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INTEGRAL REPRESENTATIONS ON CONVEX SEMIGROUPS

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0. Introduction.

Integral representations, especially if they are unique, play an important rôle in many parts of analysis. When collecting examples one realizes the existence of at least two larger subsets of theorems in this area, one dealing with the representation of positive definite (and related) functions on semigroups as mixtures of (semigroup-) characters, as f.ex. developed in great detail in [1], the other having to do with the representation of positive linear functionals as mixtures of multiplicative linear functionals. It is the purpose of the present paper to show how theorems of the latter type may be deduced straightforwardly from the main results of the former one. The key observation – enabling this reduction – is the fact that any affine moment function on what we shall call a *convex semigroup* (i.e. a semigroup with a compatible convex structure), is already a mixture of affine characters (Theorem 1). Combined with the fundamental theorem of Berg and Maserick about exponentially bounded positive definite functions, many important results for convex semigroups with neutral element become easily available; in particular a new – and as we believe very natural – proof of Riesz's representation theorem on compact spaces is obtained in this way.

In the last paragraph we generalize the notion of an exponentially bounded positive definite function to semigroups without neutral element and extend the Berg/Maserick theorem to this situation (Theorem 4). Applied to an algebra (as a special case of a convex semigroup) this yields the Plancherel-Godement representation for bitraces. Some examples treat different versions of Riesz's theorem on locally compact (non-compact) spaces.

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1. Affine moment functions as mixtures of affine characters.

Let $H = (H, \cdot, *)$ denote an abelian $*$-semigroup, i.e. H is a set equipped with an associative and commutative composition, written here as multiplication, and $*: H \to H$ is a map with $(xy)^* = x^*y^*$ and $(x^*)^* = x$ for $x, y \in H$. We do not require at the beginning that H has got a neutral element.

We will say that H is a *convex semigroup* if H is a convex subset of some (real or complex) linear space in such a way that the semigroup multiplication is compatible with the linear structure in the sense that $x(\alpha y + \beta z) = \alpha(xy) + \beta(xz)$ and $(\alpha x + \beta y)^* = \alpha x^* + \beta y^*$ for $x, y, z \in H$ and $\alpha, \beta \geqq 0$, $\alpha + \beta = 1$. We call H a *multiplicative cone* in case H is even a convex cone and the two equations just given hold for all $\alpha, \beta \geqq 0$. Finally H is a real (complex) algebra – always commutative! – if $x(\alpha y + \beta z) = \alpha(xy) + \beta(xz)$ and $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} z^*$ for all $x, y, z \in H$ and $\alpha, \beta \in \mathsf{R}(\mathsf{C})$. The set

$$H^* := \{\varrho : H \to \mathsf{C} \mid \varrho \not\equiv 0, \varrho(xy^*) = \varrho(x)\overline{\varrho(y)} \quad \text{for all } x, y \in H\}$$

of all *characters* (i.e. $*$-homomorphisms into $(\mathsf{C}, \cdot, -)$) is a completely regular Hausdorff space with regard to the topology of pointwise convergence; in case H contains a neutral element e, we have $\varrho(e) = 1$ for all $\varrho \in H^*$. The closed subset $H^{\circledast}$ of all *affine* characters – i.e. those $\varrho \in H^*$ for which $\varrho(\alpha x + (1-\alpha)y) = \alpha\varrho(x) + (1-\alpha)\varrho(y)$ for $x, y \in H$, $\alpha \in [0, 1]$ – will be of particular interest for us. With slightly imprecise notation, in case H is a multiplicative cone, $H^{\circledast}$ will denote all additive and positively homogeneous characters, and if H is an algebra, $H^{\circledast}$ will be the set of all linear characters. A function $\varphi : H \to \mathsf{C}$ is a *moment function* if there exists a Radon measure μ on some subspace A of H^* (in symbols: $\mu \in M_+(A)$) such that $\varphi(x) = \int_A \varrho(x) d\mu(\varrho)$ for all $x \in H$ in which case we say that φ is represented by μ. Here $\int_A |\varrho(x)| d\mu(\varrho) < \infty$ is assumed for all $x \in H$; if H contains a neutral element, then $\mu(A) < \infty$ and therefore μ can be thought of as a (finite) Radon measure on H^*.

THEOREM 1. *Any affine moment function φ on a convex semigroup is a mixture of affine characters. This holds also if we only assume that $\phi(x, y) := \varphi(xy^*)$ is bi-affine, i.e. ϕ is affine in one variable while the other remains fixed.*

More generally let $\mu \in M_+(A)$ be such that $\int |\varrho(x)|^2 d\mu(\varrho) < \infty$ for all $x \in H$ and assume that

$$\phi_z(x, y) := \int \varrho(xy^*) |\varrho(z)|^2 d\mu(\varrho)$$

is bi-affine for each $z \in H$, then already μ is concentrated on the affine characters in A.

A corresponding statement holds on multiplicative cones, respectively on algebras, if we replace "affine" by "additive and positively homogeneous", respectively "linear".

PROOF. We begin by assuming ϕ to be bi-affine. Let φ be represented by $\mu \in M_+(A)$, $\emptyset \neq A \subseteqq H^*$. Then for any $\{x_1, \ldots, x_n\} \subseteqq H$, $\{\lambda_1, \ldots, \lambda_n\} \subseteqq \mathbb{C}$ the function

$$\varrho \mapsto \left| \sum_{j=1}^{n} \lambda_j \varrho(x_j) \right|^2 = \sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \varrho(x_j x_k^*)$$

is μ-integrable. Specifically for $(\alpha, \beta) \in \{(a, b) \in \mathbb{R}_+^2 \,|\, a+b = 1\}$, respectively $(\alpha, \beta) \in \mathbb{R}_+^2\,(\mathbb{R}^2, \mathbb{C}^2)$ and $x, y \in H$ we get

$$\begin{aligned}
\infty &> \int |\varrho(\alpha x+\beta y)-\alpha\varrho(x)-\beta\varrho(y)|^2 d\mu(\varrho)\\
&= \int \{|\varrho(\alpha x+\beta y)|^2+|\alpha|^2|\varrho(x)|^2+|\beta|^2|\varrho(y)|^2+\\
&\quad +2\mathrm{Re}[-\varrho(\alpha x+\beta y)\bar{\alpha}\overline{\varrho(x)}-\varrho(\alpha x+\beta y)\bar{\beta}\overline{\varrho(y)}+\alpha\varrho(x)\bar{\beta}\overline{\varrho(y)}]\}d\mu(\varrho)\\
&= \phi(\alpha x+\beta y, \alpha x+\beta y)+|\alpha|^2\phi(x, x)+|\beta|^2\phi(y, y)+\\
&\quad +2\mathrm{Re}[-\bar{\alpha}\phi(\alpha x+\beta y, x)-\bar{\beta}\phi(\alpha x+\beta y, y)+\alpha\bar{\beta}\phi(x, y)]\\
&= 2|\alpha|^2\phi(x, x)+2|\beta|^2\phi(y, y)+\alpha\bar{\beta}\phi(x, y)+\bar{\alpha}\beta\phi(y, x)+\\
&\quad +2\mathrm{Re}[-|\alpha|^2\phi(x, x)-\bar{\alpha}\beta\phi(y, x)-\alpha\bar{\beta}\phi(x, y)-|\beta|^2\phi(y, y)+\alpha\bar{\beta}\phi(x, y)]\\
&= 0.
\end{aligned}$$

The open subset

$$G_{\alpha,\beta;x,y} := \{\varrho \in A \,|\, \varrho(\alpha x+\beta y) \neq \alpha\varrho(x)+\beta\varrho(y)\}$$

of A has therefore μ-measure zero, and so has – μ being a Radon measure –

$$G := \bigcup_{\substack{\alpha,\beta\\ x,y}} G_{\alpha,\beta;x,y}$$

where the union is taken over all $x, y \in H$ and the respective set of pairs of scalars. Since $A \cap H^{\circledast} = A \setminus G$ we are done.

Suppose now that only all the ϕ_z's are bi-affine. By what has been shown so far we get

$$\int_G |\varrho(z)|^2 d\mu(\varrho) = 0$$

for all $z \in H$, hence, denoting $\mathcal{O}_z := \{\varrho \in H^* | \varrho(z) \neq 0\}$, $\mu(G \cap \mathcal{O}_z) = 0$ for all z and therefore $\mu(G) = 0$, since $H^* = \bigcup_{z \in H} \mathcal{O}_z$.

2. Convex semigroups with neutral element.

We will begin with a few notions from harmonic analysis on semigroups. Let $S = (S, \cdot, *, e)$ be any abelian *-semigroup with neutral element e. A function $\varphi : S \to \mathbb{C}$ is called *positive definite* if

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(s_j s_k^*) \geqq 0$$

for all $n \in \mathbb{N}$, $\{s_1, \ldots, s_n\} \subseteqq S$ and $\{c_1, \ldots, c_n\} \subseteqq \mathbb{C}$. A function $\alpha : S \to \mathbb{R}_+$ is an *absolute value* iff $\alpha(e) \geqq 1$, $\alpha(st) \leqq \alpha(s)\alpha(t)$ and $\alpha(s^*) = \alpha(s)$ for all $s, t \in S$. We say that φ is *α-bounded* if $|\varphi(s)| \leqq C\alpha(s)$ for some $C \in \mathbb{R}_+$ and all $s \in S$; φ is *exponentially bounded* if it is bounded with respect to at least one absolute value. Let $\mathscr{P}(S)$ denote the set of all positive definite functions on S, $\mathscr{P}^\alpha(S)$ the subset of all α-bounded positive definite functions and $\mathscr{H}(S)$ the set of all moment functions. It is immediate that $S^* \subseteqq \mathscr{H}(S) \subseteqq \mathscr{P}(S)$, and in general $\mathscr{H}(S) \neq \mathscr{P}(S)$, but nevertheless the following very satisfactory and far reaching integral representation theorem holds (cf. [1, Theorem 4.2.6]):

THEOREM (Berg and Maserick). *If $\varphi : S \to \mathbb{C}$ is positive definite and α-bounded then there exists a uniquely determined Radon measure $\mu \in M_+(S^*)$ such that $\varphi(s) = \int \varrho(s) d\mu(\varrho)$ for $s \in S$. Furthermore the measure μ is concentrated on the compact subset*

$$S^\alpha := \{\varrho \in S^* \,|\, |\varrho| \leqq \alpha\}$$

of α-bounded characters.

If again S is a convex semigroup (respectively a multiplicative cone, respectively an algebra) we denote by $S^{@} := S^\alpha \cap S^{\circledast}$ the compact set of all α-bounded affine (respectively additive and positively homogeneous, respectively linear) characters. We get the important

COROLLARY 1. *Let $\varphi \in \mathscr{P}^\alpha(S)$ be affine (linear). Then the unique measure on S^* representing φ is concentrated on $S^{@}$.*

As a first application we shall derive a slight generalization of Raikov's theorem. Let S be a commutative seminormed algebra with unit e and an involution such that $\|s^*\| = \|s\|$.

THEOREM 2. *Any continuous positive linear functional on S (positivity meaning $L(ss^*) \geqq 0$ for all $s \in S$) is a unique mixture of continuous linear characters.*

PROOF. Given $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{C}$ put $s := \sum_{j=1}^n c_j s_j$, then

$$\sum c_j \bar{c}_k L(s_j s_k^*) = L(ss^*) \geqq 0,$$

i.e. L is positive definite. Since $\alpha(s) := \|s\|$ is an absolute value on S and $|L(s)| \leqq C\|s\|$ for some $C \geqq 0$ and all $s \in S$, L is α-bounded. The result now follows from Corollary 1.

EXAMPLE 1. (Riesz's representation theorem). Let $S = C(X)$ be the algebra of continuous real-valued functions on a compact Hausdorff space X. An easy direct argument, using only the very definition of compactness, shows that any linear character on S is a point evaluation on X, and this identification of $S^{\circledast}$ with X is even a topological one. If $L: C(X) \to \mathbb{R}$ is a positive linear functional, it is automatically continuous ($|L(f)| \leqq L(\|f\| \cdot 1) = \|f\| \cdot L(1)$), so $L(f) = \int f(x) d\mu(x)$ for a unique Radon measure μ on X.

It should be noted that this is really a new proof of Riesz's theorem which therefore may be considered as a special case of the integral representation of positive definite functions on semigroups. The proof of Berg and Maserick's theorem is based essentially on Krein-Milman's theorem and there the compactness of $M_+^1(X)$, the Radon probability measures on a compact space X, is an important ingredience in the proof. Of course this compactness is an immediate consequence of Riesz's theorem, and so it is usually presented in the literature. Some years ago Masani [7] pointed out that "the placement of the Riesz Theorem ahead of the Krein-Milman in the mathematical edifice is an architectural blemish" and gave an independent measure-theoretic proof of the compactness of $M_+^1(X)$ for compact X. Much shorter is the following argument: let X be compact and let (μ_α) be a net in $M_+^1(X)$, then for a suitable subnet (α_β) the limit

$$\nu(B) := \lim_{\beta} \mu_{\alpha_\beta}(B)$$

exists for every Borel set $B \subseteqq X$, and ν is a finitely additive probability content. Lemma 2.1.9 in [1] shows that

$$\lambda(C) := \inf\{\nu(G) | C \subseteqq G, G \text{ open}\},$$

defined for all compact subsets $C \subseteqq X$, is a Radon content, which Kisyński's fundamental measure extension theorem (cf. [1, Theorem 2.1.4]) extends (uniquely) to a Radon measure μ on X. Since X itself is compact we have $\mu(X) = \lambda(X) = \nu(X) = 1$, that is $\mu \in M_+^1(X)$, and from $\nu(G) \geqq \mu(G)$ for open subsets $G \subseteqq X$ we deduce that $\mu_{\alpha_\beta} \to \mu$ weakly (= vaguely); hence $M_+^1(X)$ is compact.

EXAMPLE 2. Let $S = C^b(X)$ be the algebra of bounded continuous real-valued functions on an arbitrary Hausdorff space X. There is a canonical mapping $\gamma : X \to S^{\circledast}$ defined by $\gamma_x(f) := f(x)$, which is obviously continuous and which is one-to-one if and only if X is completely Hausdorff (i.e. S separates the points in X); γ is a homeomorphism onto its image iff X is completely regular. The following interesting result we found in (unpublished) lecture notes of Bierstedt:

LEMMA 1. *For every Hausdorff space X the image $\gamma(X)$ is dense in $S^{\circledast}$.*

PROOF. We show first that $\varrho(f) \in \overline{f(X)}$ for $\varrho \in S^{\circledast}$, $f \in S$. If $\varrho(f) = 0 \notin \overline{f(X)}$ then $1/f \in S$ and $\varrho(f)\varrho(1/f) = 1$, a contradiction. In general we have $\varrho(f - \varrho(f)) = 0$, so $0 \in \overline{f(X) - \varrho(f)}$ or $\varrho(f) \in \overline{f(X)}$; in particular we see $|\varrho(f)| \leqq \|f\|$, showing $S^{\circledast}$ to be a subset of the unit ball in the dual of the Banach space S.

We have also $\varrho(f) \geqq 0$ for $f \geqq 0$, therefore $\varrho(|f|) \geqq 0$ and $(\varrho(|f|))^2 = (\varrho(f))^2 = \varrho(f^2)$, hence $\varrho(|f|) = |\varrho(f)|$.

Let now U be a neighbourhood of $\varrho_0 \in S^{\circledast}$, then for suitable $f_1, \ldots, f_n \in S$ and $\varepsilon > 0$

$$\{\varrho \mid |\varrho(f_j) - \varrho_0(f_j)| < \varepsilon, 1 \leqq j \leqq n\} \subseteqq U.$$

Put

$$g = \sum_{j=1}^{n} |f_j - \varrho_0(f_j)| \in S;$$

then $\varrho_0(g) = 0 \in \overline{g(X)}$, so for some $x \in X$ we have $g(x) < \varepsilon$, in particular $|f_j(x) - \varrho_0(f_j)| < \varepsilon$ for $j = 1, \ldots, n$, which means $\gamma_x \in U$.

COROLLARY 2. *Let X be a Hausdorff space and L: $C^b(X) \to \mathsf{R}$ a positive linear functional. Then there is a unique Radon measure μ on $\overline{\gamma(X)} = (C^b(X))^{\circledast}$ such that $L(f) = \int \varrho(f) d\mu(\varrho)$, $f \in C^b(X)$.*

REMARK. If X is completely regular, $S^{\circledast}$ equals βX; for let $f \in C^b(X)$ and define $\tilde{f} : S^{\circledast} \to \mathsf{R}$ by $\tilde{f}(\varrho) := \varrho(f)$, then $\tilde{f} \in C(S^{\circledast})$ and $\tilde{f}(\gamma_x) = f(x)$ for all $x \in X$.

EXAMPLE 3. (Integral representation of τ-positive functions.) Let S be any real or complex commutative algebra with unit e. A subset $\tau \subseteqq S$ is called *admissible* if

(i) $t^* = t$ for all $t \in \tau$,
(ii) $e - t \in \operatorname{cone}(\boldsymbol{S}(\tau))$ for all $t \in \tau$ (where $\boldsymbol{S}(\tau)$ denotes the (multiplicative) semigroup generated by τ, and cone (A) is the smallest convex cone containing A),
(iii) $S = \operatorname{lin}(\boldsymbol{S}(\tau))$.

A linear functional L on S is *τ-positive* iff $L|\boldsymbol{S}(\tau) \geqq 0$.

LEMMA 2. *Any τ-positive linear functional is positive, i.e.* $L(ss^*) \geqq 0$ *for each* $s \in S$.

This has been shown (without using integral representations!) by Maserick and Szafraniec [9]; their proof, involving Bernstein polynomials in two variables, is reproduced in [1, p. 125/126].

We define a function α on S by

$$\alpha(s) := \inf\{\textstyle\sum|\lambda_j| \mid s = \sum \lambda_j t_j, t_j \in \boldsymbol{S}(\tau)\}$$

which is finite by assumption. A routine argument shows $\alpha(st) \leqq \alpha(s)\alpha(t)$ and even that α is a seminorm; if $\alpha \not\equiv 0$ then $\alpha(e) \geqq 1$. Put

$$\Delta := \{\varrho \in S^{\circledast} \mid \varrho \text{ is } \tau\text{-positive}\}.$$

We have $\varrho(t) \geqq 0$ and $\varrho(e-t) \geqq 0$ for $\varrho \in \Delta$ and $t \in \tau$, hence $\varrho(t) \in [0, 1]$ for all $t \in \boldsymbol{S}(\tau)$ and therefore Δ is a compact subset of $S^{\circledast}$.

THEOREM 3 (Maserick, 1977). *Any τ-positive linear functional is a unique mixture of τ-positive linear characters.*

PROOF. Let $L: S \to \mathbf{C}$ be linear and τ-positive, then by Lemma 2, L is positive definite. It is easy to see that $e - t \in \operatorname{cone}(\boldsymbol{S}(\tau))$ holds also for elements $t \in \boldsymbol{S}(\tau)$, implying $|L(t)| \leqq L(e)$ for all $t \in \boldsymbol{S}(\tau)$. So if $s = \sum \lambda_j t_j$ with $t_j \in \boldsymbol{S}(\tau)$ we get $|L(s)| \leqq L(e) \cdot \sum|\lambda_j|$, hence is α-bounded. By Corollary 1 there is a unique measure $\mu \in M_+(S^{\circledast})$ so that $L(s) = \int \varrho(s)\,d\mu(\varrho)$, $s \in S$. It remains to see that $\operatorname{supp}(\mu) \subseteqq \Delta$.

Now for fixed $t \in \boldsymbol{S}(\tau)$ the functional $s \mapsto L(st)$ is also τ-positive and α-bounded, so that

$$L(st) = \int \varrho(s)\,d\mu_t(\varrho) = \int \varrho(s)\varrho(t)\,d\mu(\varrho), \quad s \in S,$$

where $\mu_t \in M_+(S^{\circledast})$, implying $d\mu_t(\varrho) = \varrho(t) d\mu(\varrho)$ and

$$0 \leqq (\mu_t + \mu_{t^*})(\{\varrho \mid \operatorname{Re} \varrho(t) < 0\}) = 2 \operatorname{Re} \int_{\{\varrho \mid \operatorname{Re} \varrho(t) < 0\}} \varrho(t) d\mu(\varrho),$$

hence $\mu(\{\varrho \mid \operatorname{Re} \varrho(t) < 0\}) = 0$ for all $t \in \boldsymbol{S}(\tau)$, and finally, μ being a Radon measure,

$$\mu\left(\bigcup_{t \in \boldsymbol{S}(\tau)} \{\varrho \mid \operatorname{Re} \varrho(t) < 0\} \right) = 0.$$

If, however, $\operatorname{Re} \varrho(t) \geqq 0$ for all $t \in \boldsymbol{S}(\tau)$ then already $\varrho(t) \geqq 0$ for $t \in \boldsymbol{S}(\tau)$, so that indeed μ is concentrated on Δ.

Let us mention a typical application of Maserick's theorem. If S is the algebra of polynomials in k real variables, then linear functionals L on S are in one-to-one correspondence with functions

$$f : \mathsf{N}_0^k \to \mathsf{R} \text{ via } L(x^n) = f(n), \quad n \in \mathsf{N}_0^k.$$

The set

$$\tau := \left\{ x_1, x_2, \ldots, x_k, 1 - \sum_{i=1}^{k} x_i \right\}$$

is admissible and $\varrho \in \Delta$ is equivalent with $\varrho(x^n) = a^n = \Pi a_j^{n_j}$ where $a_1, \ldots, a_k \geqq 0$ and $1 - \sum a_j \geqq 0$, so τ-positivity of $L \simeq f$ characterises the moment functions on the simplex

$$K := \left\{ a \in \mathsf{R}_+^k \;\middle|\; \sum_{i=1}^{k} a_i \leqq 1 \right\}.$$

EXAMPLE 4. Let T be an abelian $*$-semigroup with neutral element and consider $S = \operatorname{Mol}_+^1(T)$, the set of all probability measures on T with finite support. With respect to convolution and the induced involution $\sigma^* :=$ image of σ under the given involution on T, S becomes a convex semigroup. Let $\varrho \in S^{\circledast}$ and put $\eta(t) := \varrho(\varepsilon_t)$, $t \in T$, ε_t denoting the Dirac measure in t. Then $\eta \in T^*$ and $\varrho(\sigma) = \int \eta \, d\sigma$ for $\sigma \in S$, so that $S^{\circledast}$ may be identified with T^*. Our main result implies that any positive definite bounded affine function $\phi : \operatorname{Mol}_+^1(T) \to \mathsf{C}$ has a unique representation

$$\phi(\sigma) = \int_{\hat{T}} \left(\int_T \eta \, d\sigma \right) d\mu(\eta)$$

where $\mu \in M_+(\hat{T})$, $\hat{T}$ denoting the set of all bounded characters on T. If for example $T = \mathsf{R}_+$ and ϕ is furthermore continoous, then ϕ has a unique continuous extension to $M_+^1(\mathsf{R}_+)$, given by

$$\phi(\sigma) = \int_0^\infty \int_0^\infty e^{-\lambda t} d\sigma(t) d\mu(\lambda), \quad \sigma \in M_+^1(\mathsf{R}_+)$$

where μ is a finite measure on R_+.

EXAMPLE 5. Let $\mathscr{A}$ be an algebra of subsets of $\Omega \neq \emptyset$. The set

$$S := \{f : \Omega \to [0,1] \,|\, |f(\Omega)| < \infty, \{f = x\} \in \mathscr{A} \quad \text{for all } x \in [0,1]\}$$

of elementary $[0,1]$-valued $\mathscr{A}$-measurable functions is a convex semigroup. A finitely additive probability content (charge) ν induces the affine positive definite function $\varphi(f) = \int f d\nu$ on S. Since φ is bounded, it is a unique mixture of multiplicative charges; equivalently

$$\nu(A) = \int_M \varrho(A) d\mu(\varrho), \quad A \in \mathscr{A}$$

where $M := \{\varrho : \mathscr{A} \to [0,1] \,|\, \varrho$ is additive, $\varrho(A \cap B) = \varrho(A)\varrho(B)$ for all $A, B \in \mathscr{A}$, $\varrho(\Omega) = 1\}$.

3. Convex semigroups without neutral element.

Let $H = (H, \cdot, *)$ denote an abelian $*$-semigroup without neutral element, written multiplicatively, however, without assuming for the time being any additional linear structure. Motivated by the notion of a bitrace on an algebra we shall say that $\phi : H \times H \to \mathsf{C}$ is a *positive definite bi-function* if ϕ is a positive definite kernel with the property $\phi(xy, z) = \phi(x, y^* z)$ for $x,y,z \in H$. If α is any absolute value on H (that is $\alpha : H \to \mathsf{R}_+$ is submultiplicative and $\alpha(x^*) = \alpha(x)$ for all $x \in H$) then ϕ is called *α-bounded* if for some nonnegative function b on H we have

$$\sqrt{\phi(xy, xy)} \leqq \alpha(x) b(y), \quad x,y \in H. \tag{1}$$

(Note that in case H contained a neutral element e, we had $\phi(x,y) = \varphi(xy^*)$ with $\varphi : H \to \mathsf{C}$ defined by $\varphi(x) = \phi(x, e)$, and because of

$$|\varphi(x)| \leqq [\phi(x,x)\phi(e,e)]^{1/2} \leqq b(e)[\phi(e,e)]^{1/2}\alpha(x),$$

φ would be α-bounded in the sense defined before.) Again we will say that ϕ is *exponentially bounded* if ϕ is α-bounded for at least one absolute value α. With some symbol $e \notin H$ we put $S := H \cup \{e\}$ and make S to a semigroup with neutral element by the obvious definitions $ex = xe = x$, $ee = e^* = e$. Any absolute value α on H extends to one on S by $\alpha(e) := 1$, so that S^α again is a compact subset of S^*, containing the special character $\theta := 1_{\{e\}}$, which by the way is an absorbing element in the dual semigroup S^*, that is $\theta\varrho = \theta$ for all $\varrho \in S^*$. The locally compact space $H^\alpha := S^\alpha \setminus \{\theta\}$, consisting of all α-bounded characters on H, not identically zero, will get importance in the sequel. Without restriction we shall always assume that the bounding function b in (1) fulfills $b(y) \geqq [\phi(y, y)]^{1/2}$ for all $y \in H$, extending by this device the validity of (1) to $x \in S$, $y \in H$.

THEOREM 4. *For any α-bounded positive definite bi-function ϕ on $H \times H$ there exists a unique Radon measure $\mu \in M_+(H^\alpha)$ such that*

$$\phi(xy, z) = \int \varrho(xyz^*)d\mu(\varrho), \quad x,y,z \in H. \tag{2}$$

The inequalities

$$\int |\varrho(x)|^2 d\mu(\varrho) \leqq \phi(x, x), \quad x \in H \tag{3}$$

and

$$[\phi(xy, xy)]^{1/2} \leqq \alpha(x)[\phi(y, y)]^{1/2}, \quad x,y \in H \tag{4}$$

are fulfilled.

Let $K \subseteq \mathbf{C}^H$ denote the reproducing kernel Hilbert space of ϕ and let $\Phi: H \to K$ be the canonical map given by $(\Phi(x))(y) := \phi(x, y)$. Then if $\Phi(HH)$ is total in K one also has

$$\phi(x, y) = \int \varrho(xy^*)d\mu(\varrho), \quad x,y \in H. \tag{5}$$

(*This holds in particular if $HH = H$, but is contained in this case already in* (2).)

PROOF. Let $\{x_1, \ldots, x_n\} \subseteq H$ and $\{c_1, \ldots, c_n\} \subseteq \mathbf{C}$ be given. Define $\varphi: S \to \mathbf{C}$ by

$$\varphi(s) := \sum_{j,k=1}^{n} c_j \bar{c}_k \phi(sx_j, x_k);$$

we shall see that $\varphi \in \mathscr{P}^\alpha(S)$: for $\{s_1, \ldots, s_m\} \subseteq S$ and $\{d_1, \ldots, d_m\} \subseteq \mathbf{C}$ we have

$$\sum_{p,q=1}^{m} d_p\overline{d_q}\varphi(s_p s_q^*) = \sum_{j,k=1}^{n} \sum_{p,q=1}^{m} c_j d_p \overline{c_k d_q} \phi(s_p s_q^* x_j, x_k)$$

$$= \sum_{j,k=1}^{n} \sum_{p,q=1}^{m} c_j d_p \overline{c_k d_q} \phi(x_j s_p, x_k s_q) \geqq 0,$$

and

$$|\varphi(s)| \leqq \sum_{j,k=1}^{n} |c_j||c_k|[\phi(sx_j, sx_j)]^{1/2}[\phi(x_k, x_k)]^{1/2}$$

$$\leqq \alpha(s)\left\{\sum_{j=1}^{n} |c_j| b(x_j)\right\}\left\{\sum_{k=1}^{n} |c_k|[\phi(x_k, x_k)]^{1/2}\right\}.$$

By Berg's and Maserick's theorem there is a unique measure in $M_+(S^\alpha)$ representing φ. Applying this fact for $n = 1, 2$ we find families of measures $\mu_x, \nu_{x,y}, \eta_{x,y} \in M_+(S^\alpha)$ such that

$$\phi(sx, x) = \int \varrho(s) d\mu_x(\varrho),$$

$$\phi(sx, x) + \phi(sy, y) - \phi(sx, y) - \phi(sy, x) = \int \varrho(s) d\nu_{x,y}(\varrho),$$

$$\phi(sx, x) + \phi(sy, y) + i[\phi(sx, y) - \phi(sy, x)] = \int \varrho(s) d\eta_{x,y}(\varrho)$$

for $x, y \in H$ and $s \in S$.

We define complex-valued Radon measures $\mu_{x,y}$ by

$$\mu_{x,y} := \tfrac{1}{2}(\mu_x + \mu_y - \nu_{x,y}) + \frac{i}{2}(\mu_x + \mu_y - \eta_{x,y})$$

and obtain

$$\int \varrho(s) d\mu_{x,y}(\varrho) = \tfrac{1}{2}[\phi(sx, y) + \phi(sy, x)] + \frac{i}{2}(-i)[\phi(sx, y) - \phi(sy, x)]$$

$$= \phi(sx, y),$$

so in particular $\mu_{x,x} = \mu_x$. The equality

$$\text{(6)} \qquad \int \varrho(sxy^*) d\mu_{u,v}(\varrho) = \phi(sxy^*u, v) = \phi(sxu, yv) = \phi(suv^*x, y)$$

$$= \int \varrho(suv^*) d\mu_{x,y}(\varrho)$$

gives (for $x = y$, $u = v$)

$$|\varrho(x)|^2 d\mu_u(\varrho) = |\varrho(u)|^2 d\mu_x(\varrho).$$

Hence, denoting $\mathcal{O}_x := \{\varrho \in S^\alpha | \varrho(x) \neq 0\}$,

$$\frac{d\mu_u(\varrho)}{|\varrho(u)|^2} = \frac{d\mu_x(\varrho)}{|\varrho(x)|^2} \quad \text{on} \quad \mathcal{O}_x \cap \mathcal{O}_u.$$

Thus we obtain a (unique) Radon measure $\mu \in M_+(H^\alpha)$ with

$$d\mu(\varrho) = \frac{d\mu_x(\varrho)}{|\varrho(x)|^2} \quad \text{on} \quad \mathcal{O}_x, \quad x \in H,$$

taking into account that $\{\mathcal{O}_x | x \in H\}$ is an open covering of H^α. The equation (6) for $u = v$ implies $\varrho(xy^*)d\mu_u(\varrho) = |\varrho(u)|^2 d\mu_{x,y}(\varrho)$, so

$$\varrho(xy^*)d\mu(\varrho) = d\mu_{x,y}(\varrho)$$

on $\mathcal{O}_u$, all $u \in H$, hence on the whole space H^α.

For $x,y,z \in H$ we have

$$\phi(xy,z) = \int_{S^\alpha} \varrho(x)d\mu_{y,z}(\varrho) = \int_{H^\alpha} \varrho(x)d\mu_{y,z}(\varrho)$$

$$= \int_{H^\alpha} \varrho(x)\varrho(yz^*)d\mu(\varrho) = \int_{H^\alpha} \varrho(xyz^*)d\mu(\varrho)$$

as asserted.

Inequality (3) follows from

$$\phi(x,x) = \mu_x(S^\alpha) = \mu_x(\{\theta\}) + \mu_x(H^\alpha) \geqq \mu_x(H^\alpha) = \int |\varrho(x)|^2 d\mu(\varrho),$$

and then

$$\phi(xy,xy) = \int_{H^\alpha} |\varrho(x)|^2 |\varrho(y)|^2 d\mu(\varrho) \leqq \alpha^2(x) \int |\varrho(y)|^2 d\mu(\varrho) \leqq \alpha^2(x)\phi(y,y)$$

which shows (4).

For $\{x_1, \ldots, x_n\} \subseteq H$ and $\{c_1, \ldots, c_n\} \subseteq \mathbf{C}$ the complex-valued Radon measure $\sum_{j,k=1}^{n} c_j \bar{c}_k \mu_{x_j, x_k}$ has the Laplace transform $\sum c_j \bar{c}_k \phi(sx_j, x_k)$ which is positive definite, as we saw above. Hence this measure is in fact positive. Therefore

$$\phi(x, y) - \int \varrho(xy^*) d\mu(\varrho) = \mu_{x,y}(S^\alpha) - \mu_{x,y}(H^\alpha) = \mu_{x,y}(\{\theta\})$$

is also a positive definite kernel. This yields – generalizing (3) – the inequality

$$\int \left| \sum_{j=1}^{n} c_j \varrho(x_j) \right|^2 d\mu(\varrho) \leqq \left\| \sum_{j=1}^{n} c_j \Phi(x_j) \right\|_K^2, \tag{7}$$

the difference between both sides being $\sum_{j,k=1}^{n} c_j \bar{c}_k \mu_{x_j, x_k}(\{\theta\})$.

Suppose now $\Phi(HH)$ to be total in K. Let $x \in H$ and $\varepsilon > 0$. Then for suitable $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\} \subseteq H$ and $\{c_1, \ldots, c_n\} \subseteq \mathbf{C}$ we have

$$\delta := \left\| \Phi(x) - \sum_{j=1}^{n} c_j \Phi(x_j y_j) \right\|_K < \varepsilon.$$

The triangle inequality in normed spaces and (7) give us

$$\left| \sqrt{\int |\varrho(x)|^2 d\mu(\varrho)} - \sqrt{\int |\sum c_j \varrho(x_j y_j)|^2 d\mu(\varrho)} \right|$$

$$\leqq \sqrt{\int |\varrho(x) - \sum c_j \varrho(x_j y_j)|^2 d\mu(\varrho)} \leqq \delta$$

as well as

$$\left| \sqrt{\phi(x, x)} - \sqrt{\sum_{j,k=1}^{n} c_j \bar{c}_k \phi(x_j y_j, x_k y_k)} \right|$$

$$= \left| \|\Phi(x)\|_K - \left\| \sum_{j=1}^{n} c_j \Phi(x_j y_j) \right\|_K \right| \leqq \delta.$$

Hence, observing $\sum c_j \bar{c}_k \phi(x_j y_j, x_k y_k) = \int |\sum c_j \varrho(x_j y_j)|^2 d\mu(\varrho)$, we obtain

$$\left| \sqrt{\int |\varrho(x)|^2 d\mu(\varrho)} - \sqrt{\phi(x, x)} \right| < 2\varepsilon.$$

Since ε was arbitrary we get

$$\phi(x, x) = \int |\varrho(x)|^2 d\mu(\varrho),$$

that is, $\mu_{x,x}(\{\theta\})(=\mu_x(\{\theta\})) = 0$. Therefore, using once more positive definiteness, $\mu_{x,y}(\{\theta\}) = 0$ for all $x,y \in H$, and we conclude

$$\phi(x, y) = \int \varrho(xy^*)d\mu(\varrho).$$

We still have to show the unicity of μ. Thus suppose $\mu' \in M_+(H^\alpha)$ also represents ϕ. Then for $x,y \in H$

$$\phi(yx, x) = \int \varrho(y)|\varrho(x)|^2 d\mu(\varrho) = \int \varrho(y)|\varrho(x)|^2 d\mu'(\varrho),$$

which implies $|\varrho(x)|^2 d\mu(\varrho) = |\varrho(x)|^2 d\mu'(\varrho)$ on H^α. Hence $\mu = \mu'$ on $\mathcal{O}_x$ for all $x \in H$ and so finally $\mu = \mu'$.

REMARKS 1. Equation (5) does not always hold: let H be the *additive* semigroup $[1, \infty[$ (and $S = \{0\} \cup [1, \infty[$). The kernel $\phi(x, y) = 1_{\{2\}}(x+y)$ is a bounded positive definite bi-function for which $\mu_{1,1} = \varepsilon_\theta$. Here $\alpha \equiv 1$,

$$S^\alpha \equiv \hat{S} = \{\exp(-\lambda \cdot)|0 \leqq \lambda \leqq \infty\} \simeq [0, \infty]$$

where $\theta = 1_{\{0\}}$ corresponds to $\lambda = \infty$, so $H^\alpha = \hat{S} \setminus \{\theta\} \simeq [0, \infty[$. A representation of the form

$$\phi(x, y) = \int \exp(-\lambda(x+y))d\mu(\lambda)$$

is, however, definitely impossible. Nevertheless $\phi(x+y, z) = 0$ is represented by the zero measure.

2. Although we have assumed H to be without neutral element, no use of this fact has been made. Therefore the Berg/Maserick theorem is formally contained in Theorem 4. In case H has already a neutral element H^α itself is compact, θ being an isolated point in S^α.

3. The condition $HH = H$ means that H contains no *maximal elements* (cf. Choquet [3, p. 262]). This holds obviously if H contains a neutral element. In many cases this property will be easy to check, but it may also be a non-trivial fact, as e.g. for $H = L^1(G)$ under convolution – G denoting a non-discrete locally compact abelian group – where the result $HH = H$ is known as Cohen's factorisation theorem.

4. If $\phi(x, y) = \varphi(xy^*)$ for some positive definite function $\varphi: HH \to \mathbf{C}$, then ϕ is a positive definite bi-function, since

$$\phi(xy, z) = \varphi(xyz^*) = \varphi(x(y^*z)^*) = \phi(x, y^*z).$$

Theorem 4 shows in particular that in case $\Phi(HH)$ is total in K, any exponentially bounded positive definite bi-function is obtained in this way (that is $\phi(x, y) = \phi(u, v)$ if $xy^* = uv^*$), a property not clear at the outset. In general there are positive definite bi-functions not factorisable over a positive definite function, cf. [4, problème XV.9.2]).

5. The measure μ in the representation of Theorem 4 is finite if and only if ϕ has a positive definite α-bounded extension to S, i.e. there is some $\varphi \in \mathscr{P}^\alpha(S)$ such that $\phi(xy, z) = \varphi(xyz^*)$ for $x,y,z \in H$.

6. The first version of Theorem 4 has been given in [10], however only for semigroups with identical involution and only for the special absolute value $\alpha \equiv 1$; cf. also Chapter 8 in [1], where functions bounded with regard to this α were called *quasibounded*. Stochel [11] proved the representation (5) under the stronger boundedness condition $\phi(xy, xy) \leqq c(x)\phi(y, y)$ for some non-negative function c on H. Defining

$$\alpha^2(x) := \min\{\beta \geqq 0 \,|\, \phi(xy, xy) \leqq \beta\phi(y, y) \quad \text{for all } y \in H\}$$

it is immediately seen that α is submultiplicative, and ϕ is α-bounded with the special bounding function $b(y) = [\phi(y, y)]^{1/2}$. It is a non-trivial fact that as a consequence of Theorem 4 any α-bounded positive definite bi-function ϕ allows finally this bound. Stochel's proof is based on a general spectral theorem.

From now on we shall again assume that H denotes a convex semigroup, without neutral element.

THEOREM 5. *Let ϕ be an α-bounded bi-affine positive definite bi-function on $H \times H$, where H is a convex semigroup. Then the unique measure $\mu \in M_+(H^\alpha)$ representing ϕ is concentrated on $H^{\circledast}$, the non-zero affine α-bounded characters on H.*

A corresponding result holds for multiplicative cones and for algebras.

The proof is an immediate consequence of Theorems 1 and 4.

In case H is a commutative algebra, a kernel ϕ with the properties stated in Theorem 5 is usually called a *bitrace* (sometimes with the additional requirement $\phi(y^*, x^*) = \phi(x, y)$ which holds necessarily, as the above proof shows), and in this case the representation of Theorem 5 in the form of equation (5) is known as the *Plancherel-Godement theorem*, cf. [6] or [4, Théorème 15.9.2.]. Bauer [2, Satz 2] was the first who proved a special case of Theorem 5 in the form of equation (2), for point separating real sub-algebras of $C^0(X)$, X denoting a locally compact space. I am indebted to him for pointing out his result, which gave me the idea to prove the representation (2) for general abelian semigroups without any further assumption.

In the examples which follow some other versions of Riesz's representation theorem will be derived in a straightforward manner.

EXAMPLE 6. Let X be locally compact and non-compact and consider the algebra $H = C^0(X)$ of all real-valued continuous functions on X vanishing at infinity. Here, as in the next two examples, the involution is the identity. We have $HH = H$, since e.g. $f = \sqrt{|f|}\cdot(\sqrt{|f|}\cdot \operatorname{sgn}(f))$.

Let $L: H \to \mathsf{R}$ be a positive linear functional. Then L is positive definite and because of $L(f^2g^2) \leqq \|g\|^2 L(f^2)$ it is also α-bounded with $\alpha(g) = \|g\|$. Therefore

$$L(f) = \int \varrho(f)d\mu(\varrho), \quad f \in H,$$

for some $\mu \in M_+(H^{\circledast})$ by Theorem 5. Let $\varrho \in H^{\circledast}$. Then ϱ is a non-zero multiplicative linear functional on H. Denote by $Y := X \cup \{\omega\}$ the one-point compactification of X and define $\tilde{\varrho}: C(Y) \to \mathsf{R}$ by

$$\tilde{\varrho}(\tilde{f}) := \varrho(\tilde{f}|X - \tilde{f}(\omega)) + \tilde{f}(\omega).$$

An easy calculation shows $\tilde{\varrho}$ to be multiplicative and linear, too, so for some $y \in Y$ we have $\tilde{\varrho}(\tilde{f}) = \tilde{f}(y)$ for all $\tilde{f} \in C(Y)$. The case $y = \omega$ is ruled out, since ϱ is not identically zero. Therefore $y \in X$, and in this way we have identified $H^{\circledast}$ with X, i.e.

$$L(f) = \int_X f(x)d\mu(x), \quad f \in H,$$

where $\mu \in M_+(X)$. Since L is finite on all functions in $C^0_+(X)$, it is easy to see that $\mu(X) < \infty$.

EXAMPLE 7. Instead of $C^0(X)$ let us consider $H = C^c(X)$, the continuous functions with compact support. Here again $HH = H$ and a positive linear functional L on H is automatically α-bounded, again with $\alpha(g) = \|g\|$. We will see that any $\varrho \in H^{\circledast}$ has a unique extension $\tilde{\varrho} \in (C^0(X))^{\circledast}$. Indeed, given $f \in C^0(X)$ · there is a sequence $\{f_n\} \subseteqq C^c(X)$ converging uniformly to f. Then

$$|\varrho(f_n) - \varrho(f_m)| = |\varrho(f_n - f_m)| \leqq \|f_n - f_m\| \to 0$$

as $n,m \to \infty$, so $\tilde{\varrho}(f) := \lim \varrho(f_n)$ exists (and is independent of the chosen sequence $\{f_n\}$), and certainly $\tilde{\varrho}$ is also multiplicative and linear. Again we may therefore identify $H^{\circledast}$ with the given space X and get the well-known and important representation

$$L(f) = \int f d\mu, \quad f \in C^c(X),$$

where this time the Radon measure $\mu \in M_+(X)$ may of course be unbounded.

Our last example will concern the algebra $C(X)$ of all real-valued continuous functions on some Hausdorff space X. This is of course a semigroup *with* neutral element but we have to use some of the preceding results of this paragraph as well. To begin with we will give a general result whose proof is due to J. P. R. Christensen.

THEOREM 6. *Let S be any algebra of real-valued functions such that $f \in S$ and $h \in C(\mathsf{R})$ implies $h \circ f \in S$. Then any positive linear functional on S is represented by a unique Radon measure with compact support on $S^{\circledast}$.*

PROOF. Let us first consider the case $S = C(X)$, X denoting a locally compact and σ-compact Hausdorff space. From Example 6 we know that the restriction of a given positive linear functional L on S to $C^0(X)$ is given by a bounded Radon measure μ on X. If $f \in C_+(X)$ and $K \subseteqq X$ is compact, choose $g \in C^c(X)$ such that $1_K \leqq g \leqq 1$. Then

$$\int_K f d\mu \leqq \int fg d\mu = L(fg) \leqq L(f),$$

hence $\int f d\mu \leqq L(f) < \infty$. Since X is σ-compact there is a strictly positive function $p \in C^0(X)$. For $\varepsilon > 0$ there is a $g \in C^c(X)$ such that $1_{\{p \geqq \varepsilon\}} \leqq g \leqq 1$ implying

$$L(f) = L(fg) + L(f(1-g)) = \int fg d\mu + L(f(1-g)) \leqq \int f d\mu + \varepsilon L(f/p),$$

whence $L(f) \leqq \int f d\mu$, that is L is indeed represented by μ. Using once more the σ-compactness of X the support of μ turns out to be compact, cf. [4, 13.19.3].

Let now S be any algebra of real functions, stable under the composition with continuous functions on the real line. We must show that a given positive linear functional L on S is exponentially bounded. Fix $f \in S$ and consider $\varphi_f : C(\mathsf{R}) \to \mathsf{R}$, defined by $\varphi_f(h) := L(h \circ f)$. Since the real line is σ-compact, there is a measure $\mu_f \in M_+(\mathsf{R})$ with compact support representing φ_f. Put

$$\alpha(f) := \sup\{|t| \mid t \in \operatorname{supp}(\mu_f)\}.$$

With $h_k(t) := t^k$ we have for $f, g \in S$

$$\begin{aligned}\alpha(fg) &= \lim_{n\to\infty} \sqrt[2n]{\int t^{2n} d\mu_{fg}(t)} \\ &= \lim_{n\to\infty} \sqrt[2n]{L(h_{2n}\circ(fg))} \\ &= \lim_{n\to\infty} \sqrt[2n]{L(f^{2n}\cdot g^{2n})} \\ &\leqq \lim_{n\to\infty} \sqrt[4n]{L(f^{4n})}\sqrt[4n]{L(g^{4n})} \\ &= \alpha(f)\alpha(g)\end{aligned}$$

where we used the Cauchy-Schwarz inequality for L as well as the fact, that L^p-norms approach the L^∞-norm on any finite measure space. Furthermore $\mu_f(\mathsf{R}) = L(1)$ for all $f \in S$ and

$$|L(f)| = |L(h_1 \circ f)| = \left| \int_{-\infty}^{\infty} t d\mu_f(t) \right| \leqq \int_{-\infty}^{\infty} |t| d\mu_f(t) \leqq L(1)\alpha(f).$$

We see that α is an absolute value on S with respect to which L is bounded. The proof is finished by an application of Corollary 1.

EXAMPLE 8. Consider $S = C(X)$, X denoting some Hausdorff space. Again $\gamma: X \to S^{\circledast}$ denotes the canonical map associating with a point $x \in X$ the corresponding evaluation, i.e. $\gamma_x(f = f(x)$. Just as in Example 2, $\gamma(X)$ is dense in $S^{\circledast}$, and this even in a stronger sense: imitating the proof of Lemma 1 one gets $\varrho(f) \in f(X)$ for $\varrho \in S^{\circledast}$ and $f \in S$, and given some $\varrho \in S^{\circledast}$ there is for any sequence $f_1, f_2, \ldots \in S$ a point $x \in X$ with $\varrho(f_j) = f_j(x)$, $j = 1, 2, \ldots$. In case X is completely regular, $S^{\circledast}$ may be identified with *Hewitt's real-compactification* vX of X – up to a homeomorphism the unique real-compact space containing X as a dense subset to which every $f \in C(X)$ may be continuously extended (by $\bar{f}(\varrho) := \varrho(f)$ of course); cf. [5, 3.11 and 3.12]. As a corollary of Theorem 6 we may state that any positive linear functional L on $C(X)$ is represented by

$$L(f) = \int_{vX} \bar{f}(x) d\mu(x)$$

where μ is a Radon measure with compact support on vX.

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