MATH. SCAND. 61 (1987), 112–116

THE RESTRICTION ALGEBRA $A(\Gamma)$ FOR CURVES $\Gamma \subset \mathsf{R}^n$

ENJI SATO

Let $E \subset \mathsf{R}^n$ be a compact set, $A(\mathsf{R}^n) = \mathscr{F}L^1(\mathsf{R}^n)$ the Fourier algebra on R^n, and $A(E) = A(\mathsf{R}^n)/I(E)$ with the norm $\|\cdot\|_{A(E)}$, where $I(E)$ is the ideal in $A(\mathsf{R}^n)$ of all functions, vanishing on E. Also let $\psi = (\psi_1, \ldots, \psi_m)$ belong to $C^\infty(\mathsf{R}^n, \mathsf{R}^m)$, for $\eta, \xi \in \mathsf{R}^m$, $\eta \cdot \xi$ denote Euclid inner product on R^m, and correspondingly $\eta \cdot \psi(x) = \sum_{j=1}^m \eta_j \psi_j(x)$. Further let $|\eta| = (\sum_{j=1}^m |\eta_j|^2)^{1/2}$ for $\eta \in \mathsf{R}^m$.

Now for $n = 1$ and a compact interval $I \subset \mathsf{R}$, D. Müller [3] investigated the asymptotic behaviour of

$$\int \theta(x) \exp(2\pi i \eta \cdot \psi(x)) dx$$

as $\eta = (\eta_1, \ldots, \eta_m)$ tends to infinity for $\theta \in C^\infty(I, \mathsf{R})$ and $\operatorname{supp}\theta \subset \operatorname{int} I$. Then he obtained the following result.

THEOREM 1 (cf. [3, Corollary 1]). *The following conditions are equivalent*:

(i) *For each compact interval* $J \subset \operatorname{int} I$, *there are constants* $C_1 > 0$, $C_2 > 0$ *such that for all* $\eta \in \mathsf{R}^m$ $C_1(1+|\eta|)^{1/2} \leqq \|\exp(i\eta \cdot \psi)\|_{A(J)} \leqq C_2(1+|\eta|)^{1/2}$.

(ii) $\psi_1, \ldots, \psi_m$ *are linearly independent modulo affine linear functions on every nonempty compact subinterval of* $\operatorname{int} I$.

We shall give a generalization of the above result by the method of Y. Domar [1], [2].

DEFINTION 2. Let $I \subset \mathsf{R}$ be a compact interval with $\operatorname{int} I \neq \emptyset$. Then we define a curve

$$\Gamma_I = \{\gamma(x) = (x, g(x)) \mid x \in I\} \subset \mathsf{R}^n$$

with $g \in C^\infty(I, \mathsf{R}^{n-1})$.

DEFINITION 3. For $\psi = (\psi_1, \ldots, \psi_m)$, $\psi_1, \ldots, \psi_m$ are linearly independent modulo affine linear functions on Γ_I, if

$$\eta \cdot \psi(\gamma(x)) = c_1 g_1(x) + \ldots + c_{n-1} g_{n-1}(x) + c_n x + c_{n+1}$$

Received April 20, 1986.

on I, where $\eta \in \mathbb{R}^m$, $g = (g_1, \ldots, g_{n-1})$, $c_j \in \mathbb{R}$, $j = 1, \ldots, n+1$ implies $\eta = 0$ and $c_j = 0$, $j = 1, \ldots, n+1$.

REMARK 4. Under the notation of Definition 2, the next results are equivalent:

(i) $\gamma'(x), \ldots, \gamma^{(n)}(x)$ are linearly independent for all $x \in I$.
(ii) The torsion $\tau(x) = \det(g_j^{(k+1)}(x))_{k,j=1,\ldots,n-1} \neq 0$ for all $x \in I$.

THEOREM 5. *Let I be a compact interval, and*

$$\Gamma_I = \{\gamma(x) = (x, g(x)) \mid x \in I\},$$

where $g \in C^\infty(I, \mathbb{R}^{n-1})$, and $g = (g_1, \ldots, g_{n-1})$. Also we assume

$$\tau(x) = \det(g_j^{(k+1)}(x))_{k,j=1,\ldots,n-1} \neq 0$$

for all $x \in I$. Then for any $\psi \in C^\infty(\Gamma_I, \mathbb{R}^m)$ (i.e. *each component of $\psi \circ \gamma$ is a C^∞-function on I*), *there exists a constant $C > 0$ such that*

$$\|\exp(i\eta \cdot \psi)\|_{A(\Gamma_I)} \leqq C(1 + |\eta|)^{1/(n+1)} \quad \textit{for all } \eta \in \mathbb{R}^m.$$

PROOF. Let ε be a positive number. Then we may assume that γ is a function on $[-\varepsilon, 1+\varepsilon]$ and $\gamma([0,1]) = \Gamma_I$. Also we define $f_\eta = \eta \cdot \psi / |\eta|$, and we obtain $\exp(i\eta \cdot \psi) = \exp(i|\eta| f_\eta)$. Then by the method of [1], we can prove the above result. We omit the details.

THEOREM 6. *In the notation of Theorem 5, we assume both the condition $\tau(x) \neq 0$ for all $x \in I$ and that $\psi_1, \ldots, \psi_m$ are linearly independent modulo affine function on Γ_J for all $J \subset I_0$, where I_0 is some subinterval of I. Then there exists a constant $C > 0$ such that*

$$\|\exp(i\eta \cdot \psi)\|_{A(\Gamma_I)} \geqq C(1 + |\eta|)^{1/(n+1)} \quad \textit{for all } \eta \in \mathbb{R}^m.$$

PROOF. We will use the method in the proof of [2, Theorem 2.1]. We define $h_\eta(x) = \eta \cdot \psi(\gamma(x))$ for $\eta \in S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ and $x \in I$. By $\tau(x) \neq 0$ for all $x \in I$ and Definition 3,

$$A_\eta(x) = \det \begin{pmatrix} g_1''(x), \ldots, g_{n-1}''(x), h_\eta''(x) \\ \cdots\cdots\cdots\cdots \\ g_1^{(n+1)}(x), \ldots, g_{n-1}^{(n+1)}(x), h_\eta^{(n+1)}(x) \end{pmatrix}$$

satisfies $A_\eta(x_0) \neq 0$ for at least some $x_0 \in I_0$ (cf. [2, Theorem 2.1]). Since $A_\eta(x)$ is a continuous function on $S^{n-1} \times I$, there exist two positive numbers ε and δ such that for an arbitrary $\eta \in S^{n-1}$, there exists an interval $J_\eta (|J_\eta| = 2\varepsilon)$ such that $|A_\eta(x)| \geqq \delta$ for all $x \in J_\eta$.

Now let ϕ be in $C^\infty(I, \mathbb{R})$ with $\operatorname{supp}\phi \subset (-\varepsilon, \varepsilon)$, $\phi \geqq 0$ and $\int\phi = 1$. For any $\eta \in \mathbb{R}^m$ ($\neq 0$), we define $\eta' = \eta/|\eta|$. So we choose $J_{\eta'}$, with the above property. Also let $\tilde{\phi}$ be a translation of ϕ with $\operatorname{supp}\tilde{\phi} \subset J_{\eta'}$, and $d\mu_{\eta'}(x, g(x)) = \tilde{\phi}(x)dx$ in the Borel measures on $\mathbb{R}^n$. By the definition of $\mu_{\eta'}$

$$(1) \qquad 1 = \int_{\Gamma_I} \exp(i\eta \cdot \psi(x_1, \ldots, x_n)) \exp(-i\eta \cdot \psi(x_1, \ldots, x_n)) d\mu_{\eta'}(x_1, \ldots, x_n)$$

$$\leqq \|\exp(i\eta \cdot \psi)\|_{A(\Gamma_I)} \|\exp(-i\eta \cdot \psi) d\mu_{\eta'}\|_{PM(\mathbb{R}^n)},$$

where $PM(\mathbb{R}^n)$ is the space of pseudomeasures, with the norm defined as the L^∞ norm of the Fourier transform of its elements. Then it is sufficient to prove

$$(2) \qquad \|\exp(-i\eta \cdot \psi) d\mu_{\eta'}\|_{PM(\mathbb{R}^n)} = O(|\eta|^{-1/(n+1)}).$$

Now we estimate

$$(3) \qquad = \sup\{| \int \exp(i(-|\eta| \cdot (\eta/|\eta|) \cdot \psi(x, g) - ux - v \cdot g)\phi(x) dx|$$

$$|u \in \mathbb{R}, v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}\}.$$

Let $\eta \neq 0$ be fixed, $s = |\eta| + |u| + |v_1| + \ldots + |v_{n-1}|$, and

$$\tilde{k}(x) = -|\eta|/s \cdot \eta' \cdot \psi(\gamma(x)) - u/s \cdot x - v/s \cdot g(x)$$

on $J_{\eta'}$. Also $k(x)$ be a translation of $\tilde{k}(x)$ such that $\operatorname{supp} k \subset (-\varepsilon, \varepsilon)$. Then we obtain

$$\int \exp(is\tilde{k}(x))\tilde{\phi}(x) dx = \int \exp(isk(x))\phi(x) dx.$$

On the other hand, since we get that

$$k'' = -|\eta|/s \cdot h''_{\eta'} - v/s \cdot g'', \ldots, k^{(n+1)} = -|\eta|/s \cdot h^{(n+1)}_{\eta'} - v/s \cdot g^{(n+1)},$$

we obtain

$$\begin{pmatrix} g''_1, \ldots, g''_{n-1}, h''_{\eta'} \\ \\ \cdots \\ \\ g^{(n+1)}_1, \ldots, g^{(n+1)}_{n-1}, h^{(n+1)}_{\eta'} \end{pmatrix} \begin{pmatrix} -v_1/s \\ -v_2/s \\ \cdots \\ -v_{n-1}/s \\ -|\eta|/s \end{pmatrix} = \begin{pmatrix} k'' \\ \\ \cdots \\ \\ k^{(n+1)} \end{pmatrix}$$

Therefore it is easy that there exists an absolute constant C such that

$$|v_1/s| + \ldots + |v_{n-1}/s| + |\eta/s| \leqq C\{|k''(x)| + \ldots + |k^{(n+1)}(x)|\}$$

for all $x \in (-\varepsilon, \varepsilon)$. Moreover it is established that

$$|k'(x)| + \ldots + |k^{(n+1)}(x)| \leqq |\eta/s|(|(\chi \circ \gamma)'(x)| + \ldots + \\ + |(\psi \circ \gamma)^{(n+1)}(x)|) + |u|/s + |v/s|(|g'(x)| + \ldots + |g^{(n+1)}(x)|)$$

for all $x \in (-\varepsilon, \varepsilon)$. Hence there exist two positive numbers C_1 and C_2 such that

$$0 < C_1 < |k'(x)| + \ldots + |k^{(n+1)}(x)| < C_2$$

for all $x \in (-\varepsilon, \varepsilon)$, and C_j $(j = 1, 2)$ are independent from η.

Then by a result of J. E. Björk (cf. [2, Lemma 1.6]), there exists a positive number C such that

$$\left| \int \exp(isk(x))\phi(x)dx \right| \leqq C(1 + |\eta|)^{-1/(n+1)}$$

for all η, and C is independent from η.

Hence we obtain

$$\|\exp(i\eta \cdot \psi)\|_{A(\Gamma_I)} \geqq C(1 + |\eta|)^{1/(n+1)}$$

for all $\eta \in \mathbf{R}^m$.

By Theorems 5 and 6, we can clearly prove the following result.

COROLLARY 7. *Under the conditions of Theorem* 5, *the following two conditions are equivalent*:

(i) *For an interval* $J \subset \operatorname{int} I$ *there exist two positive numbers* C_1 *and* C_2 *such that*

$$C_1(1 + |\eta|)^{1/(n+1)} \leqq \|\exp(i\eta \cdot \psi)\|_{A(\Gamma_I)} \\ \leqq C_2(1 + |\eta|)^{1/(n+1)}$$

for all $\eta \in \mathbf{R}^m$.

(ii) $\psi_1, \ldots, \psi_m$ *are linearly independent modulo affine linear functions on* Γ_J *for every closed interval* $J \subset \operatorname{int} I$.

ACKNOWLEDGEMENT. The author wishes to thank the referee for his kind suggestions.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
YAMAGATA UNIVERSITY
YAMAGATA, 990
JAPAN