}$. Let $f \in \operatorname{Prop} X$. If f fixes a point, then $f \in \operatorname{Aut} X$.*

PROOF. If f fixes x, then $\hat{f}$ fixes $\varphi(x)$. This implies, by Corollaries 2 and 5, that $\hat{f}$ is univalent. Then f is too.

3.1.1. Our standing hypothesis is that $g \in \operatorname{Prop} \boldsymbol{D}$. Let $g \notin \operatorname{Aut} \boldsymbol{D}$, and let g fix p, $p \in \boldsymbol{D}$. Let $\zeta \in \boldsymbol{D}$, and put

$$\Delta_\zeta = \bigcup_{l=0}^{\infty} \{g_l = \zeta\}.$$

LEMMA 9. *If $g(z)$ is conjugate to a power of z, let $\zeta \neq p$. Then Δ_ζ is infinite.*

PROOF. Let k be a positive integer. If $\zeta \neq p$, and if x is a point with $g_k(x) = \zeta$, then $x, g(x), g_2(x), \ldots, g_{k-1}(x), \zeta$ are distinct. E.g., if $g_3(x) = g_7(x) = g_4(g_3(x))$, then $g_3(x) = p$ because $g \notin \operatorname{Aut} \boldsymbol{D}$, hence $\zeta = g_{k-3}(p) = p$.

If $\zeta = p$, let y be such that $y \neq \zeta$ while $g(y) = \zeta$. (There is such a point because otherwise $g(z)$ would be conjugate to a power of z.) Let $g_k(x) = y$. Then $x, g(x), g_2(x), \ldots g_{k-1}(x)$, y are distinct.

LEMMA 10. *If $g(z)$ is conjugate to a power of z, let $\zeta \neq p$. Then Δ_ζ does not satisfy the Blaschke condition*, i.e.,

$$\sum_{z \in \Delta_\zeta} (1 - |z|) = \infty.$$

PROOF. If g' vanishes nowhere in Δ_ζ, let $\xi = \zeta$. Otherwise, let $x_1, \ldots, x_t$ be the points, in Δ_ζ, where g' vanishes, let $m(1), \ldots, m(t)$ be integers such that $g_{m(1)}(x_1) = \ldots = g_{m(t)}(x_t) = \zeta$, and let $\xi \in \Delta_\zeta$ with $\xi \neq g_l(x_s)$ if $0 \leqq l \leqq m(s)$ and $1 \leqq s \leqq t$. There is such a point ξ because, by Lemma 9, Δ_ζ is infinite. Then

(i) Δ_ζ contains Δ_ξ because $\xi \in \Delta_\zeta$;
(ii) g' vanishes nowhere in Δ_ξ.

To prove (ii), let $x_1 \in \Delta_\xi$. Then $\xi = g_l(x_1)$ say, which means $\xi = g_{l-m(1)}(\zeta)$ because $l > m(1)$. On the other hand, because $\xi \in \Delta_\zeta$, $\zeta = g_k(\xi)$ say, hence

$$\zeta = g_{k+l-m(1)}(\zeta) \quad \text{and} \quad \xi = g_{l-m(1)+k}(\xi).$$

But $\zeta \neq \xi$ while $l - m(1) + k > 0$. This proves (ii) because $g \notin \operatorname{Aut} \boldsymbol{D}$.

Let Δ_ξ satisfy the Blaschke condition. Then there is a Blaschke product B that vanishes to order one everywhere in Δ_ξ while vanishing to order zero elsewhere. Put $\theta = B(g)$. Then θ, like B, is a Blaschke product. Let $\theta(x) = 0$. Then $g(x) \in \Delta_\xi$, hence $x \in \Delta_\xi$, hence $\theta'(x) \neq 0$ because $\theta'(x) = B'(g(x))g'(x)$. In words, the zeros of θ are simple and each is a zero of B. Then $B/\theta \in \mathcal{O}(\boldsymbol{D})$, hence

$$|B/\theta| \leqq 1 \tag{3.18}$$

because $|B| < 1$. The inequality (3.18) means $|B| \leqq |B(g)|$, hence $|B| \leqq |B(g_l)|$.

Then $|B| \leqq |B(p)|$ because the iterates of g converge in $\boldsymbol{D}$ to the point fixed by g. This proves that Δ_ξ does not satisfy the Blaschke condition. Then by (i), neither does Δ_ζ.

COROLLARY 6. *If g is an $\hat{f}$, then Δ does not satisfy the Blaschke condition.*

PROOF. If $x \in \Delta$, then by the $g = \hat{f}$ test, each point of the sequence

$$g(x), g_2(x), \ldots, g_l(x), \ldots$$

is in Δ. This implies that $g_l(x) = p$ for some l because Δ is discrete in the disc while, once more, the iterates of g converge there to the point fixed by g. In other words,

$$p \in \Delta \quad \text{and} \quad \Delta \subset \Delta_p. \tag{3.19}$$

The previous lemma means in part this: to prove the corollary it is enough to prove there is a point ζ such that Δ contains Δ_ζ. There are two possibilities. The first is that g' vanishes to odd order nowhere in Δ_p, the second is g' vanishes to odd order somewhere in Δ_p.

(i) g' vanishes to odd order nowhere in Δ_p. Then $\Delta = \Delta_p$.

To prove this, let $x \in \Delta_p$. Let k be the first integer with $g_k(x) \in \Delta$. If $k \geqq 1$, then $g(g_{k-1}(x)) \in \Delta$. But g' vanishes to even order at the point $g_{k-1}(x)$ because this point, like x, is in Δ_p, hence by the "$g = \hat{f}$ test", $g_{k-1}(x) \in \Delta$. Thus $k = 0$, that is, $x \in \Delta$.

(ii) g' vanishes to odd order somewhere in Δ_p. Let $x_1, \ldots, x_t$ be the points, in Δ_p, where g' vanishes to odd order, let m be an integer such that $g_m(x_1) = \ldots = g_m(x_t) = p$, and let $\zeta \in \Delta$ with $\zeta \neq g_l(x_s)$ if $0 \leqq l \leqq m$ and $1 \leqq s \leqq t$. Then Δ contains Δ_ζ.

To prove this, we first prove $\zeta \neq g_l(x_s)$ if $l \geqq 0$ and $1 \leqq s \leqq t$. If $g_l(x_s) = \zeta$, then $l > m$, hence $\zeta = g_{l-m}(p) = p$. But $\zeta \neq p$.

Let $x \in \Delta_\zeta$. Then $\zeta = g_l(x)$ say. Once more, let k be the first integer with $g_k(x) \in \Delta$. If $k \geqq 1$, then by the "$g = \hat{f}$ test", g' vanishes to odd order at $g_{k-1}(x)$. Then $g_{k-1}(x) = x_1$ say, hence $\zeta = g_{l-k+1}(x_1)$ because $l \geqq k-1$. This proves $k = 0$.

One has, in the symbols of the proof of the corollary,

$$\Delta = \Delta_p \setminus \left(\bigcup_{s=1}^{t} \Delta_{x_s} \right). \tag{3.20}$$

The identity means in part that g determines Δ if g is an $\hat{f}$. How can one exploit this, or for that matter, how can one exploit (3.20)?

We may paraphrase the corollary in this way.

THEOREM 4. *Let Δ satisfy the Blaschke condition. Equivalently, let $H^\infty(X)$ separate points in X. Let $f \in \operatorname{Prop} X$. If f fixes a point, then $f \in \operatorname{Aut} X$.*

PROOF. By the hypothesis on f, $\hat{f}$ fixes a point in $\boldsymbol{D}$, hence by the hypothesis on Δ, f is univalent.

Here is the fourth theorem on $\operatorname{Prop} X$. We might have proved this at the outset.

THEOREM 5. *Let $f \in \operatorname{Prop} X$ but $\notin \operatorname{Aut} X$. Then f fixes at most one point. If f fixes a point, then the point is a branch point. (Neither of these need hold if $f \in \operatorname{Aut} X$.)*

PROOF. Let f fix x and put $p = \varphi(x)$. Then $\hat{f}$ fixes p, hence by (3.19), $p \in \Delta$ because $\hat{f} \notin \operatorname{Aut} \boldsymbol{D}$. This proves the second assertion of the theorem. For the first assertion, let f fix y and put $q = \varphi(y)$. Then $\hat{f}$ fixes q as well as p, hence $q = p$ because otherwise $\hat{f}(z)$ would be z everywhere, but f is neither ι nor σ. That is, $\varphi(y) = \varphi(x)$, hence by the second assertion, $y = x$.

3.2. The theorems are pretty good. What we mean is this:

A. There is an X, whose N is infinite, that has a proper holomorphic map of itself of valence 2 that fixes a point.

B. There is an X, whose N is infinite, whose T is not the circle, and whose Δ satisfies the Blaschke condition, that has a proper holomorphic map of itself of valence 2.

C. There is an X, whose T is the circle, and whose Δ does not satisfy the Blaschke condition, that has a proper holomorphic map of itself of valence 2 that fixes no point.

All three have a period 2 automorphism that fixes two points, neither point being a branch point, and a period 2 automorphism that fixes no point.

We will omit the proofs of the first two. Here is the proof of the third. We will work not in the disc $\boldsymbol{D}$, but in the right half plane $\boldsymbol{H}$. This is all right, because if (X, φ) is a two-sheeted covering of $\boldsymbol{H}$, and if φ_1 is the Cayley transform, $(\varphi-1)/(\varphi+1)$, of φ, then (X, φ_1) is a two-sheeted covering of $\boldsymbol{D}$, and vice versa. Let

$$g(z) = z + \frac{1}{z}.$$

Then g maps $\boldsymbol{H}$ bivalently onto itself.

LEMMA 11 (Fatou [1]). *There is a θ that is holomorphic in $\boldsymbol{H}$, that is not a constant, and that is g-invariant. In symbols,*

$$\theta \in \mathcal{O}(\boldsymbol{H}),\ \theta \notin \mathrm{C}, \quad \textit{and} \quad \theta(g) = \theta.$$

PROOF. The key to θ is this: iterate the square of g. We begin though by iterating g. Let

$$z_0 = z,\ z_{l+1} = z_l + \frac{1}{z_l},\ A_l = \frac{1}{z_l \bar{z}_l}, \quad \text{and} \quad x_l + iy_l = z_l.$$

Then

$$x_{l+1} = x_l(1+A_l), \tag{3.21}$$

hence $x_l \uparrow s$. It is understood that $x_0 > 0$. If $s < \infty$, then by (3.21) once more, $A_l \to 0$, hence $y_l \to t$ with $-\infty < t < \infty$ because

$$y_{l+1} = y_l(1-A_l). \tag{3.21A}$$

Then $A_l \to |s+it|^{-2}$, but this is not 0. This proves $s = \infty$. Thus

$$\sum_{k=0}^{\infty} A_k = \infty \tag{3.22}$$

because

$$x_{l+1} = x_0 \prod_{k=0}^{l} (1+A_k),$$

while

$$A_k \to 0 \tag{3.23}$$

because $A_k \leqq 1/x_k^2$. By (3.22) and (3.23), $y_l \to 0$ because

$$y_{l+1} = y_0 \prod_{k=0}^{l} (1-A_k).$$

Then

$$y_l/x_l \to 0. \tag{3.24}$$

By (3.21) and (3.21A),

$$|y_{l+1}/x_{l+1}| < |y_l/x_l|. \tag{3.25}$$

Let $f(w) = w+1/w+2$. Then f maps the slit plane $\{z^2 : z \in \boldsymbol{H}\}$ bivalently onto itself. One has $g_l(z)^2 = f_l(z^2)$, but we do not use this if $l \geqq 2$. Anyway, passing to f and its iterates amounts to squaring the iterates of g.

Let

$$w_0 = w,\ w_{l+1} = w_l + \frac{1}{w_l} + 2,\ B_l = \frac{1}{w_l \bar{w}_l}, \quad \text{and} \quad u_l + iv_l = w_l.$$

Then, like y_{l+1},

$$v_{l+1} = v_0 \prod_{k=0}^{l} (1-B_k), \tag{3.26}$$

while, unlike x_{l+1},

$$u_{l+1} = u_l(1+B_l)+2. \tag{3.27}$$

Let $u_0 > 0$. Then by (3.27), $u_l > 2l$, hence

$$B_l < 1/4l^2. \tag{3.28}$$

By (3.26) and (3.28), the harmonic functions v_l converge uniformly in the half plane $u > 0$, to V say. Then

$$V = v_1 \prod_{k=1}^{\infty} (1-B_k).$$

The infinite product is positive, hence the sign of V changes with that of v_1. This proves V is not a constant. We have $v_{l+1} = v_l(f)$ because $w_{l+1} = w_l(f)$, hence $V = V(f)$. Thus if F is holomorphic in $u > 0$ with $\operatorname{im} F = V$ there, then $F(f) = F+\gamma$ with $\gamma \in \mathbb{R}$. In other symbols,

$$F(w+1/w+2) = F(w)+\gamma.$$

Let $\boldsymbol{H}^{1/2}$ be the sector of opening $\pi/2$, i.e.,

$$\boldsymbol{H}^{1/2} = \{x+iy : |y| < x\},$$

and put $G(z) = F(z^2)$ if $z \in \boldsymbol{H}^{1/2}$. By (3.25), $z+1/z$ is in $\boldsymbol{H}^{1/2}$ if z is. Then

$$G(z+1/z) = F(z^2+1/z^2+2) = F(z^2)+\gamma = G(z)+\gamma.$$

Thus in $\boldsymbol{H}^{1/2}$,

$$G(g_l) = G + l\gamma \quad \text{if} \quad l = 0, 1, 2, \ldots . \tag{3.29}$$

We may combine (3.29) with (3.24) to continue G to $\boldsymbol{H}$. Put $X_l = \{g_l \in \boldsymbol{H}^{1/2}\}$. Then in $X_k \cap X_l$, $G(g_k) - k\gamma = G(g_l) - l\gamma$, while by (3.24), $\boldsymbol{H}$ is the union of the X_l. This continues G to $\boldsymbol{H}$ if in X_l we let $G = G(g_l) - l\gamma$. Then $G \in \mathcal{O}(\boldsymbol{H})$ with $G(g) = G + \gamma$.

We now come to θ. If $\gamma = 0$, put $\theta = G$, while if $\gamma \neq 0$, put $\theta = e^{2\pi i G/\gamma}$. Then $\theta \in \mathcal{O}(\boldsymbol{H})$ with $\theta(g) = \theta$. Finally, θ is not a constant since V is not.

Fatou's proof, which is of greater subtlety, gives more. Namely, the value of γ, which is 2.

To ξ in $\boldsymbol{H}$, or for that matter in $\boldsymbol{P}$, corresponds its g-orbit Δ_ξ. This is to say,

$$\Delta_\xi = \bigcup_{k,l=0}^{\infty} \{g_k = g_l(\xi)\}.$$

(The Δ_ξ in 3.1.1 is just a piece of the Δ_ξ here.) Then

$$\Delta_\xi = \{g \in \Delta_\xi\}, \tag{3.30}$$

in words used before, Δ_ξ is completely g-invariant. It is the least set containing ξ that is.

COROLLARY 7. *Let $\xi \in \boldsymbol{H}$. Then Δ_ξ is discrete in $\boldsymbol{H}$.*

PROOF. The set where $\theta = \theta(\xi)$, which is discrete in $\boldsymbol{H}$, contains Δ_ξ.

LEMMA 12. *Δ_ξ does not satisfy the Blaschke condition.*

PROOF. By (3.22),

$$\sum_{k=0}^{\infty} \frac{4x_k}{|z_k + 1|^2} = \infty$$

because $x_k \geqq 1$ if k is large.

LEMMA 13 (Fatou [1], Julia [2]). *Δ_ξ is dense in the imaginary line.*

PROOF (in brief). Let $z_0 \in i\mathbb{R} \setminus \Delta_\infty$, and let D be an open disc of center z_0. By (3.21A), $A_l \nrightarrow 0$, hence by Montel,

$$\bigcup_{l=0}^{\infty} g_l(D)$$

meets Δ_ξ. Then D does too.

We now come to the X in the statement C. Let $\xi \in \boldsymbol{H}$ with $Z \notin \Delta_\xi$. Then

$$g' \text{ vanishes nowhere in } \Delta_\xi. \tag{3.31}$$

By Corollary 7, there is a γ in $\mathcal{O}(\boldsymbol{H})$ that vanishes to order one everywhere in Δ_ξ while vanishing to order zero elsewhere. Let X be the Riemann surface of $\sqrt{\gamma}$. Then X is a two-sheeted covering, of $\boldsymbol{H}$, whose Δ is Δ_ξ. By (3.30) and (3.31), there is an f in $\operatorname{Prop} X$ with $\hat{f} = g$. Then f is of valence 2 and fixes no point. Put $G(z) = 1/z$. Then $g(G) = g$, hence Δ is completely G-invariant, hence by the "$g = \hat{f}$ test" once more, there is an F in $\operatorname{Prop} X$ with $\hat{F} = G$. Then F and $F(\sigma)$ are period 2 automorphisms. One of these fixes two points, neither point being a branch point, the other fixes no point. Finally, Δ does not satisfy the Blaschke condition, this is Lemma 12, while in the disc $\boldsymbol{D}$, $T = \partial \boldsymbol{D}$ by Lemma 13.

3.3. *Three problems.* We state each in terms of the two-sheeted covering X, but by Theorem 1, plus the "$g = \hat{f}$ test", all are problems on finite Blaschke products.

Problem 1. Can we have $f, g \in \operatorname{Prop} X$ but $\notin \operatorname{Aut} X$ with g fixing a point and f fixing no point?

Problem 2. Let χ be the number of points fixed by the proper maps that are not automorphisms, i.e., χ is the cardinality of

$$\{x \in X : \exists f \in \operatorname{Prop} X \text{ but } \notin \operatorname{Aut} X \text{ with } f(x) = x\}.$$

By Theorem 5, χ is at most countably infinite. What are the possible values of χ? One can have $\chi \geqq 2$, but we do not know, for example, if 1, 2, or ω is possible.

The underlying problem is one of size. Namely, how large can the semigroup $\operatorname{Prop} X \setminus \operatorname{Aut} X$ be? I think the answer is "not very". This would mean that in Problem 1 there is no such pair (f, g), while in Problem 2, χ is finite.

Let f and g be holomorphic maps, of $\boldsymbol{D}$ to itself, such that g is proper while f fixes no point. Suppose that to each positive integer l there is a positive integer m such that the lth iterate of f followed by the mth iterate of g fixes the origin, in symbols, $g_m(f_l(0)) = 0$. Is this possible? It seems unlikely, especially if g fixes the origin and f is proper. This, i.e., "not possible if $g(0) = 0$ and $f \in \operatorname{Prop} \boldsymbol{D}$", would imply that in the first problem the answer is no.

This is a good place to prove that $\operatorname{Prop} X$ is countable, which means that $\operatorname{Prop} X$ is countably infinite if it is not equal to $\operatorname{Aut} X$. In other words, in terms of cardinality, Prop X is as small as it can be provided it is not equal to Aut X.

Let $g \in \operatorname{Prop} \boldsymbol{D}$ and put $k =$ the valence of g. If $\xi \in \boldsymbol{D}$, let $(g = \xi)$ be the unordered k-tuple of points where $g = \xi$. Thus if the point x is such that $g(x) = \xi$ with multiplicity l, then x is listed l times in the tuple $(g = \xi)$.

LEMMA 14. *Let $f, f_1 \in \operatorname{Prop} X$ and let $\xi \in \boldsymbol{D}$. If the tuples $(\hat{f} = \xi)$ and $(\hat{f}_1 = \xi)$ are equal, then*

$$\frac{\xi - \hat{f}}{1 - \bar{\xi}\hat{f}} = \mu \frac{\xi - \hat{f}_1}{1 - \bar{\xi}\hat{f}_1}$$

where μ is a root of unity.

PROOF. Let A be the period 2 automorphism of $\boldsymbol{D}$ that takes the origin to the point ξ, in symbols, $A(w) = (\xi - w)/(1 - \bar{\xi}w)$. The hypothesis implies that $A(\hat{f}) = \mu A(\hat{f}_1)$ where $|\mu| = 1$. It is to be proved that μ is a root of unity.

Put $g = A(\hat{f}(\mathrm{A}))$, $g_1 = A(\hat{f}_1(A))$, $\varphi_1 = A(\varphi)$, and $\Delta_1 = A(\Delta)$. Then

$$g = \mu g_1, \tag{3.32}$$

while $g(\varphi_1) = A(\hat{f}(\varphi)) = A(\varphi(f)) = \varphi_1(f)$, which means that g is an $\hat{f}$. Likewise, g_1 is an $\hat{f}$. By the "$g = \hat{f}$ test", there is at most a finite number of points ζ, in Δ_1, such that Δ_1 does not meet $\{g_1 = \zeta\}$. Call these points, if any, $\zeta_1, \ldots, \zeta_t$. Let $\theta \in \Delta_1$ with $|\theta| \neq |\zeta_k|$ if $1 \leqq k \leqq t$. Because θ is not a ζ_k, there is a y in Δ_1 with $g_1(y) = \theta$, but then by (3.32), $\mu\theta = g(y)$, hence $\mu\theta \in \Delta_1$. Then $\mu^2\theta \in \Delta_1$, etc. This proves, if $\theta \neq 0$, that μ is a root of unity because Δ_1 is discrete in the disc.

THEOREM 6. Prop X *is countable.*

PROOF. If $\xi \in \Delta$, let Δ^ξ be the set of those unordered tuples, $(\hat{f} = \xi)$, whose components belong to Δ. Then Δ^ξ is countable because Δ is, in symbols,

$$\Delta^\xi = \{(\hat{f}_1 = \xi), (\hat{f}_2 = \xi), \ldots\}. \tag{3.33}$$

Let A, once more, be the period 2 automorphism of $\boldsymbol{D}$ that takes the origin to the point ξ.

Let $f \in \operatorname{Prop} X$. By the "$g = \hat{f}$ test", there is a ξ in Δ with $(\hat{f} = \xi)$ in Δ^ξ, but then the tuple $(\hat{f} = \xi)$ is equal to one of the tuples in the right side of (3.33), to $(\hat{f}_1 = \xi)$ say. By the lemma, $\hat{f} = A(\mu A(\hat{f}_1))$ where μ is a root of unity.

PROBLEM 3. If $0 \leqq t < 1$, let $\mu(t)$ be the number of points in Δ bounded by t.

Then

$$\int_0^1 \mu(t)dt = \sum_{z \in \Delta} (1-|z|).$$

Because of Theorem 4, we ask if

$$\operatorname{Prop} X = \operatorname{Aut} X \tag{3.34}$$

if μ is not too large? E.g., is there a p, $1 < p < \infty$, such that (3.34) holds if

$$\int_0^1 \mu(t)^p dt < \infty\,?$$

Failing this, what if

$$\int_0^1 2^{\mu(t)} dt < \infty\,?$$

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