MATH. SCAND. 61 (1987), 68–92

PARACOMMUTATORS OF SCHATTEN – VON NEUMANN CLASS S_p, $0 < p < 1$

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1. Introduction.

In their paper [3], Janson and Peetre consider the paracommutator defined by

$$(1) \qquad (T_b^{st}f)\hat{}(\xi) = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t \hat{f}(\eta)d\eta$$

and obtain a series of results on L^2-boundedness and S_p-estimates for $1 \leqq p \leqq \infty$. In this paper we study corresponding S_p-estimates for $0 < p < 1$. For the notion of the Schatten – von Neumann class S_p, see McCarthy [4]. In the case $0 < p < 1$, S_p is not a Banach space, only a quasi-Banach space. For it,

$$(2) \qquad \|T_1 + T_2\|_{S_p}^p \leqq \|T_1\|_{S_p}^p + \|T_2\|_{S_p}^p$$

holds. We shall repeatedly use this fact.

We shall give the assumptions on $A(\xi,\eta)$ in terms of $V_p(E \times F)$ defined below instead of $M(E \times F)$ in [3].

Definition 1. If $E, F \subset \mathbf{R}^d$, then we define

$$(3) \qquad \begin{aligned} &V_p(E \times F) = \{K(\xi,\eta): K(\xi,\eta) = \textstyle\sum \lambda_i f_i(\xi) g_i(\eta),\ f_i, g_i \text{ measurable},\\ &|f_i(\xi)| \leqq 1 \text{ for } \xi \in E,\ |g_i(\eta)| \leqq 1 \text{ for } \eta \in F,\ \textstyle\sum |\lambda_i|^p < \infty\} \end{aligned}$$

and

$$\|K\|_{V_p(E\times F)} = \inf(\textstyle\sum|\lambda_i|^p)^{1/p}$$

the infimum being taken over all such decompositions in (3).

For $0 < p \leqq 1$, $V_p(E \times F)$ is well defined, because $\sum|\lambda_i| \leqq (\sum|\lambda_i|^p)^{1/p}$, so $\|K\|_{V_p(E\times F)} < \infty$ implies that the series $\sum \lambda_i f_i(\xi) g_i(\eta)$ converges absolutely and uniformly.

Received May 14, 1986.

REMARK 1. In fact, we may assume that $|f_i(\xi)| \leqq 1$ and $|g_i(\eta)| \leqq 1$ hold only almost everywhere (a.e.) on E and F instead of $|f_i(\xi)| \leqq 1$ and $|g_i(\eta)| \leqq 1$ on E and F in Definition 1. The results of Theorems 1–3 below still hold, provided $\varepsilon \to 0$ is along a sequence, and $\varepsilon \boldsymbol{m}$ are replaced by some points $\eta^\varepsilon_{\boldsymbol{m}} \in Q^\varepsilon_{\boldsymbol{m}}$. But for the sake of simplicity, we prefer Definition 1 in the above form.

It is easy to see that for $0 < p_1 \leqq p_2 \leqq 1$, $V_{p_1} \subset V_{p_2} \subset V_1 \subset M$, where V_1 is the tensor product $L^\infty(E) \hat{\otimes} L^\infty(F)$ and $M(E \times F)$ is the space of Schur multipliers, see Janson and Peetre [3].

Similarly, corresponding to Lemma 3.1 of [3], for $V_p(E \times F)$ we have

PROPOSITION 1. *If* $\varphi(\xi, \eta) \in V_p(E \times F)$ *and* $K(\xi, \eta) \in S_p(E \times F)$. *Then* $\varphi K \in S_p(E \times F)$ *and*

$$\|\varphi K\|_{S_p(E \times F)} \leqq \|\varphi\|_{V_p(E \times F)} \|K\|_{S_p(E \times F)}, \quad 0 < p \leqq 1. \tag{4}$$

PROOF. For any $\varepsilon > 0$, let $\varphi(\xi, \eta) = \sum \lambda_i f(\xi) g_i(\eta)$, where

$$|f_i(\xi)| \leqq 1, \quad |g_i(\eta)| \leqq 1, \quad \sum |\lambda_i|^p \leqq (\|\varphi\|_{V_p(E \times F)} + \varepsilon)^p.$$

Then we have

$$\begin{aligned} \|\varphi K\|^p_{S_p(E \times F)} &\geqq \sum |\lambda_i|^p \|f_i(\xi) K(\xi, \eta) g_i(\eta)\|^p_{S_p(E \times F)} \\ &\leqq \sum |\lambda_i|^p \|K\|^p_{S_p(E \times F)} \\ &\leqq (\|\varphi\|_{V_p(E \times F)} + \varepsilon)^p \|K\|^p_{S_p(E \times F)}. \end{aligned}$$

So (4) holds.

REMARK 2. $V_p(E \times F)$ is a quasi-Banach algebra but not a Banach algebra, as $S_p(E \times F)$ is a quasi-Banach space but not a Banach space, for $0 < p < 1$. $\| \cdot \|^p_{V_p}$ induces a metric as $\| \cdot \|^p_{S_p}$ does. So the results analogous to Lemmas 3.3, 3.4, and 3.11 in [3] do not hold for V_p when $0 < p < 1$.

As in [3], let Δ_k denote the set $\{\xi \in \mathbf{R}^d : 2^k \leqq |\xi| \leqq 2^{k+1}\}$ and $\tilde{\Delta}_k = \Delta_{k-1} \cup \Delta_k \cup \Delta_{k+1}$. Now we list some assumptions on A which will be used in the theorems below.

A0: There exists an $r > 1$ such that $A(r\xi, r\eta) = A(\xi, \eta)$.

A_p1: $\|A\|_{V_p(\Delta_j \times \Delta_k)} \leqq C$, for all $j, k \in \mathbf{Z}$.

$\mathrm{A}_p3(\alpha)$: There exist $\alpha > 0$ and $0 < \delta < \frac{1}{2}$ such that

$$\|A\|_{V_p(B \times B)} \leqq C(r/|\xi_0|)^\alpha,$$

for every ball $B = B(\xi_0, r)$ with the centre ξ_0 and radius $r < \delta|\xi_0|$.

$A_p 4\frac{1}{2}$: For every $\xi_0 \neq 0$, there exist $\eta_0 \in \mathbb{R}^d$ and $\delta > 0$ such that, with $B_0 = B(\xi_0 + \eta_0, \delta|\xi_0|)$ and $D_0 = B(\eta, \delta|\xi_0|)$, $A(\xi,\eta)^{-1} \in V_p(B_0 \times D_0)$.

$A_p 9(\alpha_0)$: $A(\xi,\eta)$ satisfies $A_p 1$ and $A_p 3(\alpha)$. Furthermore, for every $\varepsilon > 0$ small enough, let $\{Q^\varepsilon_{\boldsymbol{m}}\}_{\boldsymbol{m} \in Z^d}$ be a family of disjoint cubes with centres $\varepsilon \boldsymbol{m}$ and sides ε, let $\tilde{Q}^\varepsilon_{\boldsymbol{m}} = 3Q^\varepsilon_{\boldsymbol{m}}$ and let

$$A_\varepsilon(\xi,\eta) = \sum_{\substack{\boldsymbol{m} \in Z^d \\ 0 \notin Q^\varepsilon_{\boldsymbol{m}}}} A(\xi, \varepsilon \boldsymbol{m}) \chi_{Q^\varepsilon_{\boldsymbol{m}}}(\eta), \quad K_\varepsilon(\xi,\eta) = A(\xi,\eta) - A_\varepsilon(\xi,\eta).$$

Then

$$\|K_\varepsilon\|_{V_p(\Delta_l \times \Delta_k)} \leqq C(\varepsilon/2^k)^{\alpha_0}, \qquad \text{for every } l \in Z,\ k > \log_2 \varepsilon,$$

and

$$\|K_\varepsilon\|_{V_p(B \times B)} \leqq C(\varepsilon/|\xi_0|)^{\alpha_0}(r/|\xi_0|)^{\alpha - \alpha_0}, \quad \text{for every } B = B(\xi_0, r)$$

with $\varepsilon < r < \delta|\xi_0|$.

$A10(\alpha)$: For any $0 \neq \theta \in \mathbb{R}^d$, there exist a positive number $\delta < \frac{1}{2}$ and a subset V_θ of $\mathbb{R}^d$ such that if N_r denotes the number of integer points contained in $V_\theta \cap B_r$, where $B_r = B(0,r)$, then

$$\overline{\lim_{r \to \infty}} \frac{N_r}{r^d} > 0,$$

and for every $\boldsymbol{n} \in V_\theta$,

$$\left\| \frac{1}{A(\cdot + \boldsymbol{n} + \theta, \cdot + \boldsymbol{n})} \right\|_{M(B \times B)} \leqq C|\boldsymbol{n}|^\alpha, \quad \text{where } B = B(0,\delta).$$

REMARK 3. The assumption A0 is about the homogeneity of A. The assumption $A_p 1$ is about the boundedness of A just like A1 in [3]. $A_p 1$ implies A1 in [3] and hence it implies that $A \in L^\infty(\mathbb{R}^{2d})$. The assumption $A_p 3$ is about the order of the zero at the diagonal $\{\xi = \eta\}$ of A just like A3 in [3]. The assumption $A_p 4\frac{1}{2}$ is about non-degeneracy of A just like $A4\frac{1}{2}$ in [7]. It is stronger than A4 in [3] but weaker than the one in Timotin [10] and [11]; for example, the kernel $A(\xi,\eta)$ of commutator, see Example 2 below, satisfies $A_p 4\frac{1}{2}$ but $A \notin C^\infty(\mathbb{R}^{2d} \setminus \{0\})$. The assumption $A_p 9$ is about the smoothness of A on all of $\mathbb{R}^{2d}$. It is not necessary for the S_p-estimates if $1 \leqq p \leqq \infty$, but when $0 < p < 1$, we need an assumption such as $A_p 9$. The assumption $A10(\alpha)$ again is about the order of the zero at the diagonal $\{\xi = \eta\}$ of A. $A_p 3(\alpha)$ says that the order is $\geqq \alpha$, $A10(\alpha)$ says that the order is $\geqq \alpha$. $A10(\alpha)$ will be used to characterize the "Janson-Wolff phenomenon". It should be noticed that in the assumption A10(α), we use $M(B \times B)$ regardless of p.

Sometimes we write $T_b^{st}(A)$ to emphasize the kernel A.

The main results of this paper are the following four theorems.

THEOREM 1. *Suppose that* $0 < p < 1$, $s,t > -d/2$, $\alpha > s+t+d/p$ *and suppose further that* $A(\xi, \eta)$ *satisfies* A_p1 *and* $A_p3(\alpha)$. *Then* $b \in B_p^{s+t+d/p}$ *implies that* $T_b^{st} \in S_p$ *and*

$$(5) \qquad \|T_b^{st}\|_{S_p} \leqq C\|b\|_{B_p^{s+t+d/p}}.$$

THEOREM 2. *Suppose that* $0 < p < 1$, $s,t > -d/2$ *and suppose further that* $A(\xi, \eta)$ *satisfies* A0, A_p1 *and* $A_p4\frac{1}{2}$. *Then the a priori inequality*

$$(6) \qquad \|b\|_{B_p^{s+t+d/p}} \leqq C\|T_b^{st}\|_{S_p}$$

holds for every $b \in B_p^{s+t+d/p}$.

THEOREM 3. *Suppose that* $A(\xi, \eta)$ *satisfies* A0, A_p1, $A_p3(\alpha)$, $A_p4\frac{1}{2}$, $A_p9(\alpha_0)$ *and suppose further that* $\alpha > \alpha_0 > 0$, $0 < p < 1$, $s,t > -d/2$ *and* $s+t+d/p < \alpha$. *Let*

$$A_\varepsilon^\theta(\xi, \eta) = \sum_{\substack{m \in Z^d \\ 0 \notin \tilde{Q}_m^\varepsilon}} A(\xi, \varepsilon(m+\theta))\chi_{Q_{m+\theta}^\varepsilon}(\eta).$$

Then for $b \in S'(\mathrm{R}^d)$ *with* $\hat{b}$ *with compact support* $\subset \mathrm{R}^d \setminus \{0\}$, $T_b^{st} \in S_p$ *and* $T_b^{st}(A_\varepsilon^\theta) \in S_p$ *uniformly in* $\varepsilon \leqq \varepsilon_0$ *and* $|\theta| \leqq \sqrt{d}/3$ *imply that* $b \in B_p^{s+t+d/p}$ *and that* (6) *holds.*

THEOREM 4. *Suppose that* $A(\xi, \eta)$ *satisfies* A10(α) *and suppose further that* $0 < p \leqq d/(\alpha - s - t)$, $b \in S'(\mathrm{R}^d)$ *with* $\hat{b}$ *with compact support* $\subset \mathrm{R}^d$ *such that* $T_b^{st} \in S_p$. *Then* b *must be a polynomial.*

REMARK 4. The results of Theorems 3 and 4 are not as good as one would like. This is mainly because the analogue of Lemma 3.3 in [3] is false for V_p, when $p < 1$, so the restriction that $\hat{b}$ has compact support $\subset \mathrm{R}^d \setminus \{0\}$ or R^d cannot easily be removed. But from the proof of Theorem 1, see section 4 below, we see that under the hypotheses of Theorem 1, with $b_N = \sum_{-N}^{N} b * \psi_k$, we have $T_{b_N}^{st} \in S_p$ and $T_{b_N}^{st} \to T_b^{st}$ in the norm S_p. Let us define $T_b^{st} \in S_p$ in this way for $b \in S'(\mathrm{R}^d)$, of course, this is different from the natural definition of $T_b^{st} \in S_p$, and let us denote $T_b^{st} \in S_p$ strongly. Then, using Corollary 2 in section 4, we obtain formally good-looking results as follows.

COROLLARY 1. *Suppose that* $A(\xi,\eta)$ *satisfies* A0, A_p1, $A_p3(\alpha)$, $A_p4\frac{1}{2}$, $A_p9(\alpha_0)$ *and* $A10(\alpha)$ *and suppose further that* $\alpha \geqq \alpha_0 > 0$, $0 < p < 1$, $s,t > -d/2$, $\alpha > s+t$. *Then*

1) *if* $p > d/(\alpha - s - t)$, $T_b^{st} \in S_p$ *strongly and* $T_{b_N}^{st}(A_\varepsilon^\theta) \in S_p$ *strongly and uniformly in* $\varepsilon \leqq \varepsilon_0$ *and* $|\theta| \leqq \sqrt{d}/3$ *if and only if* $b \in B_p^{s+t+d/p}$,

2) *if* $p \leqq d/(\alpha - s - t)$, $T_b^{st} \in S_p$ *strongly if and only if b is a polynomial.*

These theorems and corollary look somewhat complicated, but they cover at least paraproducts, higher order commutators of singular integral operators and some pseudo-differential operators, which are the cases of main interest. For Hankel operators, or equivalently for the one-dimensional commutators $[b, H]$, Peller [6] and Semmes [9] have obtained S_p-estimates, for $0 < p < 1$. So our results are generalizations of their results. In fact, our methods for proving Theorem 2 are close to those of Peller [6] and Semmes [9]. Their results can be obtained from Theorems 1 and 2. More generally, we consider $T_b^{st} = D^s[\ldots,[b,H_1],\ldots,H_d]D^t$, where b is a function on $\mathbf{R}^d$, H_i is defined by

$$H_i f(x) = \text{p.v.}\ \frac{1}{2\pi}\int_{\mathbf{R}} \frac{f(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_d)}{x_i - y_i}\,dy_i, \quad 1 \leqq i \leqq d.$$

It has the Fourier kernel

$$A(\xi,\eta) = C\prod_{i=1}^{d}\,[I(\xi_i > 0 > \eta_i) - I(\xi_i < 0 < \eta_i)].$$

This kernel satisfies A0, A_p1, $A_p3(\infty)$, $A_p4\frac{1}{2}$. So the conclusions of Theorems 1 and 2 hold for it. Using an argument of Semmes ([9, pp. 261–265]), we get that if $0 < p < 1$, $s,t > -d/2$, then

$$D^s[\ldots,[b,H_1],\ldots,H_d]D^t \in S_p$$

if and only if $b \in B^{s+t+d/p}$ and furthermore, $\|D^s[\ldots,[b,H_1],\ldots,H_d]D^t\|_{S_p}$ is comparable to $\|b\|_{B_p^{s+t+d/p}}$. Here $D^s[\ldots,[b,H_1],\ldots,H_d] \in S_p$ is defined in the natural way.

The phenomena of Theorems 3 and 4 do not appear for Hankel operators or one-dimensional commutators but they appear for general kernels $A(\xi,\eta)$. Thus we need other methods to deal with them. Our method to deal with the "Janson-Wolff phenomenon" in Theorem 4 is close to that of [2].

The proofs of Theorems 1, 2, and 4 are given in sections 4–6, respectively.

We omit the proof of Theorem 3, referring Peng [8]. In section 2 we examine some examples and in section 3 we present some lemmas which will be used in sections 4–6.

Results similar to Theorems 1 and 2 have also been given by Timotin [11].

2. Examples.

To show that a function belongs to $V_p(E \times F)$, the Fourier series expansion is often an efficient tool.

PROPOSITION 2. *Let* $\mathrm{T}^d = [-\pi, \pi]^d$, $u = [d(2/p-1)]+1$. *Suppose that* $A \in C^u(\mathrm{T}^d \times \mathrm{T}^d)$ *and* $\operatorname{supp} A \subset \operatorname{Int}(\mathrm{T}^d \times \mathrm{T}^d)$.

Then $A \in V_p(\mathrm{T}^d \times \mathrm{T}^d)$ *and*

$$\|A\|_{V_p(\mathrm{T}^d \times \mathrm{T}^d)} \leqq C \sum_{0 \leqq |\alpha|+|\beta| \leqq u} \sup|D_\xi^\alpha D_\eta^\beta A|. \tag{7}$$

PROOF. Use the Fourier series expansion

$$A(\xi, \eta) = \sum_{(\boldsymbol{m}, \boldsymbol{n}) \in Z^{2d}} \hat{A}(\boldsymbol{m}, \boldsymbol{n}) e^{i\boldsymbol{m} \cdot \xi} e^{i\boldsymbol{n} \cdot \eta}, \quad \xi, \eta \in \mathrm{T}^d.$$

Thus

$$\begin{aligned}
\|A\|^p_{V_p(\mathrm{T}^d \times \mathrm{T}^d)} &\leqq \sum_{(\boldsymbol{m}, \boldsymbol{n}) \in Z^{2d}} |\hat{A}(\boldsymbol{m}, \boldsymbol{n})|^p \\
&\leqq \left\{ \sum_{(\boldsymbol{m}, \boldsymbol{n})} (1+|\boldsymbol{m}|^2+|\boldsymbol{n}|^2)^u |\hat{A}(\boldsymbol{m}, \boldsymbol{n})|^2 \right\}^{p/2} \cdot \\
&\quad \cdot \left\{ \sum_{(\boldsymbol{m}, \boldsymbol{n})} (1+|\boldsymbol{m}|^2+|\boldsymbol{n}|^2)^{-u(p)/(2-p)} \right\}^{(2-p)/2} \\
&= C \left\{ \sum_{0 \leqq |\alpha|+|\beta| \leqq u} C_{\alpha, \beta} \|D_\xi^\alpha D_\eta^\beta A\|_2^2 \right\}^{p/2}.
\end{aligned}$$

Therefore

$$\|A\|_{V_p(\mathrm{T}^d \times \mathrm{T}^d)} \leqq C \sum_{0 \leqq |\alpha|+|\beta| \leqq u} \sup|D_\beta^\alpha D_\eta^\beta A|.$$

Using Proposition 2, we can obtain the following three propositions.

PROPOSITION 3. *Let* $A \in C^u(\tilde{\Delta}_k \times \tilde{\Delta}_l)$. *Then*

$$(8) \qquad \|A\|_{V_p(\Delta_k \times \Delta_l)} \leqq C \sup_{0 \leqq |\alpha|+|\beta| \leqq u} \sup_{\substack{\xi \in \tilde{\Delta}_k \\ \eta \in \tilde{\Delta}_l}} |\xi|^{\alpha} |\eta|^{\beta} |D_\xi^\alpha D_\eta^\beta A(\xi, \eta)|.$$

PROPOSITION 4. *Let* $B = B(\xi_0, r)$. *Then*

$$(9) \qquad \|A\|_{V_p(B \times B)} \leqq C \sup_{0 \leqq |\alpha|+|\beta| \leqq u} r^{|\alpha|+|\beta|} \sup_{\xi, \eta \in B(\xi_0, 2r)} |D_\xi^\alpha D_\eta^\beta A(\xi, \eta)|.$$

PROPOSITION 5. *Suppose that* $k \geqq 1$ *and* $m \geqq \max(u, k)$. *Suppose further that* $r < \frac{1}{2}|\xi_0|$ *and* $A \in C^m(B(\xi_0, 2r) \times B(\xi_0, 2r))$ *with* $D^\alpha A(\xi_0, \xi_0) = 0$, *when* $|\alpha| \leqq k-1$. *Then*

$$\|A\|_{V_p(B \times B)} \leqq C(r/|\xi_0|)^k \sup_{|\alpha| \leqq m} \sup_{\xi, \eta \in B(\xi_0, 2r)} |\xi_0|^{\alpha} |D^\alpha A(\xi, \eta)|.$$

Now we examine some examples.

EXAMPLE 1. Nth *order commutators of singular integral operators.* When $d = 1$, the singular integral operator K is a scalar multiple of the Hilbert transform, so the Nth order commutator has the kernel.

$$A(\xi, \eta) = C(I(\xi > 0 > \eta) - I(\xi < 0 < \eta))^N.$$

($I(\ldots)$ denotes the indicator function, see [3].) It is clear that in this case $A(\xi, \eta)$ satisfies A0 for any $r > 0$, A_p1, $A_p3(\infty)$, $A_p4\frac{1}{2}$, $A_p9(1)$, but not A10 for any $\alpha > 0$. So the conclusions of Theorems 1 and 2 hold for it, and the results of Semmes [9] can be easily obtained. The "Janson-Wolff phenomenon" described in Theorem 4 does not appear for it.

When $d \geqq 2$, let K_i denote a Calderón-Zygmund transform, i.e. the principal value convolution with a kernel K_i whose Fourier transform $\hat{K}_i$ is homogeneous of degree 0, $C^\infty(\mathbb{R}^d \setminus \{0\})$ and has vanishing spherical mean values. The Nth order commutator $[K_1, \ldots, [K_{N,b}] \ldots]$ has its kernel $A(\xi, \eta) = \prod_{i=1}^N [\hat{K}_i(\xi) - \hat{K}_i(\eta)]$. It is easy to check that in this case $A(\xi, \eta)$ satisfies A0 for any $r > 0$, A_p1, $A_p3(N)$, and $A_p9(1)$. If A satisfies the non-degeneracy condition:

$$(*) \quad \text{if } \prod_{i=1}^N (\hat{K}_i(\xi+\theta) - \hat{K}_i(\xi)) = 0 \quad \text{for all } \xi \text{ then } \theta = 0, \text{then } A \text{ satisfies } A_p4\tfrac{1}{2}.$$

If A satisfies the non-degeneracy condition:

$$(**) \quad \text{if } \prod_{i=1}^N D_\theta \hat{K}_i(\xi) = 0 \quad \text{for all } \xi \text{ then } \theta = 0, \text{ then } A \text{ satisfies } A10(N).$$

It is obvious that $(**) \Rightarrow (*)$, so if A satisfies $(**)$, then A satisfies

$A_p 4\frac{1}{2}$ and A10(N). In this case, all of the conclusions of Theorems 1–4 and Corollary 1 hold.

EXAMPLE 2. *Paraproducts.* The name "paraproduct" denotes an idea rather than a unique definition; several versions exist and can be used for the same purposes. For example, consider the paracommutator with the kernel

$$A(\xi,\eta) = \varphi(|\xi|/|\xi-\eta|),$$

where $\varphi \in C^\infty(0,\infty)$, $\varphi = 1$ on $(0,\delta)$ and $\varphi = 0$ on $(1-\delta,\infty)$ for some $\delta > 0$. It is easy to check that in this case $A(\xi,\eta)$ satisfies A0 for any $r > 0$, $A_p 1$, $A_p 3(\infty)$, $A_p 4\frac{1}{2}$, $A_p 9(1)$, but does not satisfy A10 for any $\alpha > 0$. So the conclusions of Theorems 1–3 hold and the "Janson-Wolff phenomenon" described in Theorem 4 does not appear for it.

EXAMPLE 3. *$A(\xi,\eta)$ smooth.* Suppose that $A \in C^\infty(\mathrm{R}^{2d}\setminus\{0\})$ and that, for each multi-index α, there exist a constant C_α such that

$$|D^\alpha A(\xi,\eta)| \leqq C_\alpha(|\xi|+|\eta|)^{-|\alpha|}$$

and a positive integer N such that

$$D^\alpha A(\xi,\eta) = 0 \quad \text{for} \quad |\alpha| \leqq N-1.$$

This is a kind of pseudo-differential operators studied by Coifman and Meyer [1]. Using Propositions 2–5, it is easy to check that $A(\xi,\eta)$ satisfies $A_p 1$, $A_p 3(N)$, and $A_p 9(1)$. If $A(\xi,\eta)$ satisfies A0 for some $r > 1$ and A4 (see [3]), then it is not too hard to check that $A(\xi,\eta)$ satisfies $A_p 4\frac{1}{2}$. If further $A(\xi,\eta)$ satisfies

$$(\triangle) \text{ for each } \theta \neq 0 \text{ there exists } \xi_1 \neq 0 \text{ with } D_\theta^N A(\xi_1,\xi_1) \neq 0,$$

then $A(\xi,\eta)$ satisfies $A10(N)$.

So the conclusions of Theorems 1–4 and Corollary 1 hold for such paracommutators.

3. Some lemmas.

LEMMA 1. *If $0 < p < 1$, $T, S \in S_p$, then*

$$\|T+S\|_{S_p}^p \leqq \|T\|_{S_p}^p + \|S\|_{S_p}^p,$$

and equality holds if and only if $T^*TS^*S = 0$. (Cf. McCarthy [4].)

LEMMA 2. *If $\{E_k\}_{k\in Z}$, $\{F_k\}_{k\in Z}$ are sets of disjoint subsets of R^d such that $E_k \cap F_l = \emptyset$ for $k \neq l$, let Q_k, P_k denote the projections from $L^2(\mathrm{R}^d)$ into*

$L^2(E_k)$, $L^2(F_k)$. *Then*

$$\left\|\sum_{k\in Z} Q_k T P_k\right\|_{S_p}^p = \sum_{k\in Z} \|Q_k T P_k\|_{S_p}^p, \tag{10}$$

holds for $T \in S_p(\mathrm{R}^d \times \mathrm{R}^d)$.

This is a consequence of Lemma 1.

LEMMA 3. *If* $F_1 \cap F_2 = \phi$, $A \in V_p(E \times F_1)$, $A \in V_p(E \times F_2)$. *Then* $A \in V_p(E \times (F_1 \cup F_2))$ and

$$\|A\|_{V_p(E\times(F_1 \cup F_2))} \leqq C(\|A\|_{V_p(E\times F_1)} + \|A\|_{V_p(E\times F_2)}). \tag{11}$$

This is obvious.

LEMMA 4. *Let* $\chi \in C_0^\infty(\mathrm{R}^d)$, $\operatorname{supp}\chi \subset \tilde{\Delta}_0$, $\chi(\xi) = 1$ *on* Δ_0, *and* N *be a fixed integer. Then*

$$\|\chi(\xi-\eta)\|_{V_p(\Delta_k\times\Delta_l)} \leqq C(N) \quad \textit{for } k,l \leqq N$$

and

$$\|\chi(\xi-\eta)\|_{V_p(B\times B)} \leqq C(r) \quad \text{for all } B = B(\xi_0, r).$$

These are consequences of Propositions 3–4 in section 2.

LEMMA 5. *Let* Ω *be a compact subset of* R^d, $0 < p < 1$. *Then, for every* $r < p$, *there exists a constant* C *such that*

$$\sup_{z\in\mathrm{R}^d} \frac{|\varphi(x-z)|}{1+|z|^{d/r}} \leqq C[M|\varphi|^r(x)]^{1/r} \tag{12}$$

holds for all $\varphi \in L_p^\Omega = \{\varphi \in L^p : \operatorname{supp}\hat{\varphi} \subset \Omega\}$. (Cf. Triebel [12, p. 16 and p. 22].)

LEMMA 6. *Let* $0 < p < \infty$, $a > 0$. *For any* $a' > a$, *there exist two positive constants* C_1 *and* C_2 *such that*

$$C_1\left(\sum_{k\in Z^d} |\varphi(k/a')|^p\right)^{1/p} \leqq \|\varphi\|_p \leqq C_2\left(\sum_{k\in Z^d} |\varphi(k/a')|^p\right)^{1/p} \tag{13}$$

holds for all $\varphi \in \{\varphi \in S' : \operatorname{supp}\hat{\varphi} \subset B(0,a)\}$.

This is the theorem of Plancherel and Polya, see Triebel [12, p. 19–20].

LEMMA 7. *Let* $b \in L_p^{B(0,R/2)}$, $0 < p < 1$, *and let* $(\hat{b})_e(\xi)$ *denote the periodic extension of* $\hat{b}(\xi)$ *with the period* $2\pi\mathrm{R}$. *Then*

$$\|(\hat{b})_e(\xi-\eta)\|_{V_p(\mathrm{R}^d\times\mathrm{R}^d)} \leqq \mathrm{R}^{d/p-d}\|b\|_p. \tag{14}$$

PROOF. By the homogeneity, it suffices to show (14) for $R = 1$. In that case, $\operatorname{supp}\hat{b} \subset B(0, \frac{1}{2})$, and we extend $\hat{b}(\xi)$ to a periodic function $(\hat{b})_e(\xi)$ with the period 2π and expand it into a Fourier series

$$(\hat{b})_e(\xi) = \sum_{k \in Z^d} b(k) e^{ik\cdot\xi}.$$

Thus

$$(\hat{b})_e(\xi - \eta) = \sum_{k \in Z^d} b(k) e^{ik\cdot\xi_e - ik\cdot\eta}.$$

Since $|e^{ik\cdot\xi}| = |e^{-ik\cdot\eta}| = 1$ for $\xi \in \mathbb{R}^d$, $\eta \in \mathbb{R}^d$, by Lemma 6,

$$\left(\sum_{k \in Z^d} |b(k)|^p\right)^{1/p} \leqq C\|b\|_p.$$

Hence (14) holds.

4. Proof of Theorem 1.

Let $\psi \in S(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\psi} \subset \tilde{\Delta}_0$ and if $\xi \neq 0$, then $\sum_{k=-\infty}^{\infty} \hat{\psi}_k(\xi) = 1$ with $\hat{\psi}_k(\xi) = \hat{\psi}(2^{-k}\xi)$. Thus we have

$$b = \sum_{k=-\infty}^{\infty} b * \psi_k.$$

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi(\xi) = 1$ for $\xi \in \tilde{\Delta}_0$ and

$$\operatorname{supp}\chi \subset \{\tfrac{1}{2} - \varepsilon \leqq |\xi| \leqq 2 + \varepsilon\} \quad \text{for some } 0 < \varepsilon < \tfrac{1}{4}.$$

Put $\chi_k(\xi) = \chi(2^{-k}\xi)$. By Lemma 1, we have

$$\|T_b^{st}\|_{S_p}^p \leqq \sum_{k=-\infty}^{\infty} \|T_{b*\psi_k}^{st}\|_{S_p}^p. \tag{15}$$

Note that $(b * \psi_k)\hat{}\,(\xi - \eta)A(\xi, \eta) = (b * \psi_k)\hat{}_e(\xi - \eta)\chi_k(\xi - \eta)A(\xi, \eta)$. By Proposition 1, we get

$$\begin{aligned}\|T_{b*\psi_k}^{st}\|_{S_p}^p &= \|(b * \psi_k)\hat{}_e(\xi - \eta)\chi_k(\xi - \eta)A(\xi, \eta)|\xi|^s|\eta|^t\|_{S_p}^p \\ &\leqq \|(b * \psi_k)\hat{}_e(\xi - \eta)\|_{V_p(\mathbb{R}^d \times \mathbb{R}^d)}^p \|\chi_k(\xi - \eta)A(\xi, \eta)|\xi|^s|\eta|^t\|_{S_p}^p.\end{aligned}$$

By Lemma 7, we know that

$$\|(b * \psi_k)\hat{}_e(\xi - \eta)\|_{V_p(\mathbb{R}^d \times \mathbb{R}^d)}^p \leqq C 2^{kd(1-p)} \|b * \psi_k\|_p^p.$$

It suffices to show that

$$\|\chi_k(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p}^p \leqq C2^{dkp+skp+tkp}. \tag{16}$$

In view of the homogeneity of the assumptions on A, it suffices to show (16) for $k = 0$, i.e.

$$\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p}^p \leqq C. \tag{17}$$

To show (17), we use the analogue of the argument in [3] for $p = 1$. First of all, by Lemma 4, we have

$$\begin{aligned}
&\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\Delta_k\times\Delta_l)}^p \\
&\leqq \|\chi(\xi-\eta)\|_{V_p(\Delta_k\times\Delta_l)}^p\|A(\xi,\eta)\|_{V_p(\Delta_k\times\Delta_l)}^p\||\xi|^s|\eta|^t\|_{S_p(\Delta_k\times\Delta_l)}^p \\
&\leqq C\||\xi|^s\|_{L^2(\Delta_k)}^p\||\eta|^t\|_{L^2(\Delta_l)}^p \\
&\leqq C2^{kp(s+d/2)}2^{lp(t+d/2)}
\end{aligned} \tag{18}$$

for $k,l \leqq N$.

For $\boldsymbol{k}\in \mathbf{Z}^d$, let $Q_{\boldsymbol{k}}$ denote the cube with centre $4\boldsymbol{k}$ and side 4, and let $\tilde{Q}_{\boldsymbol{k}}$ be the concentric cube with side 9. Note that if $\operatorname{supp}\hat{f} \subset Q_{\boldsymbol{k}}$, then

$$\operatorname{supp}\int\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\hat{f}(\eta)d\eta \subset \tilde{Q}_{\boldsymbol{k}}.$$

Thus we have

$$\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{Q_{\boldsymbol{k}}}(\eta)\|_{S_p(\mathbf{R}^d\times\mathbf{R}^d)}^p = \|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\tilde{Q}_{\boldsymbol{k}}\times Q_{\boldsymbol{k}})}^p.$$

When $|\boldsymbol{k}| > 3\sqrt{d}/4\delta$, where δ is as in $\mathrm{A}_p3(\alpha)$, by Lemma 4 and $\mathrm{A}_p3(\alpha)$, we have

$$\begin{aligned}
&\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\tilde{Q}_{\boldsymbol{k}}\times Q_{\boldsymbol{k}})}^p \\
&\leqq \|\chi(\xi-\eta)\|_{V_p(\tilde{Q}_{\boldsymbol{k}}\times\tilde{Q}_{\boldsymbol{k}})}^p\|A(\xi,\eta)\|_{V_p(\tilde{Q}_{\boldsymbol{k}}\times\tilde{Q}_{\boldsymbol{k}})}^p\||\xi|^s\|_{L^2(\tilde{Q}_{\boldsymbol{k}})}^p\||\eta|^t\|_{L^2(Q_{\boldsymbol{k}})}^p \\
&\leqq C|\boldsymbol{k}|^{-p\alpha+ps+pt}.
\end{aligned} \tag{19}$$

When $|\boldsymbol{k}| \leqq 3\sqrt{d}/4\delta$, note that $\tilde{Q}_{\boldsymbol{k}} \subset \bigcup_{-\infty}^{N}\Delta_k$, where N is an integer depending only on δ and d. Thus we have

$$\begin{aligned}
&\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\tilde{Q}_{\boldsymbol{k}}\times Q_{\boldsymbol{k}})}^p \\
&\leqq \sum_{k=-\infty}^{N}\sum_{l=-\infty}^{N}\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\Delta_k\times\Delta_l)}^p.
\end{aligned}$$

Using (18), we get

$$\begin{aligned}
&\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|^p_{S_p(\tilde{Q}_k\times Q_k)}\\
(20)\qquad &\leqq \sum_{k=-\infty}^{N}\sum_{l=-\infty}^{N} 2^{kp(s+d/2)}2^{lp(t+d/2)}\\
&= C.
\end{aligned}$$

Since $\alpha > s+t+d/p$, we have

$$\sum_{|\boldsymbol{k}|>3\sqrt{d}/4\delta} |\boldsymbol{k}|^{-p\alpha+ps+pt} < \infty.$$

Therefore (19) and (20) imply that

$$\begin{aligned}
&\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|^p_{S_p(\mathbf{R}^d\times\mathbf{R}^d)}\\
&\leqq \sum_{\boldsymbol{k}\in\mathbf{Z}^d}\|\chi(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\|^p_{S_p(\tilde{Q}_k\times Q_k)}\\
&\leqq C\sum_{|\boldsymbol{k}|>3\sqrt{d}/4\delta}|\boldsymbol{k}|^{-p\alpha+ps+pt}+\sum_{|\boldsymbol{k}|\leq 3\sqrt{d}/4\delta} C\\
&= C.
\end{aligned}$$

This completes the proof of Theorem 1.

COROLLARY 2. *Under the same assumption as in Theorem* 1. *If* $b\in B_p^{s+t+d/p}$ *with* $\operatorname{supp}\hat{b}\subset\mathbf{R}^d\setminus\{0\}$, *then* $T_b^{st}(A_\varepsilon^\theta)\in S_p$ *and*

$$\|T_b^{st}(A_\varepsilon^\theta)\|_{S_p}\leqq C\|b\|_{B_p^{s+t+d/p}}$$

holds uniformly in $\varepsilon\leqq\varepsilon_0$ *and* $|\theta|\leqq\sqrt{d}/3$, *for some* $\varepsilon_0>0$.

PROOF. We may assume that $\operatorname{supp}\hat{b}\subset\{|\xi|\geqq 2^{-N_0}\}$, for some large number $N_0>0$. Let $\varepsilon<\varepsilon_0=2^{-N_0}/\sqrt{d}$. By the proof of Theorem 1, it suffices to show that A_ε^θ satisfies A_p1 and that for $B=B(\xi_0,r)$ with $2^{-N_0}<r<\delta/2|\xi_0|$, holds

$$\|A_\varepsilon^\theta\|_{V_p(B\times B)}\leqq C(r/|\xi_0|)^\alpha.$$

In fact, if $2^{k+1}<\varepsilon/2$,

$$\|A_\varepsilon^\theta\|_{V_p(\Delta_l\times\Delta_k)}=0;$$

if $2^{k+1} \geqq \varepsilon/2$,

$$\|A_\varepsilon^\theta\|_{V_p(\Delta_l \times \Delta_k)} = \Big\| \sum_{\substack{m \in Z^d \\ \Delta^k \cap Q_{m+\theta}^\varepsilon \neq \emptyset}} A(\xi, \varepsilon(m+\theta))\chi_{Q_{m+\theta}^\varepsilon}(\eta) \Big\|_{V_p(\Delta_l \times \Delta_k)}$$

$$\leqq \Big\| \sum_{\substack{\bar{m} \in Z^d \\ \Delta^k \cap Q_{m+\theta}^\varepsilon \neq \emptyset}} A(\xi, \varepsilon(m+\theta))\chi_{Q_{m+\theta}^\varepsilon}(\eta) \Big\|_{V_p(\Delta_l \times \tilde{\Delta}_k)}$$

$$\leqq \|A\|_{V_p(\Delta_l \times \tilde{\Delta}_k)} \leqq C \quad \text{(by Lemma 3)}.$$

So A_ε^θ satisfies A_p1. If $2^{-N_0} < r < d/2|\xi_0|$, then

$$\|A_\varepsilon^\theta\|_{V_p(\times B)} = \Big\| \sum_{\substack{m \in Z^d \\ B \cap Q_{m+\theta}^\varepsilon \neq \emptyset}} A(\xi, \varepsilon(m+\theta))\chi_{Q_{m+\theta}^\varepsilon}(\eta)\|_{V_p(B \times B)}$$

$$\leqq \Big\| \sum_{\substack{m \in Z^d \\ B \cap Q_{m+\theta}^\varepsilon \neq \emptyset}} A(\xi, \varepsilon(m+\theta))\chi_{Q_{m+\theta}^\varepsilon}(\eta)\|_{V_p(\bar{B} \times \bar{B})}$$

(where $\bar{B} = B(\xi_0, r+\varepsilon\sqrt{d})$)

$$\leqq \|A\|_{V_p(\bar{B} \times \bar{B})}$$

$$\leqq C\left(\frac{r+2^{-N}}{|\xi_0|}\right)^\alpha \leqq C\left(\frac{r}{|\xi_0|}\right)^\alpha \quad (\text{because } r+2^{-N_0} \leqq 2r < \delta|\xi_0|).$$

5. Proof of Theorem 2.

For the sake of simplicity, we assume that $r = 2$ in A0. It is easy to show that $A_p4\frac{1}{2}$ is equivalent to the following statement.

For every $\xi_0 \neq 0$, there exist $\eta_0 \in \mathbb{R}^d$ and $\delta > 0$ with $\eta_0 \notin \{0, -\xi_0\}$ and $\delta < \frac{1}{4}\min(|\xi_0+\eta_0|, |\eta_0|, 1)$ such that, with $B_0 = B(\xi_0+\eta_0, \delta|\xi_0|)$ and $D_0 = B(\eta_0, \delta|\xi_0|)$, $A(\xi, \eta)^{-1} \in V_p(B_0 \times D_0)$.

By the compactness of Δ_0, there exist finite sets of points $\{\xi_0^{(j)}\}_{j=1}^J$ in Δ_0 and $\{\eta_0^{(j)}\}_{j=1}^J$, with corresponding open balls $B(\xi_0^{(j)}, \delta^{(J)})$ and $B(\eta_0^{(j)}, \delta^{(J)})$, such that $\eta_0^{(j)} \neq 0$, $\eta_0^{(j)} \neq -\xi_0^{(j)}$,

$$\bigcup_{j=1}^J B(\xi_0^{(j)}, \delta^{(J)}) \supset \Delta_0, \quad \delta^{(J)} < \tfrac{1}{4}\min(|\xi_0^{(j)}+\eta_0^{(j)}|, |\eta_0^{(j)}|, 1)$$

and, with $B_j = B(\xi_0^{(j)}+\eta_0^{(j)}, \delta^{(J)})$ and $D_j = B(\eta_0^{(j)}, \delta^{(J)})$,

$$A^{-1} \in V_p(B_j \times D_j).$$

We choose the positive functions $h'_j(\xi)$ and $h_j(\eta)$ such that $h'_j, h_j \in C_0^\infty(\mathbf{R}^d)$, $\operatorname{supp} h'_j = \bar{B}_j$, $h'_j(\xi) > 0$ on B_j, $\operatorname{supp} h_j = \bar{D}_j$ and $h_j(\eta) > 0$ on D_j. Let

$$\hat{\psi}(\xi) = \sum_{j=1}^{J} \int |\xi+\eta|^s |\eta|^t h'_j(\xi+\eta) h_j(\eta) d\eta. \tag{21}$$

Then $\hat{\psi} \in C_0^\infty(\mathbf{R}^d)$, $\operatorname{supp} \hat{\psi} \subset \{\frac{1}{2} \leqq |\xi| \leqq 2+\frac{1}{2}\}$ and $\hat{\psi}(\xi) \geqq C > 0$ on Δ_0. Thus ψ can be used to define the norm of $B_p^{s+t+d/p}$.

Let $\hat{\psi}' \in C_0^\infty(\mathbf{R}^d)$ with support $\subset \{\frac{1}{8} \leqq |\xi| \leqq 4\}$ and $\hat{\psi}'(\xi) = 1$ on $\{\frac{1}{4} \leqq |\xi| \leqq 3\}$. Thus ψ' can be used to define the norm of $B_p^{s+t+d/p}$ also, and

$$\|\cdot\|_{B_p^{s+t+d/p}(\psi)} \approx \|\cdot\|_{B_p^{s+t+d/p}(\psi')}.$$

Let $\hat{\psi}_k(\xi)$, $\hat{\psi}'_k(\xi)$ denote $\hat{\psi}(2^{-k}\xi)$, $\hat{\psi}'(2^{-k}\xi)$ respectively.

For $\eta_0^{(j)} \neq 0$, $\xi_0^{(j)}+\eta_0^{(j)} \neq 0$, there exist r_1 and r_2 with $0 < r_1 < r_2 < r_2 < \infty$ such that

$$r_1 \leqq |\eta_0^{(j)}|,\ |\xi_0^{(j)}+\eta_0^{(j)}| \leqq r_2, \quad j = 1,\dots,J.$$

For the sake of simplicity, we assume that $3/4 \leqq |\eta_0^{(j)}|, |\xi_0^{(j)}+\eta_0^{(j)}| \leqq 2\frac{1}{2}$, thus

$$\operatorname{supp} h'_j,\ \operatorname{supp} h_j \subset \{\tfrac{1}{2} \leqq /\xi| \leqq 1\tfrac{3}{4}\} = \bar{\Delta}_0, \quad \text{for} \quad j = 1,\dots,J.$$

The proof of the general case is similar.

We fix a positive integer M, which is large enough and whose choice will be specified later. We define operators T_i, $i = 0,\dots, M-1$, by

$$\begin{aligned}(T_i f)\hat{}(\xi) \\ = (2\pi)^{-d} \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int \hat{b}(\xi-\eta) A(\xi,\eta) |\xi|^s |\eta|^t \chi_{\bar{\Delta}_{lM+i}}(\xi) \chi_{\bar{\Delta}_{kM+i}}(\eta) \hat{f}(\eta) d\eta,\end{aligned} \tag{22}$$

where $\bar{\Delta}_k = 2^k \bar{\Delta}_0$. Note that $\|T_i\|_{S_p} \leqq \|T_b^{st}\|_{S_p}$, so we have

$$\sum_{i=0}^{M-1} \|T_i\|_{S_p}^p \leqq M \|T_b^{st}\|_{S_p}^p. \tag{23}$$

We put

$$T_i = T_i^{(1)} + T_i^{(2)}, \tag{24}$$

where $T_i^{(1)}$ is defined by

$$(T_i^{(1)}f)\hat{}(\xi) = (2\pi)^{-d}\sum_{k=-\infty}^{\infty}\int \hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\xi)\chi_{\bar{\Delta}_{kM+i}}(\eta)\hat{f}(\eta)d\eta \tag{25}$$

and $T_i^{(2)}$ by

$$\begin{aligned}(T_i^{(2)}f)\hat{}(\xi) &= (2\pi)^{-d}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{k-1}\int \hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{lM+i}}(\xi)\chi_{\bar{\Delta}_{kM+i}}(\eta)\hat{f}(\eta)d\eta + \\ &+ (2\pi)^{-d}\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{l-1}\int \hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{lM+i}}(\xi)\chi_{\bar{\Delta}_{kM+i}}(\eta)\hat{f}(\eta)d\eta.\end{aligned} \tag{26}$$

We estimate the "S_p-norm" of $T_i^{(1)}$ from below and the "S_p-norm" of $T_i^{(2)}$ from above separately. First of all we have

$$\begin{aligned}&\left\|\left\{\sum_{l=-\infty}^{k-1}\chi_{\bar{\Delta}_{lM+i}}(\xi)\right\}\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\right\|_{S_p}^p \\ &\leqq \|\chi_{(0,2^{(k-1)M+i+2}]}(\xi)\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p\end{aligned}$$

where $\chi_{(0,b]}(\xi)$ is the characteristic function of $\{|\xi|\leqq b\}$. If M is large enough, then

$$\hat{\psi}'_{kM+i}(\xi-\eta)=1 \quad \text{on} \quad \{|\xi|\leqq 2^{-(k-1)M+i+2}\}\times\bar{\Delta}_{kM+i}.$$

Thus we have

$$\begin{aligned}&\|\chi_{(0,2^{(k-1)M+i+2}]}(\xi)\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p \\ &= \|_{(0,2^{(k-1)M+i+2}]}(\xi)\hat{b}(\xi-\eta)\hat{\psi}'_{kM+i}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p \\ &= \|\chi_{(0,2^{(k-1)M+i+2}]}(\xi)(b*\psi'_{kM+i})\hat{}_e(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p \\ &\leqq C\|b*\psi'_{kM+i}\|_p^p 2^{(kM+i)(d-pd)}\|\chi_{(0,2^{(k-1)M+i+2}]}(\xi)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p\end{aligned}$$

(by Proposition 1 and Lemma 7).

We claim that

$$\begin{aligned}&\|\chi_{(0,2^{(k-1)M+i+2}]}(\xi)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p \\ &\leqq C2^{(kM+i)(sp+tp+dp)}2^{-MP(s+d/2)}.\end{aligned} \tag{27}$$

In fact, to show (27), by homogeneity of $A(\xi,\eta)$, it is sufficient to show it for $k = 0$, $i = 0$. In that case we have

$$\|\chi_{(0,2^{-M+2}]}(\xi)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}}(\eta)\|_{S_p}^p$$

$$\leqq \sum_{k=-1}^{1}\sum_{l=-\infty}^{-M+1}\|A(\xi,\eta)|\xi|^s|\eta|^t\|_{S_p(\Delta_l\times\Delta_k)}^p$$

$$\leqq C\sum_{k=-1}^{1}\sum_{l=-\infty}^{-M+1}2^{lp(s+d/2)}2^{kp(t+d/2)} = C2^{-Mp(s+d/2)},$$

i.e. (27) holds.

Thus we obtain

$$\left\|\left\{\sum_{l=-\infty}^{k-1}\chi_{\bar{\Delta}_{lM+i}}(\xi)\right\}\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\right\|_{S_p}^p$$

$$\leqq C\,2^{-Mp(s+d/2)}2^{(kM+i)(sp+tp+d)}\|b*\psi'_{kM+i}\|_p^p.$$

Similarly, we obtain

$$\left\|\chi_{\bar{\Delta}_{lM+i}}(\xi)\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\left\{\sum_{k=-\infty}^{l-1}\chi_{\bar{\Delta}_{kM+i}}(\eta)\right\}\right\|_{S_p}^p$$

$$\leqq C2^{-Mp(t+d/2)}2^{(lM+i)(sp+tp+d)}\|b*\psi'_{lM+i}\|_p^p.$$

Hence we get the estimate of the "S_p-norm" of $T_i^{(2)}$ from above

$$\|T_i^{(2)}\|_{S_p}^p \leqq C[2^{-Mp(s+d/2)}+2^{-Mp(t+d/2)}]\sum_{k=-\infty}^{\infty}2^{(kM+i)(sp+tp+d)}\|b*\psi'_{kM+i}\|_p^p.$$

Consequently,

$$\sum_{i=0}^{M-1}\|T_i^{(2)}\|_{S_p}^p \leqq C[2^{-Mp(s+d/2)}+2^{-Mp(t+d/2)}]\|b\|_{B^{s+t+d/p}(\psi')}^p. \tag{28}$$

Now we are going to estimate the "S_p-norm" of $T_i^{(1)}$ from below. By Lemma 2

$$\|T_i^{(1)}\|_{S_p}^p$$

$$= (2\pi)^{-dp}\sum_{k=-\infty}^{\infty}\|\chi_{\bar{\Delta}_{kM+i}}(\xi)\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p.$$

We claim that when M_1 is large enough

$$(29)\quad \begin{aligned} &\|\chi_{\bar{\Delta}_{kM+i}}(\xi)\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_{kM+i}}(\eta)\|_{S_p}^p \\ &\geqq CM_1^{-d}2^{(kM+i)(sp+tp+d)}[\|b*\psi_{kM+i}\|_p^p - M_1^{-Np}\|b*\psi'_{kM+i}\|_p^p]. \end{aligned}$$

In fact, by the homogeneity of $A(\xi,\eta)$ it is sufficient to show (29) for $k = i = 0$.

Since $\operatorname{supp} h'_j = \bar{B}_j$ and $\operatorname{supp} h_j = \bar{D}_j$, $A_p 4\frac{1}{2}$ gives that

$$\begin{aligned} &\|\hat{b}(\xi-\eta)\sum_{j=1}^{J}|\xi|^s|\eta|^t h'_j(\xi)h_j(\eta)\|_{S_p}^p \\ &\leqq \left(\max_{1\leqq j\leqq J}\|A^{-1}\|_{V_p(B_j\times D_j)}\right)^p \sum_{j=1}^{J}\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t h'_j(\xi)h_j(\eta)\|_{S_p}^p \\ &\leqq C\sum_{j=1}^{J}\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t h'_j(\xi)h_j(\eta)\|_{S_p}^p. \end{aligned}$$

Note that $\operatorname{supp} h'_j$, $\operatorname{supp} h_j \subset \bar{\Delta}_0$. We therefore get

$$\begin{aligned} &\|\hat{b}(\xi-\eta)\sum_{j=1}^{J}|\xi|^s|\eta|^t h'_j(\xi)h_j(\eta)\|_{S_p}^p \\ &\leqq C\|\hat{b}(\xi-\eta)A(\xi,\eta)|\xi|^s|\eta|^t\chi_{\bar{\Delta}_0}(\xi)\chi_{\bar{\Delta}_0(\eta)}\|_{S_p}^p. \end{aligned}$$

We consider the operator defined by

$$(Sf)(\xi) = \int\hat{b}(\xi-\eta)\sum_{j=1}^{J}|\xi|^s|\eta|^t h'_j(\xi)h_j(\eta)f(\eta)d\eta$$

as an operator from $L^2((3\mathrm{T})^d)$ to $L^2((3\mathrm{T})^d)$. It is clear that the family

$$\{e_n(\eta)\}_{n\in Z^d} = \{(6\pi)^{-d/2}e^{in\cdot\eta/3}\}_{n\in Z^d}$$

forms a complete basis of $L^2((3\mathrm{T})^d)$.

Thus we have

$$(30)\qquad Sf = \sum_{n\in Z^d}\sum_{m\in Z^d}(f,e_n)(Se_n,e_m)e_m.$$

Let M_1 be a positive integer, large enough. Let P_k denote the orthognal projection from $L^2((3\mathrm{T})^d)$ onto

$$\operatorname{span}\{e_{M_1\boldsymbol{n}+\boldsymbol{k}}\}_{\boldsymbol{n}\in Z^d}, \quad \boldsymbol{k}\in\{0,\ldots,M_1-1\}^d.$$

Thus we have

$$I = \bigoplus_{\boldsymbol{k}\in\{0,\ldots,M_1-1\}^d} P_{\boldsymbol{k}} \tag{31}$$

and

$$\sum_{\boldsymbol{k}\in\{0,\ldots,M_1-1\}^d} \|P_{\boldsymbol{k}}SP_{\boldsymbol{k}}\|_{S_p}^p \leqq M_1^d\|S\|_{S_p}^p. \tag{32}$$

We put

$$P_{\boldsymbol{k}}SP_{\boldsymbol{k}} = S_{\boldsymbol{k}}^{(1)}+S_{\boldsymbol{k}}^{(2)} \tag{33}$$

where

$$S_{\boldsymbol{k}}^{(1)}f = \sum_{\boldsymbol{n}\in Z^d}(f, e_{M_1\boldsymbol{n}+\boldsymbol{k}})(Se_{M_1\boldsymbol{n}+\boldsymbol{k}}, e_{M_1\boldsymbol{n}+\boldsymbol{k}})e_{M_1\boldsymbol{n}+\boldsymbol{k}}$$

$$S_{\boldsymbol{k}}^{(2)}f = \sum_{\boldsymbol{n}\in Z^d}\sum_{\substack{\boldsymbol{m}\in Z^d\\ \boldsymbol{m}\neq\boldsymbol{n}}}(f, e_{M_1\boldsymbol{n}+\boldsymbol{k}})(Se_{M_1\boldsymbol{n}+\boldsymbol{k}}, e_{M_1\boldsymbol{m}+\boldsymbol{k}})e_{M_1\boldsymbol{m}+\boldsymbol{k}}.$$

For $S_{\boldsymbol{k}}^{(1)}$, since

$$(Se_{M_1\boldsymbol{n}+\boldsymbol{k}}, e_{M_1\boldsymbol{n}+\boldsymbol{k}})$$

$$= \iint \hat{b}(\xi-\eta)\sum_{j=1}^{J}|\xi|^s|\eta|^t h_j'(\xi)h_j(\eta)e_{M_1\boldsymbol{n}+\boldsymbol{k}}(\eta)e_{M_1\boldsymbol{n}+\boldsymbol{k}}(-\xi)d\eta d\xi$$

(by changing variables $\xi\to\xi'+\eta$)

$$= \iint \hat{b}(\xi)\sum_{j=1}^{J}|\xi+\eta|^s|\eta|^t h_j'(\xi+\eta)h_j(\eta)e_{M_1\boldsymbol{n}+\boldsymbol{k}}(-\xi)d\eta d\xi$$

$$= C\int \hat{b}(\xi)\hat{\psi}(\xi)e^{-i(M_1\boldsymbol{n}+\boldsymbol{k})\cdot\xi/3}d\xi$$

$$= Cb*\psi(-(M_1\boldsymbol{n}+\boldsymbol{k})/3),$$

we have

$$\|S_{k}^{(1)}\|_{S_p}^p = \sum_{n\in Z^d} |(Se_{M_1n+k}, e_{M_1n+k})|^p = C \sum_{n\in Z^d} |b*\psi(-(M_1n+k)/3)|^p$$

and

$$\begin{aligned}\sum_{k\in\{0,\ldots,M_1-1\}^d} \|S_{k}^{(1)}\|_{S_p}^p &= C \sum_{k\in\{0,\ldots,M_1-1\}^d} \sum_{n\in Z^d} |b*\psi(-(M_1n+k)/3)|^p \\ &= C\sum_{n\in Z^d} |b*\psi(n/3)|^p \geqq C\|b*\psi\|_p^p \quad \text{(by Lemma 6).}\end{aligned} \tag{34}$$

For $S_k^{(2)}$, we estimate its "S_p-norm" from above,

$$\begin{aligned}&\|S_p^{(2)}\|_{S_p}^p \\ &\leqq \sum_{n\in Z^d}\sum_{\substack{m\in Z^d\\ m\neq n}} |(Se_{M_1n+k}, e_{M_1m+k})|^p \\ &= \sum_{n\in Z^d}\sum_{\substack{m\in Z^d\\ m\neq n}} \left|\iint \hat{b}(\xi-\eta)\sum_{j=1}^J |\xi|^s|\eta|^t h_j'(\xi)h_j(\eta)e_{M_1n+k}(\eta)e_{M_1m+k}(-\xi)d\xi d\eta\right|^p \\ &= \sum_{n\in Z^d}\sum_{\substack{m\in Z^d\\ m\neq n}} \left|\int \hat{b}(\xi)e_{M_1m+k}(-\xi)\int\sum_{j=1}^J |\xi+\eta|^s|\eta|^t h_j'(\xi+\eta)h_j(\eta)e_{M_1(n-m)}(\eta)d\eta d\xi\right|^p\end{aligned}$$

Let $I(\xi,\eta)$ denote $\sum_{j=1}^J |\xi+\eta|^s|\eta|^t h_j'(\xi+\eta)h_j(\eta)$, and write

$$I^{\hat{2}}(\xi,z) = \int I(\xi,\eta)e^{-iz\cdot\eta}d\eta, \quad I^{\hat{1}\hat{2}}(y,z) = \int I^{\hat{2}}(\xi,z)e^{-iy\cdot\xi}d\xi.$$

Then

$$\begin{aligned}&\|S_k^{(2)}\|_{S_p}^p \\ &\leqq C\sum_{n\in Z^d}\sum_{\substack{m\in Z^d\\ m\neq n}} \left|\int \hat{b}(\xi)\hat{\psi}'(\xi)e_{M_1m+k}(-\xi)I^{\hat{2}}(\xi, M_1(m-n)/3)d\xi\right|^p \\ &= C\sum_{n\in Z^d}\sum_{\substack{m\in Z^d\\ m\neq n}} |b*\psi'*I^{\hat{1}\hat{2}}(-(M_1n+k)/3,\ M_1(m-n)/3)|^p \\ &= C\sum_{n\in Z^d}\sum_{\substack{m\in Z^d\\ m\neq n}} \left|\int b*\psi'(-(M_1n+k)/3-y)I^{\hat{1}\hat{2}}(y, M_1(m-n)/3)dy\right|^p.\end{aligned}$$

Since $I(\xi, \eta) \in C_0^\infty$, for every fixed $N > 0$,

$$|I^{\hat{1}\hat{2}}(y, z)| \leqq C \frac{1}{1+|y|^N} |z|^{-N}.$$

Let Q'_n denote the cube with centre $-1/3\boldsymbol{n}$ and side 1/3. We choose $r < p$ and N sufficiently large. For $x \in Q'_{M\boldsymbol{n}+\boldsymbol{k}}$, by Lemma 5,

$$\left|\int b*\psi'(-(M_1\boldsymbol{n}+\boldsymbol{k})/3-y)I^{\hat{1}\hat{2}}(y, M_1(\boldsymbol{m}-\boldsymbol{n})/3)dy\right|$$

$$\leqq C\int\frac{|b*\psi'[x-(y+x+(M_1\boldsymbol{n}+\boldsymbol{k})/3)]|}{1+|y+x+(M_1\boldsymbol{n}+\boldsymbol{k})/3|^{d/r}} \times$$

$$\times \frac{1+|y+x+(M_1\boldsymbol{n}+\boldsymbol{k})/3|^{d/r}}{1+|y|^N} dy M_1^{-N}|\boldsymbol{m}-\boldsymbol{n}|^{-N}$$

$$\leqq C[M|b*\psi'|^r(x)]^{1/r} M_1^{-N}|\boldsymbol{m}-\boldsymbol{n}|^{-N}.$$

Integrating over $x \in Q'_{M_1\boldsymbol{n}+\boldsymbol{k}}$, we get

$$\left|\int b*\psi'(-(M_1\boldsymbol{n}+\boldsymbol{k})/3-y)I^{\hat{1}\hat{2}}(y, M_1(\boldsymbol{m}-\boldsymbol{n})/2\tfrac{3}{4})dy\right|^p$$

$$\leqq CM_1^{-Np}|\boldsymbol{m}-\boldsymbol{n}|^{-Np} \int_{Q'_{M_1\boldsymbol{n}+\boldsymbol{k}}} (M|b*\psi'|^r(x))^{p/r}dx.$$

Finally, we obtain

$$\|S_{\boldsymbol{k}}^{(2)}\|_{S_p}^p \leqq CM_1^{-Np} \sum_{\boldsymbol{n}\in \mathbf{Z}^d} \int_{Q'_{M_1\boldsymbol{n}+\boldsymbol{k}}} (M|b*\psi'|^r(x))^{p/r}dx.$$

and

$$\sum_{\boldsymbol{k}\in\{0,\ldots,M_1-1\}^d} \|S_{\boldsymbol{k}}^{(2)}\|_{S_p}^p$$

$$\leqq CM_1^{-Np}\int_{\mathbf{R}^d}(M|b*\psi'|^r(x))^{p/r}dx \tag{35}$$

$$\leqq CM_1^{-Np}\|b*\psi'\|_p^p.$$

Combining (32), (33), (34), and (35), we obtain

$$M_1^d\|S\|_{S_p}^p \geqq (C\|b*\psi\|_p^p - CM_1^{-Np}\|b*\psi'\|_p^p)$$

i.e. (29) holds.

Combining (28) and (29), we obtain

$$\begin{aligned} M\|T_b^{st}\|_{S_p}^p & \\ &\geqq CM_1^{-d}[\|b\|^p_{B_p^{s+t+d/p}(\psi)} - M_1^{-Np}\|b\|^p_{B_p^{s+t+d/p}(\psi')}] - \\ &\quad - C[2^{-Mp(s+d/2)} + 2^{-Mp(t+d/2)}]\|b\|^p_{B_p^{s+t+d/p}(\psi')} \\ &\geqq CM_1^{-d}[\|b\|^p_{B_p^{s+t+d/p}} - M_1^{-Np}\|b\|^p_{B_p^{s+t+d/p}}] - \\ &\quad - C[2^{-Mp(s+d/2)} + 2^{-Mp(t+d/2)}]\|b\|^p_{B_p^{s+t+d/p}}. \end{aligned}$$

We now choose M_1 and M large enough. Thus we finally obtain

$$C\|T_b^{st}\|_{S_p}^p \geqq \|b\|^p_{B^{s+t+d/p}}.$$

Theorem 2 has been proved.

6. Proof of Theorem 4.

We give the proof of Theorem 4 only in the case $p < 1$. For the case $p \geqq 1$, Theorem 4 can be improved, see Corollary 3 below.

If b is not a polynomial, there exists $0 \neq \theta \in \operatorname{supp} \hat{b}$. Without loss of generality, we assume that $|\theta| = 1$. By A10(α), we find $\delta > 0$ and a subset V_θ of $\mathbb{R}^d$ such that

$$\overline{\lim_{r\to\infty}} \frac{N_r}{r^d} > 0 \tag{36}$$

and for every $\boldsymbol{n} \in V_\theta$,

$$\left\| \frac{1}{A(\cdot + \boldsymbol{n} + \theta, \cdot + \boldsymbol{n})} \right\|_{M(B_\delta \times B_\delta)} \leqq C|\boldsymbol{n}|^\alpha, \quad \text{where } B_\delta = B(0, \delta). \tag{37}$$

Let $a_{\boldsymbol{n}}$ denote $\sup|\langle T_b^{st}\psi\ \sigma\ \varphi\rangle|$, where φ and ψ range over all functions with $\|\varphi\|_2$, $\|\psi\|_2 \leqq 1$, $\operatorname{supp}\hat{\varphi} \subset B(\boldsymbol{n}+\theta, \delta)$ and $\operatorname{supp}\hat{\psi} \subset B(\boldsymbol{n}, \delta)$.

If g and h are C^∞ functions with $\|g\|_2 = 1/C_s$, $\|h\|_2 = 1/C_t$ ($C_s > 0$ depends on S, $C_t > 0$ depends on t; see below), $\operatorname{supp} g$, $\operatorname{supp} h \subset B(0, \delta)$, then we have, for any fixed $\boldsymbol{n} \in V_\theta$ with $|\boldsymbol{n}| > 6$,

$$\iint \hat{b}(\xi+\theta-\eta)(g(\xi)h(\eta)d\xi d$$

$$= \left\| \hat{b}(\xi+\theta-\eta)A(\xi+\boldsymbol{n}+\theta, \eta+\boldsymbol{n})|\xi+\boldsymbol{n}+\theta|^s|\eta+\boldsymbol{n}|^t \cdot A(\xi+\boldsymbol{n}+\theta, \eta+\boldsymbol{n})^{-1} \times \right.$$

$$\times |\xi+\boldsymbol{n}+\theta|^{-s}|\eta+\boldsymbol{n}|^{-t}g(\xi)h(\eta)d\xi d\eta.$$

Since $A(\xi+\boldsymbol{n}+\theta,\ \eta+\boldsymbol{n})^{-1} \in M(B_\delta \times B_\delta)$, it has the representation

$$A(\xi+\boldsymbol{n}+\theta,\ \eta+\boldsymbol{n})^{-1}\chi_{B_\delta}(\xi)\chi_{B_\delta}(\eta) = \int_\Omega \beta(\xi,\omega)\gamma(\eta,\omega)d\mu(\omega)$$

where

$$\|\beta\|_{L^\infty(B_\delta\times\Omega)}, \|\gamma\|_{L^\infty(B_\delta\times\Omega)} \leqq 1, \quad \mu(\Omega) \leqq C|\boldsymbol{n}|^\alpha.$$

Let

$$\beta'(\xi,\omega) = \beta(\xi,\omega)|\xi+\boldsymbol{n}+\theta|^{-s}(|\boldsymbol{n}|-2)^s,$$

$$\gamma'(\eta,\omega) = \gamma(\eta,\omega)|\eta+\boldsymbol{n}|^{-t}(|\boldsymbol{n}|-2)^t.$$

and

$$\mu'(\omega) = \mu(\omega)(|\boldsymbol{n}|-2)^{-s-t}.$$

Then

$$\|\beta'\|_{L^\infty(B_\delta\times\Omega)} \leqq C_s, \|\gamma'\|_{L^\infty(B_\delta\times\Omega)} \leqq C_t, \quad \mu'(\Omega) \leqq C_{st}|\boldsymbol{n}|^{\alpha-s-t}.$$

Thus we obtain

$$\left| \iint \hat{b}(\xi+\theta-\eta)g(\xi)h(\eta)d\xi d\eta \right|$$

$$= \left| \iiint_\Omega \hat{b}(\xi+\theta-\eta)A(\xi+\boldsymbol{n}+\theta, \eta+\boldsymbol{n})|\xi+\boldsymbol{n}+\theta|^s|\eta+\boldsymbol{n}|^t g(\xi)\beta'(\xi,\omega)h(\eta) \times \right.$$

$$\left. + \gamma'(\eta,\omega)d\xi d\eta d\mu'(\omega) \right|$$

$$\leqq \int_\Omega d\mu'(\omega) \left| \iint \hat{b}(\xi-\eta) A(\xi,\eta) |\xi|^s |\eta|^t (h\gamma')(\eta-\boldsymbol{n},\omega)(g\beta')(\eta-\boldsymbol{n}-\theta,\omega)\, d\xi d\eta \right|$$

$$\leqq \mu'(\Omega) \sup_{\substack{\|\varphi\|_2, \|\psi\|_2 \leq 1 \\ \operatorname{supp}\hat{\psi} \subset B(\boldsymbol{n}+\theta,\delta) \\ \operatorname{supp}\hat{\varphi} \subset B(\boldsymbol{n},\delta)}} |T_b^{st}\psi, \varphi\rangle| = \mu'(\Omega) a_{\boldsymbol{n}} \leqq C_{st} |\boldsymbol{n}|^{\alpha-s-t} a_{\boldsymbol{n}}.$$

Since $\theta \in \operatorname{supp} \hat{b}$ we can find g and h such that

$$\left| \iint \hat{b}(\xi+\theta-\eta) g(\xi) h(\eta) d\xi d\eta \right| > 0,$$

thus we get

$$a_{\boldsymbol{n}} \geqq C |\boldsymbol{n}|^{-\alpha+s+t} \quad \text{for} \quad \boldsymbol{n} \in V_\theta \quad \text{and} \quad |\boldsymbol{n}| > 6.$$

We claim that

$$\|T_b^{st}\|_{S_p}^p \geqq C \sum_{\boldsymbol{n} \in V_\theta} a_{\boldsymbol{n}}^p. \tag{38}$$

Then, by (36),

$$\|T_b^{st}\|_{S_p}^p \geqq C \sum_{\substack{\boldsymbol{n} \in V_\theta \\ |\boldsymbol{n}| > 6}} |\boldsymbol{n}|^{(-\alpha+s+t)p} = \infty$$

this contradicts $T_b^{st} \in S_p$. This contradiction shows that b mus be a polynomial.

To show (38), we assume that $\operatorname{supp} \hat{b} \subset \{|\xi| \leqq M-2\}$, where $M > 2$ is a positive integer.

Let $V_{\boldsymbol{r}} = \{\boldsymbol{n} \in V_\theta : \boldsymbol{n} = M\boldsymbol{k}+\boldsymbol{r}\}$ for $\boldsymbol{r} \in \{0, 1, \ldots, M-1\}^d$, let $P_{\boldsymbol{r}}^{(\boldsymbol{k})}$ denote the projection from $L^2(\mathrm{R}^d)$ to $L^2(B(M\boldsymbol{k}+r,\delta))$, let $Q_{\boldsymbol{r}}^{(\boldsymbol{k})}$ denote the projection from $L^2(\mathrm{R}^d)$ to $L^2(B(M\boldsymbol{k}+\boldsymbol{r}+\theta,\delta))$, and write

$$P_{\boldsymbol{r}} = \sum_{M\boldsymbol{k}+\boldsymbol{r} \in V_\theta} P_{\boldsymbol{r}}^{(\boldsymbol{k})},$$

$$Q_{\boldsymbol{r}} = \sum_{M\boldsymbol{k}+\boldsymbol{r} \in V_\theta} Q_{\boldsymbol{r}}^{(\boldsymbol{k})}.$$

Then we have

$$\sum_{\boldsymbol{r} \in \{0,\ldots,M-1\}^d} \|Q_{\boldsymbol{r}} T_b^{st} P_{\boldsymbol{r}}\|_{S_p}^p \leqq M^d \|T_b^{st}\|_{S_p}^p.$$

We note that

$$Q_r T_b^{st} P_r = \sum_{Mk+r \in V_\theta} Q_r^{(k)} T_b^{st} P_r^{(k)}, \tag{39}$$

because when $k \neq j$, $Mk+r \in V_\theta$ and $Mj+r \in V_\theta$,

$$(Q_r^{(k)} T_b^{st} P_r^{(j)} f)\hat{}(\xi)$$

$$= \int \chi_{B(Mk+r+\theta,\delta)}(\xi) \hat{b}(\xi-\eta) A(\xi,\eta) |\xi|^s |\eta|^t \chi_{B(Mj+r,\delta)}(\eta) \hat{f}(\eta) d\eta.$$

Also if $\xi \in B(Mk+r+\theta,\delta)$, $\eta \in B(Mj+r,\delta)$, then

$$|\xi-\eta| = |\xi-(Mk+r+\theta)-\eta+(Mj+r)+\theta+M(k-j)|$$

$$\geqq M|k-j|-|\theta|-2\delta$$

$$> M-2.$$

Thus, since $\operatorname{supp}\hat{b} \subset \{|\xi| \leqq M-2\}$, we have

$$Q_r^{(k)} T_b^{st} P_r^{(j)} = 0.$$

Therefore, by Lemma 2, we have

$$\|Q_r T_b^{st} P_r\|_{S_p}^p$$

$$= \sum_{Mk+r \in V_\theta} \|Q_r^{(k)} T_b^{st} P_r^{(k)}\|_{S_p}^p$$

$$= \sum_{Mk+r \in V_\theta} \|\hat{b}(\xi-\eta) A(\xi,\eta) |\xi|^s |\eta|^t \chi_{B(Mk+r+\theta,\delta)}(\xi) \chi_{B(Mk+r,\delta)}(\eta)\|_{S_p}^p$$

$$\geqq \sum_{Mk+r \in V_\theta} \|\hat{b}(\xi-\eta) A(\xi,\eta) |\xi|^s |\eta|^t\|_{S_\infty(B(Mk+r+\theta,\delta)\times B(Mk+r,\delta))}^p$$

$$= \sum_{Mk+r \in V_\theta} a_{Mk+r}^p.$$

Finally, we get

$$\|T_b^{st}\|_{S_p}^p \geqq N^{-d} \sum_{r\in\{0,\ldots,M-1\}^d} \sum_{Mk+r\in V_\theta} a_{Mk+r}^p = M^{-d} \sum_{n\in V_\theta} a_n^p,$$

i.e. (38) holds.

This completes the proof of Theorem 4.

Corollary 3. *Suppose* $A(\xi,\eta)$ *satisfies* A10(α), $1 \leqq p \leqq d/(\alpha-s-t)$ *and* $b\in S'(\mathsf{R}^d)$ *such that* $T_b^{st}\in S_p$. *Then* b *must be a polynomial.*

Proof. When $p \geqq 1$, (38) always holds. Noting that the argument in the proof of Theorem 4 up to (38) does not need the assumption that b is such that $\hat{b}$ has compact support, it follows that Corollary 3 holds.

Acknowledgement. This paper is a part of the author's Ph.D. thesis, written at the University of Stockholm under the guidance of S. Janson, to whom he expresses his thanks.

REFERENCES

1. R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-differentiels*, (Astérisque 57), Société Mathématique de France, Paris, 1978.
2. S. Janson and J. Peetre, *Higher order commutators of singular integral operator*, in *Interpolation Spaces and Allied Topics in Analysis* (Proc. Lund, 1983) eds. M. Cwikel and J. Petre, (Lecture Notes in Math. 1070), pp. 125–142, Springer-Verlag, Berlin - Heidelberg - New York, 1984.
3. S. Janson and J. Peetre, *Paracommutators – boundedness and Schatten-von Newmann properties*, Report No. 15, Department of Mathematics, Stockholm, 1985.
4. C. A. McCarthy, C_p, Israel J. Math. 5 (1967), 249–271.
5. J. Peetre, *New Thoughts on Besov Spaces*, (Duke Univ. Math. Ser. 1), Duke University, Durham, N.C., 1976.
6. V. V. Peller, *Hankel operator of the Schatten-von Neumann class* S_p, $0 < p < 1$, Preprints E-6-82, LOMI, Leningrad, 1982.
7. L. Z. Peng, *On the compactness of paracommutators*, Report No. 17 Department of Mathematics, Stockholm, 1985,
8. L. Z. Peng, *Contributions to certain problems in paracommutators*, Doctoral Dissertation, Stockholm, 1986.
9. S. Semmes, *Trace ideal criteria for Hankel operators and applications to Besov space*, Integral Equations Operator Theory 7 (1984), 242–281.
10. A. Timotin, *A note on* C_p*-estimates for certain kernels*, Preprint No. 47, INCREST, Bucharest, 1984.
11. A. Timotin, C_p*-estimates for certain kernels: The case* $0 < p < 1$, Preprint No. 220, INCREST, Bucharest, 1985.
12. H. Triebel, *Theory of Functions Spaces*, (Monographs in Math. 78), Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.

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