MATH. SCAND. 61 (1987), 117–133

CHARACTERIZATION OF THE PREDUAL AND IDEAL STRUCTURE OF A JBW*-TRIPLE

GÜNTHER HORN

In recent years, a certain category of normed Jordan triple systems called JB*-triples has been an object of study in both complex analysis and functional analysis. A standard example of a JB*-triple system is a norm-closed subspace of the space of all bounded linear operators on a complex Hilbert space which is also closed under the Jordan triple product $\{xy^*z\} := \frac{1}{2}(xy^*z + zy^*x)$. Therefore, JB*-triples generalize C*-algebras. The importance of the category of JB*-triples in complex analysis stems from its equivalence with the category of bounded symmetric domains with base point in complex Banach spaces [17]. A detailed presentation of this theory is contained in [29]. In the context of functional analysis, JB*-triples arise naturally in the solution of the contractive projection problem for C*-algebras in [9] (see also [18]).

In this paper, we study JBW*-triples, i.e., JB*-triples which are dual Banach spaces, in analogy to the theory of JBW*-algebras [13]. After the presentation of preparatory material in section 1 and section 2 we will characterize the predual of a JBW*-triple by various conditions and prove its uniqueness in section 3. The main result of section 4 relates the ideal structure of a JBW*-triple to that of the JBW*-algebra determined by a complete tripotent.

In a forthcoming paper, we will prove a coordinatization theorem for JBW*-triples and we will use it to obtain a classification of JBW*-triples of type I.

Since the completion of this study which is contained in the author's dissertation (1984), great progress has been made in the theory of JB*-triples in [7], [2] (separate weak-*-continuity of the product of a JBW*-triple) and [11]. The results in [2] and [11], however, may not be used to simplify proofs given here as they in turn make use of results presented here and in the mentioned forthcoming paper.

Received May 6, 1986.

1. Jordan-*-triples

Our standard reference for Jordan algebras is [15], for Jordan triple systems it is [22] and [23].

A Jordan-*-triple is a complex vector space U with a sesquilinear map $U \times U \to \mathrm{End}(U)\colon (x, y) \to x \,\square\, y^*$ such that

(1.1) the triple product $\{xy^*z\} := x \,\square\, y^*(z)$ is symmetric in x and z,

(1.2) $\{uv^*\{xy^*z\}\} = \{\{uv^*x\}y^*z\} - \{x\{vu^*y\}^*z\} + \{xy^*\{uv^*z\}\}$
for all $u,v,x,y,z \in U$.

Let $Q(x,z)y := \{xy^*z\}$, $Q(x) := Q(x,x)$ for all $x,y,z \in U$. Every Jordan-*-algebra is a Jordan-*-triple in the product

(1.3) $\{xy^*z\} := (x \circ y^*) \circ z - (x \circ z) \circ y^* + (y^* \circ z) \circ x.$

(1.4) A Jordan-*-triple U is called *abelian* if $x \,\square\, y^*$ and $u \,\square\, v^*$ commute or equivalently, if $\{xy^*\{uv^*w\}\} = \{\{xy^*u\}v^*w\}$ for all $x,y,u,v,w \in U$.

A subtriple generated by a single element is always abelian.

(1.5) A non-zero *tripotent* (i.e., an element e with $Q(e)e = e$) in a Jordan-*-triple U induces a decomposition of U into the eigenspaces of $e \,\square\, e^*$, the *Peirce decomposition* $U = U_1(e) \oplus U_{1/2}(e) \oplus U_0(e)$ where

$$U_k(e) := \{z \in U \mid \{ee^*z\} = kz\} \quad \text{for } k = 0, \tfrac{1}{2}, 1.$$

$U_k := U_k(e)$ is called the *Peirce-k-space* of e.

For Peirce-k-spaces, the following multiplication rules hold:

(1.6) $\{U_1 U_0^* U\} = \{U_0 U_1^* U\} = 0,$

(1.7) $\{U_i U_j^* U_k\} \subset U_{i-j+k}$, where $i,j,k \in \{0, \tfrac{1}{2}, 1\}$ and

$U_l :=$ for $l \neq 0, \tfrac{1}{2}, 1$.

In particular, Peirce-k-spaces are subtriples.

(1.8) The projection $p_k^e = p_k$ of U onto U_k with $p_k(z) = 0$ for $z \in U_j$, $j \neq k$ is called the *Peirce-k-projection* of e.

p_k^e is a polynomial over the integers in $e \,\square\, e^*$. Furthermore, $p_1^e = Q(e)^2$. This and (1.7) yield $U_1(e) = Q(e)U$.

(1.9) Two tripotents e and f are called *compatible* if p_j^e and p_k^f commute for all $j,k \in \{0, \frac{1}{2}, 1\}$.

(1.10) If $f \in U_k(e)$ for some $k \in \{0, \frac{1}{2}, 1\}$, then e and f are compatible. This follows from (1.8) together with (1.2).

In particular, e and f are compatible if they are orthogonal, i.e., if $\{ee^*f\} = 0$ which implies $e \square f^* = f \square e^* = 0$.

A finite compatible family $e_1, \ldots, e_n$ of tripotents has a joint Peirce decomposition

$$U = \bigoplus_{k_i = \{0, \frac{1}{2}, 1\}} U_{k_1}(f_1) \cap \cdots \cap U_{k_n}(f_n).$$

(1.11) A tripotent e is called *complete* if $U_0(e) = 0$. e is called *unitary* if $U = U_1(e)$.

(1.12) The *Peirce spaces with respect to an orthogonal family* $\mathscr{E} = (e_i)_{i \in I}$ of tripotents are defined by

$$U_{ii} := U_1(e_i),\ U_{ij} := U_{1/2}(e_i) \cap U_{1/2}(e_j) \quad (i \neq j),$$

$$U_{i0} := U_{0i} := U_{1/2}(e_i) \cap \bigcap_{j \neq i} U_0(e_j),\ U_{00} := \bigcap_{i \in I} U_0(e_i).$$

$\mathscr{E}$ is called *complete* if $U_{00} = 0$.

The sum P of the Peirce spaces is direct. If $\mathscr{E}$ is finite then $P = U$.

(1.13) $\{U_{ij}U_{jk}^*U_{kl}\} \subset U_{il} \quad (i,j,k,l \in I \,\dot\cup\, \{0\})$.
Products of Peirce spaces which cannot be written in this form vanish.

1.14) LEMMA. *Let e, f be tripotents in U.*

(1) $f \in U_1(e)$ *implies* $U_1(f) \subset U_1(e)$ and $U_0(e) \subset U_0(f)$.

(2) $f \in U_1(e)$ *and* $e \in U_1(f)$ *imply* $U_k(e) = U_k(f)$ *for every* $k = 0, \frac{1}{2}, 1$.

PROOF. (1) $U_1(f) = Q(f)U \subset U_1(e)$ by (1.7), $(f \square f^*)U_0(e) = 0$ by (1.6).
(2) By (1), $U_k(e) = U_k(f)$ for $k = 0, 1$. The compatibility of e and f (1.10) then yields $U_{1/2}(e) = U_{1/2}(f)$.

2. JB*-triples.

A JB*-*triple* is a Jordan-*-triple U endowed with a complete norm such that the triple product is jointly continuous, $z \square z^*$ is a hermitian operator with positive spectrum and

(2.1) $\|\{zz^*z\}\| = \|z\|^3$ for all $z \in U$.

(2.1) is equivalent to

(2.2) $\|z \square z^*\| = \|z\|^2$ for all $z \in U$ (cf. [17, (5.3)]).

Closed subtriples of JB*-triples and l^∞-sums of JB*-triples are again JB*-triples.

Any JB*-algebra (cf. [30], is a JB*-triple in the product (1.3) ([29, 20.35]). So in particular, every C*-algebra is a JB*-triple in the product $\{xy^*z\} := \frac{1}{2}(xy^*z + zy^*x)$.

(2.3) Conversely, if e is a tripotent in a JB*-triple U, *then* $U_1(e)$ is a JB*-algebra with product $x \circ y := \{xe^*y\}$ and involution $x^* := \{ex^*e\}$. (Cf. [6, (2.2)] and [19, (3.7)]).

(2.4) PROPOSITION. *The surjective isometries of* JB*-*triples are precisely the algebraic isomorphisms.*

PROOF. Let $f: U_1 \to U_2$ be an algebraic isomorphism of the JB*-triples U_1 and U_2 (no continuity assumed). Then $\sigma(z \square z^*) = \sigma(f(z) \square f(z)^*)$ and so

$$\|z\|^2 = \|z \square z^*\| = \sup \sigma(z \square z^*) = \sup \sigma(f(z) \square f(z)^*) = \cdots = \|f(z)\|^2$$

for all $z \in U_1$ because for hermitian operators norm and spectral radius coincide ([27, Proposition 2]). The converse follows from [17, (5.5)].

(2.5) The complete tripotents of a JB*-triple U coincide with the complex and the real extreme points of the closed unit ball of U (cf. [19, (3.5)] and [6, (14.1)]).

(2.6) Let e be a tripotent in a JB*-triple. Then the Peirce projections of e are contractive. If e is complete they are hermitian.

(Use the fact that $\exp(it\, e \square e^*)$ is an isometry for all real t (cf. [10, 1.2]).)

(2.7) Let e and f be tripotents in U. f is said to be an *e-projection* if f is a projection in the Jordan-*-algebra $U_1(e)$ (in the sense of (2.3)). If f is in the center of $U_1(e)$ it is called a *central e-projection.*

3. JBW*-triples – characterizations of the predual.

A JB*-triple need not have any tripotents. However, if the JB*-triple is a dual Banach space then it follows from (2.5) and the Krein-Milman theorem that there exist "many" tripotents (cf. (3.11)).

(3.1) DEFINITION. A JB*-triple U is a JBW*-triple if U (as a Banach space) has a predual U_* such that

(3.2) the triple product is separately $\sigma(U, U_*)$-continuous. $\sigma(U, U_*)$ will be also denoted by w^*.

In (3.21) it will be shown that a JBW*-triple has a unique predual in the following sense:

(3.3) A Banach space E is said to have a *unique* predual $F \subset E^*$, if F is the only closed subspace of E^* which is a predual of E in the canonical duality. It should be noted, however, that a weaker notion of uniqueness of the predual is also used in the literature (see e.g. [12]).

(3.4) If E_i are Banach spaces with unique preduals F_i $(i = 1, 2)$ then every surjective isometry $j: E_1 \to E_2$ is $\sigma(E_1, F_1) - \sigma(E_2, F_2)$-continuous.

(3.5) REMARK. Barton and Timoney have recently shown [2] that (3.2) is a consequence of (3.1). However, in their proof they use (3.20) so that (3.2) cannot be omitted at this stage.

(3.6) A JBW*-algebra (i.e., a JB*-algebra with a predual) is a JBW*-triple in the product (1.3) as follows from [26, Lemma 2.2] and [8, Corollary 3.3].

Further examples of JBW*-triples can be obtained from

(3.7) A $\sigma(U, U_*)$-closed subtriple V of a JBW*-triple U (with predual U_*) is a JBW*-triple with predual U_*/V° (where V° is the polar of V in U_*), and from

(3.8) If $(U_i)_{i \in I}$ is a family of JBW*-triples then $U := \oplus_{i \in I}^{\infty} U_i$ is a JBW*-triple.

(3.9) LEMMA. *If e is a tripotent in a* JBW*-*triple U then the Peirce projections of e are $\sigma(U, U_*)$-continuous.*

PROOF. The Peirce projections of e are polynomials in $e \square e^*$. So (3.9) follows from (3.2).

(3.10) If e is a tripotent in a JBW*-triple U then $U_1(e)$ is a JBW*-algebra (by means of (2.3)).

PROOF. This follows from (3.9).

With respect to the local properties of a JBW*-triple one obtains the following lemma:

(3.11) LEMMA. *An abelian, $\sigma(U, U_*)$-closed subtriple W of a* JBW*-*triple U is isometrically isomorphic to a commutative* W*-algebra (endowed with the *product* (1.3)). *In particular, the set of tripotents is norm-total in U.*

PROOF. It follows from (2.5) and the Krein-Milman theorem that W contains a tripotent e which is complete in W. Because W is abelian one has

$$\{ee^*\{ee^*z\}\} = \{\{ee^*e\}e^*z\} = \{ee^*z\} \quad \text{for all } z \in W,$$

i.e., $e \square e^*|_W$ is an idempotent map and therefore $W \cap U_{1/2}(e) = 0$. So W is an associative w^*-closed *-subalgebra of the JBW*-algebra $U_1(e)$ (in the sense of (2.3)), i.e., W is a commutative W*-algebra with unit e. The w^*-closed subtriple generated by a single element is abelian by (1.4) and (3.4). In W*-algebras, the set of projections is norm-total ([24, 1.11.3]). This proves the second assertion.

(3.12) LEMMA. *If U is a* JBW*-*triple then*

(1) *for every $z \in U$ there is a complete tripotent $e \in U$ such that $z \in U_1(e)$ and $z = \{ez^*e\}$,*

(2) *for every orthogonal family $(f_j)_{j \in J}$ of tripotents in U there is a complete tripotent f in U such that f_j is a f-projection for all $j \in J$.*

PROOF. The proofs of (1) and of (2) are parallel: The subtriple V_z (V respectively) generated by z (by $\{f_j | j \in J\}$ respectively) is abelian. By Zorn's lemma there is an abelian subtriple W_z (W respectively) containing V_z (V respectively) which is maximal with respect to inclusion. By (3.2), W_z (W respectively) is w^*-closed.

By (3.11) we can assume that W_z and W are commutative W*-algebras.

Let $z = u|z|$ be the polar decomposition of z in the W*-algebra W_z. Then one checks immediately that $e := 1_{W_z} - uu^* + u$ is a tripotent which is unitary in W_z and satisfies $z = \{ez^*e\}$. Let

$$f := 1_W - \sum_{j \in J} f_j f_j^* + \sum_{j \in J} f_j.$$

The sums exists in the W*-algebra W with respect to the w^*-topology. Using (3.4), it is easily checked that f is a tripotent which is unitary in W and that f_j is a f-projection for all $j \in J$.

We show finally that e and f are complete tripotents in U: Suppose this is false. Then there is a $0 \neq x_z \in U_0(e)$ (a $0 \neq x \in U_0(f)$ respectively). But by (1.4) and (1.6) the subtriple generated by $\{x_z\} \subset W_z$ (by $\{x\} \subset W$ respectively) is abelian. This contradicts the maximality of W_z (of W respectively).

(3.13) COROLLARY. *An orthogonal family $\mathscr{F} := (f_j)_{j \in J}$ of tripotents in a*

JBW-triple U is summable with respect to $\sigma(U, U_*)$. $g := \sum_{j \in J} f_j$ is a tripotent and f_j is a g-projection for all $j \in J$.*

$\mathcal{F}$ is a complete orthogonal family if and only if g is a complete tripotent.

PROOF. By (3.12) there is a tripotent f in U such that $\mathcal{F}$ is an orthogonal family of projections in the JBW*-algebra $U_1(f)$. Therefore $\mathcal{F}$ is summable in the w^*-topology. The next two statements follow from (3.2).

Finally, if g is complete and $z \in U_{00}$ (for the notations see (1.12)) then

$$\{gg^*z\} = \sum_{j \in J} \{f_j f_j^* z\} = 0,$$

so $z = 0$.

Conversely, let $\mathcal{F}$ be complete. Because $f_j \in U_1(g)$ for all $j \in J$ one has $U_0(g) \subset \bigcap_{j \in J} U_0(f_j) = 0$ by (1.14)(1).

(3.14) A bounded linear map p on a Banach space U is called a *projection on U* if $p^2 = p$. Two projections p and q on U are *orthogonal* if $pq = qp = 0$.

(3.15) LEMMA. *Let U be a JBW*-triple, let $\mathcal{F} := (f_j)_{j \in J}$ be an orthogonal family of tripotents in U. Then the Peirce sum with respect to $\mathcal{F}$ (see (1.12)) is $\sigma(U, U_*)$-dense in U.*

More precisely: There are unique weak--continuous, pairwise orthogonal projections p_{ij} on U onto the Peirce spaces U_{ij} $(i,j) \in J \mathbin{\dot\cup} \{0\})$ which are given by*

$$\begin{aligned} p_{kk} &= p_1^{f_k} = Q(f_k)^2, && \text{where } k,l \in J,\ k = 1, \\ p_{kl} &= p_{1/2}^{f_k} p_{1/2}^{f_l} = 4Q(f_k, f_l)^2, && \text{and } f := \sum_{j \in J} f_j, \\ p_{k0} &= p_{1/2}^{f} p_{1/2}^{f_k}, \\ p_{00} &= p_0^{f}. \end{aligned}$$

Every $z \in U$ lies in the w^*-closed subspace spanned by

$$\{p_{ij}(z) \mid i,j \in J \cup \{0\}\}.$$

PROOF. f exists by (3.13). Because $\mathcal{F} \cup \{f\}$ is a compatible family (1.9), p_{ij} is a projection for all $i,j \in J \cup \{0\}$ which is w^*-continuous by (3.9).

Obviously, $p_{kl}(U) = U_{kl}$ holds for all $k,l \in J$.

"$p_{l0}(U) \subset U_{l0}$": For $z \in p_{l0}(U), g := \sum_{\substack{k \in J \\ k \neq l}} f_k$ we have

$$\{gg^*z\} = \{ff^*z\} - \{f_l f_l^* z\} = \tfrac{1}{2}z - \tfrac{1}{2}z = 0,$$

therefore $z \in U_0(f_k)$ for all $k \in J$, $k \neq l$ by (1.14)(1).

"$U_{10} \subset p_{10}(U)$": $\{ff^*z\} = \{f_l f_l^* z\} = \frac{1}{2}z$ holds for all $z \in U_{10}$.

"$p_0(U) \subset U_{00}$": Holds by (1.14)(1).

"$U_{00} \subset p_0(U)$": $\{ff^*z\} = \sum_{j \in J} \{f_j f_j^* z\} = 0$ by (3.2).

This shows that $p_{kl}(U) = U_{kl}$ for all $k, l \in J \cup \{0\}$. The projections p_{ij} ($i, j \in J \cup \{0\}$) commute and the Peirce-sum is direct, so the projections are pairwise orthogonal.

For the last assertion we may assume without loss of generality that $z \in U_m(f)$ for some $m \in \{0, \frac{1}{2}, 1\}$.

"$m = 1$": From (1.13) (for the orthogonal family $\{f_i, f_j\}$) follows $Q(f_i, f_j)U \subset U_{ij}$, so $Q(f_k, f_l)Q(f_i, f_j) = 0$ for $\{k, l\} \neq \{i, j\}$ again by (1.13). Therefore

$$z = Q(f)^2 z = \sum_{i \in J} \sum_{j \in J} c_{ij} Q(f_i, f_j)^2 z = \sum_{i \in J} \sum_{j \in J} c_{ij}^{-1} p_{ij}(z)$$

where $c_{ij} = 1$ for $i = j$ and $c_{ij} = 2$ for $i \neq j$ (summation with respect to the w^*-topology).

"$m = \frac{1}{2}$": $p_1^{f_j}(z) \in U_1(f_j) \cap U_{1/2}(f) = 0$ for all $j \in J$ because f and f_j are compatible, so $p_{j0}(z) = p_{1/2}^{f_j}(z) = 2\{f_j f_j^* z\}$. By (3.2),

$$z = 2\{ff^*z\} = 2 \sum_{j \in J} \{f_j f_j^* z\} = \sum_{j \in J} p_{j0}(z).$$

"$m = 0$": Here nothing remains to be shown.

The uniqueness of the projections follows from the weak-*-density of the Peirce-sum.

In the following, U always is a JBW*-triple with a predual U_* which satisfies (3.2).

(3.16) If e is a tripotent in U and if $f \in U_*$ then $f|_{U_1(e)} \in U_1(e)_*$ as a consequence of (3.9) and the uniqueness of the predual of the JBW*-algebra $U_1(e)$ ([8, Corollary 3.7)].

The converse also holds:

(3.17) Proposition. *If $f \in U^*$ and if $f|_{U_1(e)} \in U_1(e)_*$ for all complete tripotents e in U then $f \in U_*$.*

PROOF. Let U^1 be the closed unit ball of U. By the Krein-Šmulyan theorem it suffices to show that $f|_{U^1}$ is continuous with respect to the topology induced by $\sigma(U, U_*)$.

Let $(z_i)_{i\in I}$ be a w^*-convergent net in U^1 with $\lim(z_i)_{i\in I} =: z_0$. Let

$$A := \{g \in U^* \mid \text{there exists } x \in U^1 \text{ such that } \|g\| = g(x)\}.$$

By [3], A is norm-dense in U^*.

Let $\varepsilon > 0$. Choose $g \in A$, $x \in U^1$ and a complete tripotent e in U such that $\|f-g\| < \frac{\varepsilon}{4}$, $\|g\| = g(x)$ and $x \in U_1(e)$ (3.12). For $i \in I \cup \{0\}$ let $z_i = z_i^1 + z_i^{1/2}$ where $z_i^k \in U_k(e)$ for $k = 1, \frac{1}{2}$.

By (3.9) and the assumption for f there is an $i_0 \in I$ such that $|f(z_0^1 - z_i^1)| < \frac{\varepsilon}{2}$ for all $i \geqq i_0$. Furthermore we have $\|z_0^{1/2} - z_i^{1/2}\| \leqq 2$ for all $i \in I$ by (2.6). By [10, Proposition 1a]), $g(U_{1/2}(e)) = 0$. Therefore $|f(z_0^{1/2} - z_i^{1/2})| < \frac{\varepsilon}{2}$ for all $i \in I$. It follows that $|f(z_0 - z_i)| < \varepsilon$ for all $i \geqq i_0$.

REMARK. (3.16) and (3.17) show that a JB*-triple U has at most one predual U_* such that (3.2) is satisfied. It cannot be inferred from this, however, that U has a unique predual.

Using (3.17), it is possible to generalize to JBW*-triples a known result about W*-algebras ([28, III 3.11]). Let us first state the result for JBW*-algebras.

(3.18) PROPOSITION. *Let A be a* JBW**-algebra, let $f \in A^*$. Then the following conditions are equivalent*

(1) $f \in A_*$,

(2) $f\left(\sum_{i\in I} e_i\right) = \sum_{i\in I} f(e_i)$ *for every orthogonal family* $(e_i)_{i\in I}$ *of projections.*

PROOF. By [8, 3.7] and [13, 4.4.15] there is a central projection e in A^{**} with $(A_*)^\circ = (1-e)A^{**}$. If one defines the normal part of a functional $g \in A^*$ to be $g_n := eg$ (where $eg(z) := g(ez)$ for all $z \in A$) and the singular part of g to be $g_s := g - g_n$, the proofs of [28, III Theorem 3.8, "(i) => (ii)"] and [28, Theorem 3.11] carry over literally.

(3.19) PROPOSITION. *Let $f \in U^*$. Then the following conditions are equivalent*

(1) $f \in U_*$,

(2) $f\left(\sum_{i\in I} e_i\right) = \sum_{i\in I} f(e_i)$ *for every orthogonal family* $(e_i)_{i\in I}$ *of tripotents.*

PROOF. (2) follows from (1) by (3.13). Conversely, by (3.17) it suffices to show that $f|_{U_1(g)} \in U_1(g)_*$ for every tripotent g in U. So (1) follows from (2) by (3.17) and (3.18).

We show next that a predual of a JBW*-triple enjoys the property of being *well-framed* ("bien encadré") (cf. [12, Definition 14]) which will imply the uniqueness of the predual.

(3.20) PROPOSITION. *A predual* U_* *of a* JBW*-*triple is well-framed.*

PROOF. By (3.19), for every $f \in U^* \setminus U_*$ there is an orthogonal family $(e_i)_{i \in I}$ of tripotents with $\sum_{i \in I} e_i = :e$ and $\sum_{i \in I} f(e_i) \neq f(e)$.

Let W be the norm closed subspace of U spanned by $\{e, e_i | i \in I\}$. By (3.13), $(e_i)_{i \in I}$ is an orthogonal family of projections in the JBW*-algebra $U_1(e)$, so W is (isometrically) isomorphic to a commutative C*-algebra with unit e. We show that W does not contain a subspace isomorphic to $l^1(\mathsf{N})$: Because $l^1(\mathsf{N})$ is separable such a subspace would be contained in a closed subspace W' of W spanned by a countable subset of $\{e, e_i | i \in I\}$. It is easily checked that a closed subset of a commutative C*-algebra spanned by the unit and a countably infinite orthogonal family of projections is isomorphic to $c(\mathsf{N})$ which in turn is isomorphic to $c_0(\mathsf{N})$. But $c_0(\mathsf{N})$ does not contain a subspace isomorphic to $l^1(\mathsf{N})$ ([21, Theorem I 2.7]) so W' and hence W does not contain such a subspace.

This, together with the proof of [12, Proposition 3], shows that the closed unit ball of W is "*-admissible" ([12, Definition 13]). Using [12, Proposition 17], we obtain the desired result.

(3.21) THEOREM. *The predual of a* JBW*-*triple is unique.*

PROOF. (3.20) and [12, Theorem 15] show that U_* is unique in the sense of [12] and that every surjective isometry on U is $\sigma(U, U_*)$-$\sigma(U, U_*)$-continuous which implies the uniqueness of U_* in the sense of (3.3).

(3.22) COROLLARY. *An (algebraic) isomorphism of JBW*-triples is weak-*-continuous.*

PROOF. By (2.4), every isomorphism of a JB*-triple is isometric. So the result follows from (3.21) and (3.4).

Summing up, one obtains the following characterization of the predual of a JBW*-triple:

(3.23) THEOREM. *Let* U *be a* JBW*-*triple with predual* U_*, *let* $f \in U^*$. *Then the following conditions are equivalent*

(1) $f \in U_*$,

(2) *there is a complete tripotent e in U such that*

$$f|_{U_1(e)} \in U_1(e)_* \quad \text{and } f(U_{1/2}(e)) = 0.$$

(3) $f|_{U_1(e)} \in U_1(e)_*$ *for every complete tripotent in U.*

(4) $f\left(\sum_{i \in I} e_i\right) = \sum_{i \in I} f(e_i)$ *for every orthogonal family* $(e_i)_{i \in I}$ *of tripotents.*

(5) $f|_W$ *is weak-*-continuous for every maximal abelian subtriple W.*

PROOF. The equivalence of (1), (3), and (4) was shown in (3.16), (3.17), and (3.19).

"(1) implies (2)": The closed unit ball U^1 of U is w^*-compact. Therefore there is a $w \in U^1$ with $f(w) = \|f\|$ and by (3.12) there is a complete tripotent e with $w \in U_1(e)$. So $f(U_{1/2}(e)) = 0$ by [10, 1.2].

"(2) implies(1)": Because $U_1(e)$ is w^*-closed and has a unique predual there is a $g \in U_*$ such that $f|_{U_1(e)} = g|_{U_1(e)}$. Let p be the Peirce projection onto $U_1(e)$. Then $f = f \circ p = g \circ p \in U_*$ by (3.9).

Obviously, (1) implies (5).

"(5) implies (4)": By (3.13) an orthogonal family $(e_i)_{i \in I}$ of tripotents is summable in the weak-*-topology. The subtriple V spanned by $(e_i)_{i \in I}$ is abelian. Choose any maximal abelian (necessarily w^*-closed) subtriple W of U which contains V and apply (5).

The following proposition is, of course, a consequence of the above mentioned result of [2]. It is used in their proof, however, and is therefore not omitted.

(3.24) PROPOSITION. *If a* JB**-triple U has a unique predual* U_* *then the triple product is separately weak-*-continuous.*

PROOF. $x \square x^*$ is a hermitian operator on U for all $x \in U$. By [31, 3.4] and the uniqueness of the predual, $x \square x^*$ is w^*-continuous. By the polarization formula [16, (1.4)] $x \square y^*$ is w^*-continuous for all $x, y \in U$. In particular, the Peirce projections are w^*-continuous (see (1.8)).

Let e be a tripotent in U. Then $Q(e)^2$ is the Peirce projection onto $U_1(e)$, $Q(e)|_{U_1(e)}$ is the involution of the JBW*-algebra $U_1(e)$ and is therefore w^*-continuous, so $Q(e) = Q(e)^3$ is w^*-continuous.

Let $x \in U$. $Q(x)$ is w^*-continuous if and only if $f \circ Q(x) \in U_*$ for every $f \in U_*$. Let $f \in U_*$. By [10, Proposition 2] and the above argument, there is a tripotent e in U such that $f = f \circ Q(e)^2$. Hence it suffices to show that $Q(e)^2 Q(x)$ is w^*-continuous. We have

$$Q(y)Q(z) = 2(y \square z^*)^2 - y \square \{zy^*z\}^* \quad \text{for all } y,z \in U,$$

so $Q(y)Q(z)$ is w^*-continuous for all $y,z \in U$. Hence, $Q(e)^2 Q(x) = Q(e)(Q(e)Q(x))$ is w^*-continuous.

4. Ideals in JBW*-triples.

(4.1) A subspace J of a Jordan-*-triple U is called an *ideal* if $\{UU^*J\} + \{UJ^*U\} \subset J$. Two ideals I and J are said to be *orthogonal* if $I \cap J = 0$. In this case, $I \square J^* = J \square I^* = 0$.

For a subset X of a JBW*-triple U let $U(X)$ denote the weak-*-closed ideal in U generated by X. For $x \in U$ we write $U(x)$ instead of $U(\{x\})$. The weak-*-closed linear span of the union of a family $(X_k)_{k \in K}$ of subsets of U is denoted by $\sum_{k \in K} X_k$. We recall that for any tripotent e in U, $U_1(e)$ naturally carries the structure of a JBW*-algebra.

(4.2) THEOREM. *Let U be a* JBW*-*triple, e a complete tripotent in U. Then the map $I \to U(I)$ is a bijection from the set $\mathscr{I}_e$ of all weak-*-closed *-ideals of the* JBW*-*algebra $U_1(e)$ onto the set $\mathscr{I}$ of all weak-*-closed ideals of U, with inverse $J \to J \cap U_1(e)$. One has the following properties*

(1) $U(I \cap J) = U(I) \cap U(J) \quad (I, J \in \mathscr{I}_e)$

(2) $U\left(\sum_{k \in K} I_k\right) = \sum_{k \in K} U(I_k) \quad (I_k \in \mathscr{I}_e)$

(3) $U(z) = U_1(z) + U_{1/2}(z)$ *for every central e-projection z, and every weak-*-closed ideal in U can be uniquely written in this form.*

(4) *To every $J \in \mathscr{I}$ there is a unique complementary ideal $J^\perp \in \mathscr{I}$.*

(5) $J = (J \cap I) \oplus (J \cap I^\perp)$ *for all $I, J \in \mathscr{I}$.*

PROOF. We first show (3): Let z be a central e-projection, $w := e - z$. Consider the Peirce decomposition of U with respect to the orthogonal family $\{z, w\}$. $U_{z,w} := U_{1/2}(z) \cap U_{1/2}(w) = 0$ because z is a central e-projection. $U_{00} := U_0(z) \cap U_0(w) = 0$ because $z + w$ is a complete tripotent. Let

$$U_{z,0} := U_{1/2}(z) \cap U_0(w) = U_{1/2}(z), \quad U_{w,0} := U_{1/2}(w) \cap U_0(z) = U_{1/2}(w).$$

We show that $U_{z,0} \square U_{w,0}^* = U_{w,0} \square U_{z,0}^* = 0$. It then follows from the multiplication rules (1.13) that $U_1(z)+U_{1/2}(z)$ is an ideal with complement $U_0(z) = U_1(w)+U_{1/2}(w)$. Suppose, $a \in U_{z,0}$, $b \in U_{w,0}$ and $a \square b^* \neq 0$. By (3.11), we can assume that a is a tripotent. We have $a \perp w$, so $c := a+w$ is a tripotent. Let $b = b_0+b_{1/2}+b_1$, where $b_k \in U_k(a)$ for $k = 0, \frac{1}{2}, 1$. Then $b_1 = Q(a)^2 b = 0$ by (1.13), $b_{1/2} \in U_{w,0} \cap U_1(c)$ (because a, z and w are compatible) and $b_{1/2} \neq 0$ because $a \square b^* \neq 0$. So

$$0 \neq \{cb_{1/2}^* c\} = 2\{ab_{1/2}^* w\} \in U_{z,w}$$

by (1.13), a contradiction. Similarly, one shows $U_{w,0} \square U_{z,0}^* = 0$. Clearly, $U_1(z)+U_{1/2}(z) \subset U(z)$, so the first assertion of (3) follows.

Conversely, let J be a w^*-closed ideal in U, let $I := J \cap U_1(e)$. I is a w^*-closed $*$-ideal in the JBW*-algebra $U_1(e)$, so $I = U_1(z)$ for some central e-projection z (cf. [8, 4.3]). Clearly $U(z) \subset J$. If $U(z) \neq J$ then $U_{1/2}(e-z) \cap J \neq 0$ because J is an ideal. This implies $U_1(e-z) \cap J \neq 0$ by [10, 1.5], a contradiction. If z' is a central e-projection with $U(z) = U(z')$, then $U_1(z) = U_1(z')$, so $z = z'$. This shows (3) and (4). (5) now follows from (3) and (4). If z_I is the unique central e-projection associated with $I \in \mathscr{I}_e$ by [8, 4.3], then $U(I) = U(z_I)$, so by (3), $I \to U(I)$ is the composition of the two bijections $I \to z_I$ and $z_I \to U(z_I)$. Its inverse is $J \to J \cap U_1(e)$ as shown above.

Let $I, J \in \mathscr{I}_e$. Then

$$U_1(e) \cap (U(I) \cap U(J)) = (U_1(e) \cap U(I)) \cap (U_1(e) \cap U(J)) = I \cap J,$$

so (1) follows.

Let $I_k \in \mathscr{I}_e (k \in K)$. Then $\sum_{k\in K} I_k$ and $\sum_{k\in K} U(I_k)$ are ideals by (3.2), $U(\sum_{k\in K} I_k) \supset U(I_j)$ for all $j \in K$ and $\sum_{k\in K} I_k \subset \sum_{k\in K} U(I_k)$, so (2) follows.

(4.2)(4) has the following converse:

(4.3) Lemma. *Let U be a* JBW*-*triple, let I and J be ideals in U with $I \oplus J = U$. Then I and J are weak-*-closed.*

Proof. Let e be a complete tripotent in U (2.5), let $f+g = e$ with $f \in I$, $g \in J$. Then f and g are tripotents with $I = U_0(g)$ and $J = U_0(f)$. So (4.3) follows from (3.9).

(4.4) Lemma. *Let U be a* JB*-*triple, let I and J be closed subtriples in U with $I \oplus J = U$. Then $\|z+w\| = \max(\|z\|, \|w\|)$ for all $z \in I$, $w \in J$ if and only if I and J are ideals.*

Proof. $I \oplus^\infty J$ is a JB*-triple (operations defined compontentwise). So (4.4) follows from the fact that the algebra isomorphisms of a JB*-triple are precisely the surjective isometries (2.4).

(4.5) Lemma. *Let U be a* JBW*-*triple, let $(U_k)_{k\in K}$ be an orthogonal family* *of weak-*-closed ideals in* U. *Then* $\sum_{k\in K} U_k$ *is canonically isometrically isomorphic to* $\oplus^{\infty}{}_{k\in K} U_k$.

PROOF. Let p_k be the canonical projection of U onto U_k $(k \in K)$. p_k is contrative by (4.2)(4)and (4.4), so $\phi(x) = (p_k(x))_{k\in K}$ defines a map from $\sum_{k\in K} U_k$ into $\oplus^{\infty}{}_{k\in K} U_k$. By (2.4), it suffices to show that ϕ is an algebric triple isomorphism. Clearly,ϕ is an injective triple homomorphism. To show that ϕ is surjective, let $(x_k)_{k\in K}$ be bounded, $x_k \in U_k$ for every $k \in K$. Then the w^*-closed subtriple generated by $(x_k)_{k\in K}$ is abelian, therefore it is isomorphic to a commutative W*-algebra by (3.11). But a bounded family of elements of a W*-algebra which lie in pairwise orthogonal w^*-closed ideals is summable in the w^*-topology. So ϕ is surjective.

A Jordan-*-triple U is called *indecomposable* if $U = I \oplus J$ for ideals I and J in U implies $I = 0$ or $J = 0$.

(4.6) LEMMA. *Let* U *be a* JBW**-triple,* e *a complete tripotent in* U. *Then the following conditions are equivalent*:

(1) U *is indecomposable,*

(2) U *and* 0 *are the only weak-*-closed ideals in* U,

(3) *the* JBW**-algebra* $U_1(e)$ *is a factor* (i.e., *has trivial center*).

PROOF. (2) implies (1) by (4.3). The other implications follow from (4.2).

(4.7) DEFINITION. A JBW*-triple which satisfies one of the conditions in (4.6) is called a JBW**-triple factor.*

If one is interested in classifying JBW*-triples then one is naturally led to the following definitions:

(4.8) DEFINITION. Let U be a Jordan-*-triple, p a tripotent in U. p is called *abelian* if $U_1(p)$ is abelian in the sense of (1.4). p is called minimal if

$$U_1(p) = \mathbf{C}\cdot p.$$

(4.9) LEMMA. *An abelian tripotent* p *in a* JBW**-triple factor* U *is minimal.*

PROOF. By (3.12), there is a complete tripotent e in U such that p is an e-projection. $U_1(e)$ is a JBW*-algebra factor by (4.6), so p is minimal in $U_1(e)$ ([13, 5.2.17]). Because $U_1(p) \subset U_1(e)$ p is also minimal in U.

If Z is the center of a JBW*-algebra A and if p is an abelian projection in A then $U_1(p) = Zp$ by [13, 5.2.17]. For JBW*-triples we have the following weak analogue:

(4.10) LEMMA. *Let U be a* JBW**-triple, let p be an abelian tripotent in U with $U(p) = U$, let $(p_i)_{i\in I}$ be an orthogonal family of tripotents in U with $\sum_{i\in I} p_i = p$. Then $U = \oplus^{\infty}{}_{i\in I} U(p_i)$.*

PROOF. By (3.12), there is a complete tripotent e in U such that p is an e-projection. The central carrier of p in the JBW*-algebra $(U_1(e), \circ, {}^*)$ equals e and $z \to z \circ p$ maps the center Z of $(U_1(e), \circ, {}^*)$ isomorphically onto $U_1(p)$ by [13, 5.2.17]. So there is an orthogonal family $(z_i)_{i\in I}$ of central e-projections with $p_i = z_i \circ p$. Clearly, $Uz_i) = U(p_i)$ and by (4.2) and (4.5), the result follows.

A JBW*-algebra is of type I if its self-adjoint part is a JBW-algebra of type I i.e., if there is an abelian projection with central carrier 1. This is equivalent to the existence of an abelian projection which generates the JBW*-algebra as a w^*-closed ideal. In analogy to these notions one defines

(4.11) DEFINITION. A JBW*-triple U is *of type* I if there is an abelian tripotent p in U with $U(p) = U$.

(4.12) LEMMA. *Every w^*-closed ideal J of a* JBW**-triple U of type* I *is of type* I.

PROOF. Let p be an abelian tripotent in U with $U(p) = U$. Then the canonical projection of U onto J (4.2)(4) maps p onto an abelian tripotent q with $J = J(q)$.

(4.13) PROPOSITION. *Let U be a* JBW**-triple. Then there is a unique decomposition $U = U_I \oplus U_0$ where U_I and U_0 are ideals in U such that U_I is of type* I *and U_0 contains no non-zero abelian tripotents.*

PROOF. By Zorn's lemma, there is a maximal family $(p_i)_{i\in I}$ of abelian tripotents in U such that the ideals $U(p_i)$ $(i \in I)$ are pairwise orthogonal.

$$U_I := \sum_{i\in I} U(p_i) = U\left(\sum_{i\in I} p_i\right)$$

is of type I and by maximality, $U_0 := U_I^{\perp}$ contains no abelian tripotents.

Suppose $U = V \oplus W$ where V is of type I and W contains no non-zero abelian tripotents. Then $U_0 = (U_0 \cap V) \oplus (U_0 \cap W)$ by (4.2)(5). By (4.12), $U_0 \cap V = 0$, that is, $U_0 = W$. By the uniqueness of the complement in (4.2)(4), also $U_I = V$.

REMARK. U_0 is isomorphic to a JW*-triple, i.e., an ultra-weakly closed J*-algebra in the sense of [14]. It will follow from this, together with the classification of JBW*-triples of type I, that U can be uniquely decomposed into a special and an exceptional part. These facts cannot be proved at this stage, however, and will be shown in forthcoming papers.

(4.14) PROPOSITION. *Let U be a* JBW*-*triple. Then the following conditions are equivalent*

(1) *U is of type* I,

(2) *every non-zero, weak-*-closed ideal of U contains a non-zero abelian tripotent,*

(3) *there is a complete tripotent e in U such that $U_1(e)$ is a* JBW*-*algebra of type* I.

PROOF. The equivalence of (1) and (2) follows from (4.13) and (4.12).

"(1) implies (3)": Let p be an abelian tripotent in U with $U(p) = U$, let e be a complete tripotent in U such that p is an (abelian) e-projection (cf. (3.12)). By (4.2), the w^*-closed ideal generated by p in the JBW*-algebra $U_1(e)$ is $U_1(e)$, so (3) follows.

"(3) implies (1)": Let p be an abelian projection in the JBW*-algebra $U_1(e)$ with central carrier e. Then p is an abelian tripotent in U with $U(p) = U$ by (4.2).

ACKNOWLEDGEMENTS. The author wishes to thank his Ph.D. supervisor Professor W. Kaup for his advice and encouragement and Professor R. B. Braun for useful discussions.

REFERENCES

1. E. Alfsen, F. W. Shultz, and E. Størmer, *A Gelfand-Neumark theorem for Jordan algebras*, Adv. in Math. 28 (1978), 11–56.
2. T. Barton and R. M. Timoney, *On biduals, preduals and ideals of* JB*-*triples*, Math. Scand. 59 (1986), 177–191.
3. E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67 (1961), 97–98.
4. R. B. Braun, *Ein Gelfand-Neumark-Theorem für* C*-*alternative Algebren*, Habilitationssschrift, Tübingen, 1980.
5. R. B. Braun, *A Gelfand-Neumark theorem for* C*-*alternative algebras*, Math. Z. 185 (1984), 225–242.
6. R. B. Braun, W. Kaup, and H. Upmeier, *A holomorphic characterization of Jordan-*C*-*algebras*, Math. Z. 161 (1978), 277–290.
7. S. Dineen, *The second dual of a* JB*-*triple system*, in *Complex Analysis, Functional Analysis, and Approximation Theory*, ed. J. Mujica, (North-Holland Math. Stud. 125), pp. 67–69. North-Holland, Amsterdam - New York, 1986.
8. C. M. Edwards, *On Jordan* W*-*algebras*, Bull. Amer. Math. Soc. 104 (1980), 393–403.
9. Y. Friedman and B. Russo, *Solution of the contractive projection problem*, J. Funct. Anal. 60 (1985), 56–79.

10. Y. Friedman and B. Russo, *Structure of the predual of a* JBW*-*triple*, J. Reine Angew. Math. 356 (1985), 67–89.
11. Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for* JB*-*triples*, Duke Math. J. 53 (1986), 139–148.
12. G. Godefroy, *Parties admissible d'un espace de Banach. Applications*, Ann. Sci. École Norm. Sup. (4) 16 (1983), 109–122.
13. H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, (Monographs Stud. Math. 21), Pitman, Boston, London, Melbourne, 1984.
14. L. A. Harris, *A generalization of* C*-*algebras*, Proc. London Math. Soc. 42 (1981), 331–361.
15. N. Jacobson, *Structure and representations of Jordan algebras*, (Amer. Math. Soc. Colloq. Publ. 39), American Mathematical Society, Providence, R.I., 1968.
16. W. Kaup, *Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension*. I, Math. Ann. 257 (1981), 463–486.
17. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 183 (1983), 503–529.
18. W. Kaup, *Contractive projections on Jordan* C*-*algebras and generalizations*, Math. Scand. 54 (1984), 95–100.
19. W. Kaup and H. Upmeier, *Jordan algebras and symmetric Siegel domains in Banach spaces*, Math. Z. 157 (1977), 179–200.
20. J. L. Kelley, *General Topology*, D. van Nostrand Company, Inc., New York, Toronto, London, 1955.
21. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, (Lecture Notes in Math. 338), Springer-Verlag, Berlin - Heidelberg - New York, 1973.
22. O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, Irvine, 1977.
23. K. Meyberg, *Lectures on Jordan algebras and triple systems*, Lecture Notes, Charlottesville, 1972.
24. S. Sakai, C*-*Algebras and* W*-*Algebras*, (Ergeb. Math. Grenzgeb. 60), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
25. H. H. Schaefer, *Topological Vector Spaces*, (Graduate Texts in Math. 3), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
26. F. W. Shultz, *On normed algebras which are Banach dual spaces*, J. Funct. Anal. 31 (1979), 360–376.
27. A. M. Sinclair, *The norm of a hermitian element in a Banach algebra*, Proc. Amer. Math. Soc. 28 (1971), 446–450.
28. M. Takesaki, *Theory of Operator Algebras*, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
29. H. Upmeier, *Symmetric Banach Manifolds and Jordan* C*-*Algebras*, (North-Holland Math. Stud., Notas de Mat. (104 No. 96)), North-Holland, Amsterdam, New York, Oxford, 1985.
30. J. D. M. Wright, *Jordan* C*-*algebras*, Michigan Math. J. 24 (1977), 291–302.
31. R. Paya, J. Perez, and A. Rodriguez, *Noncommutative Jordan* C*-*algebras*, Manuscripta Math. 37 (1982), 87–120.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT TÜBINGEN
AUF DER MORGENSTELLE 10
D-7400 TÜBINGEN
WEST GERMANY