MATH. SCAND. 61 (1987), 134–148

A PARTIAL CLASSIFICATION RESULT FOR NONCOMMUTATIVE TORI

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Abstract.

It is shown that if two simple noncommutative tori are isomorphic via a $*$-isomorphism which preserves a certain maximal abelian C*-subalgebra, then the two antisymmetric bicharacters defining them are isomorphic.

Introduction.

In [3] the question was raised whether two non-degenerate antisymmetric bicharacters have to be isomorphic if the simple C*-algebras which they give rise to by the construction of Slawny [14] are $*$-isomorphic. The famous classification results [7], [11], [13] for the rotation algebras give that this is indeed the case for antisymmetric bicharacters on Z^2.

In [3] the authors prove the conjecture in certain cases where the isomorphism between the noncommutative tori preserves a certain canonical dense $*$-algebra. The result of the present paper is in the same spirit. To state the result, let β and β_1 be two nondegenerate antisymmetric bicharacters on Z^n and Z^{n_1}, respectively, and denote by B_β and B_{β_1} the corresponding simple C*-algebras. Let H and H_1 be subgroups of Z^n and Z^{n_1} which are maximal such that

$$\beta(H,H) = 1, \quad \beta_1(H_1,H_1) = 1.$$

Assume that H and H_1 are complemented. Denote by A_H and A_{H_1} the abelian C*-subalgebras of B_β and B_{β_1} generated by $\{u_h | h \in H\}$ and $\{u_h | h \in H_1\}$, respectively.

The result is that if there is a $*$-isomorphism $\varphi: B_\beta \to B_{\beta_1}$ such that $\varphi(A_H) = A_{H_1}$, then β and β_1 are isomorphic.

The method we use is inspired by work of A. Kumjian [8] and J. Renault [12]. The idea is to find an abelian C*-subalgebra of the noncommutative

Received October 6, 1986.

torus which has the property that pure states extend uniquely, and then calculate the reduction of the dual groupoid to the spectrum of the abelian C*-subalgebra. As we shall see, this groupoid contains enough information to recover the isomorphism class of the bicharacter defining the algebra. But to get the information out of the groupoid requires a certain amount of calculation, involving group cohomology.

The method of calculating the dual groupoid, or rather a reduction of the groupoid to the spectrum of a maximal abelian C*-subalgebra, has been used before by the author, both in connection with certain inductive limit C*-algebras [15] and in connection with discrete crossed products of abelian C*-algebras by free actions [16]. To determine the groupoid, we closely follow the line of ideas used in [16], exploiting the fact that a noncommutative torus is, at least in the case we consider, a twisted discrete crossed product arising from a free action.

For other results on the classification of noncommutative tori we refer to [3], [4], [5], [6].

I would like to thank J. Tornehave and O. Kroll for helpful remarks on group cohomology and A. Kumjian for a copy of [8] back in 1984. And finally, I thank the referee for his many suggestions which have made this final version more readable.

Let G be a torsion-free abelian group. A bicharacter on G is a map

$$b : G \times G \to \boldsymbol{T} = \{z \in \mathbf{C} \mid |z| = 1\}$$

such that $b(g_1, g_2 + g_3) = b(g_1, g_2) b(g_1, g_3)$ and $b(g_1 + g_2, g_3) = b(g_1, g_3) b(g_2, g_3)$ for all $g_1, g_2, g_3 \in G$. b is antisymmetric (or symplectic) if $b(g, g) = 1$. Given any bicharacter b on G, we can define an antisymmetric bicharacter β by

$$\beta(g, h) = b(g, h)\overline{b(h, g)}, \quad h, g \in G.$$

A bicharacter b is said to be nondegenerate if $\beta(G, g) = \{1\}$ implies $g = 0$ or, equivalently, if $b(g, h) = b(h, g)\ \forall h \in G$ implies $g = 0$. For each nondegenerate bicharacter b, there is (up to $*$-isomorphism) a unique C*-algebra B generated by a set $\{u_g \mid g \in G\}$ of unitaries satisfying

$$(1) \qquad u_g u_h = b(g, h) u_{g+h}, \quad g, h \in G.$$

This was shown by Slawny in [14].

Given two abelian torsion-free groups G, G_1 with nondegenerate bicharacters b, b_1, respectively, the C*-algebras B_β and B_{β_1} are $*$-isomorphic if there is a group isomorphism $\varphi: G \to G_1$ such that

$$\beta_1(\varphi(g), \varphi(h)) = \beta(g, h), \quad g,h \in G.$$

In this case β and β_1 are said to be isomorphic. This results from [14]. In particular, we see that the C*-algebra B_β depends only on β, and not on b, thus justifying our notation.

Although it is not strictly necessary for all that follows, we will restrict our attention to the finitely generated case and take $G = \mathbf{Z}^n$, $n \in \mathbf{N}$. Also, fix a nondegenerate bicharacter b on G. A subgroup $H \subseteqq \mathbf{Z}^n$ will be called a maximal kernel group for β if H is maximal among the subgroups H_0 satisfying $\beta(H_0, H_0) = \{1\}$, that is, H is a maximal subgroup satisfying

$$b(g, h) = b(h, g), \quad h,g \in H.$$

Fix such a maximal kernel subgroup H. By [10, Proposition 3.2], there is a function $f: H \to \boldsymbol{T}$ such that

$$b(g, h) = f(g)^{-1} f(h)^{-1} f(g+h), \quad g,h \in H.$$

It follows that we can define a unitary representation w of H into B_β by

$$w_h = f(h) u_h, \quad h \in H.$$

Let A_H denote the abelian C*-subalgebra of B_β generated by $\{w_h | h \in H\}$.

There is a $*$-homomorphism $\pi: C(\hat{H}) \to A_H$ determined by the requirement $\pi(h) = w_h$, $h \in H = \hat{\hat{H}} \subseteqq C(\hat{H})$. π is dual to the map $\pi^*: \hat{A}_H \to \hat{H}$ given by

$$\pi^*(\omega)(h) = \omega(w_h), \quad \omega \in \hat{A}_H, \quad h \in H.$$

Assertion A: π is a $*$-isomorphism.

To prove this, we apply a result of Arveson [2] in a form given in [9]. By this it suffices to exhibit a faithful state ω on A_H such that $\omega(w_h) = 0$ if $h \neq 0$.

We can take ω to be the normalized trace of B_β.

Now, define a homomorphism $\lambda_H: G \to \hat{H}$ by

$$\lambda_H(g)(h) = \beta(g, h), \quad g \in G, \quad h \in H.$$

Observe that $\ker \lambda_H = H$ by the maximality of H. Define an action $\alpha: G \to \operatorname{Aut} C(\hat{H})$ by

$$\alpha_g(f)(t) = f(\lambda_H(g)t), \quad f \in C(\hat{H}), \quad t \in \hat{H}, \quad g \in G.$$

Assertion B: $\pi(\alpha_g(f)) = u_g \pi(f) u_g^*$, $g \in G$, $f \in C(\hat{H})$.

The proof of Assertion B consists of checking the identity for $f = h \in H$

$= \hat{H} \subseteq C(\hat{H})$. We find, with $\omega \in \hat{A}_H$,

$$\begin{aligned}\pi(\alpha_g(h))(\omega) &= \alpha_g(h)(\pi^*(\omega)) = h(\lambda_H(g)\pi^*(\omega)) = h(\lambda_H(g))h(\pi^*(\omega)) \\ &= \beta(g,h)\omega(\pi(h)) = \beta(g,h)f(h)u_h(\omega) \\ &= f(h)u_g u_h u_g^*(\omega) = u_g w_h u_g^*(\omega) = u_g \pi(h) u_g^*(\omega).\end{aligned}$$

This gives Assertion B.

ASSERTION C: $u_0 = \mathbf{1}$ and $u_g^* = b(g,g)u_{-g}$, $g \in \mathbf{Z}^n$.

Since

$$u_{-g}u_g = b(-g,g)u_0 = \overline{b(g,g)}u_0,$$

it suffices to show that $u_0 = \mathbf{1}$. For this, observe that $f(0)u_0 = w_0 = \mathbf{1}$ since w is a unitary representation. On the other hand, $f(0)^{-1}f(0)^{-1}f(0+0) = b(0,0) = 1$, so $f(0) = 1$, proving the assertion.

ASSERTION D: If $g \in \mathbf{Z}^n \setminus H$, $\mathrm{Ad}\, u_{g|A_H}$ defines a freely acting $*$-automorphism of A_H.

Since $\lambda_H(g) \neq 0$ in $\hat{H}$ for such g, Assertion D is an immediate consequence of Assertions B and A.

LEMMA 1. *A_H has the extension property in B_β,* i.e. *the property that every pure state of A_H has only one state extension to B_β. Furthermore,*

$$B_\beta = A_H \oplus \overline{\mathrm{span}}\{ab - ba \,|\, b \in B_\beta,\ a \in A_H\}$$

and there is exactly one conditional expectation P_H of B_β onto A_H. P_H is faithful and satisfies

$$P_H(au_g) = 0, \quad a \in A_H,\ G \in \mathbf{Z}^n \setminus H.$$

PROOF. Set

$$S = \{u_g \,|\, g \in (\mathbf{Z}^n \setminus H) \cup \{0\}\}.$$

It follows from the definition of B_β that the subspace $\mathrm{span}\, A_H S = \mathrm{span}\, S A_H$ is a dense $*$-subalgebra of B_β. Except for the faithfulness of P_H all statements of the lemma follow from the proof of [16, Proposition 4 and Lemma 11]. The faithfulness of P_H is proved be showing that

$$\{b \in B_\beta \,|\, P_H(b^*b) = 0\}$$

is a two-sided ideal in B in the same way as in [16, Lemma 13]. Since B_β is simple by [14], P_H is faithful.

The following short introduction to the dual groupoid is also found in [16]. We repeat it here for the convenience of the reader.

Let B be an arbitrary C*-algebra. The dual groupoid $G(B)$ of B consists of the extreme points in the unit ball of B^* endowed with the relative weak* topology and with a groupoid structure which we now describe.

The polar decomposition of an element $\mu \in G(B)$ produces a triple (μ_1, v, μ_2) where μ_1, μ_2 are states on B and v is a partial isometry in B^{**}. The connection between μ and (μ_1, v, μ_2) is given by

$$(*) \qquad \mu(\cdot) = \mu_1(v\,\cdot) = \mu_2(\cdot\, v)$$

where all functionals are considered as acting on B^{**}. The triple (μ_1, v, μ_2) is determined uniquely by (*) and the requirements

$$(**) \qquad v^*v = \operatorname{supp}\mu_2, \quad vv^* = \operatorname{supp}\mu_1$$

where $\operatorname{supp}\mu_i$ denotes the support projection of μ_i in B^{**}, $i = 1,2$.

Observe that μ_1, μ_2 are pure states on B since μ is extremal.

The groupoid structure of $G(B)$ is given by the formulas

$$(\mu_1, v, \mu_2)(\mu_2, u, \mu_3) = (\mu_1, vu, \mu_3)$$
$$(\mu_1, v, \mu_2)^{-1} = (\mu_2, v^*, \mu_1).$$

The range map r and the source map s of $G(B)$ are given by

$$r(\mu_1, v, \mu_2) = \mu_1,$$
$$s(\mu_1, v, \mu_2) = \mu_2.$$

This groupoid structure of $G(B)$ is compatible with the weak* topology and makes $G(B)$ into a topological groupoid.

Now, consider our "noncommutative torus" B_β with the abelian C*-subalgebra A_H. We define

$$G(B_\beta, A_H) = \{\mu = (\mu_1, v, \mu_2) \in G(B_\beta) \mid \mu_{1|A_H} \text{ and } \mu_{2|A_H} \text{ are pure}\}.$$

Then $G(B_\beta, A_H)$ is the reduction of $G(B_\beta)$ to the spectrum of A_H. It is a subgroupoid of $G(B_\beta)$, and it follows from [12] that it is a locally compact groupoid in the relative weak* topology.

Now we make the crucial assumption that H is complemented in $\mathbf{Z}^n$, i.e. that there is a subgroup $H^{\perp} \subseteqq \mathbf{Z}^n$ such that $\mathbf{Z}^n = H \oplus H^{\perp}$. We remark that this assumption is automatically satisfied if β is locally infinite in the sense that $\beta(g,h)^m = 1$ for some $m \in \mathbf{N}$ implies that $\beta(g,h) = 1$. In this case, $\mathbf{Z}^n/H$ is torsion-free so that the sequence

$$0 \to H \to \mathbf{Z}^n \to \mathbf{Z}^n/H \to 0$$

splits. Of course, $H \cong \mathbf{Z}^d, H^{\perp} \cong \mathbf{Z}^k$ for some $d,k \in \mathbf{N}$ with $d+k = n$.

It is easy to see that there has to be a maximal kernel subgroup for β which is not complemented when the range of β in $\boldsymbol{T}$ contains torsion. I do not know of an example of a nondegenerate β without any complemented maximal kernel.

We use the notation $H^{\perp}$ for the complement even though the complement is far from unique. In what follows, $H^{\perp}$ denotes an arbitrary subgroup of $\mathbf{Z}^n$ complementing H.

In this case we can give a very convenient description of $G(B_\beta, A_H)$. For every pure state ω of A_H, we use the same symbol ω for the unique state extension to B_β and the unique normal state extension to B_β^{**}. For $h \in H^{\perp}$, $\omega \in \hat{A}_H$, we define $[\omega, u_h] \in G(B_\beta, A_H)$ by

$$[\omega, u_h](b) = \omega(bu_h), \quad b \in B_\beta.$$

We find

LEMMA 2. *For every $\mu \in G(B_\beta, A_H)$ there exists a unique triple $\lambda \in \boldsymbol{T}$, $\omega \in \hat{A}_H$, $h \in H^{\perp}$, such that $\mu = \lambda[\omega, u_h]$.*

PROOF. The existence of the triple follows the lines laid out in [16]: First, show that

$$\{\mu \in G(B_\beta, A_H) | \, \mu(u_h^*) \neq 0\} = \{\mu \in G(B_\beta, A_H) | \, \exists \lambda \in \boldsymbol{T}, \ \omega \in \hat{A}_H : \mu = \lambda[\omega, u_h]\}$$

and then, show that for every $\mu \in G(B_\beta, A_H)$ there is a $h \in H^{\perp}$ such that $\mu(u_h^*) \neq 0$. The details are given in the proofs of [16, Lemmas 6 and 7].

For the uniqueness we proceed as follows. Assume $\lambda_1[\omega_1, u_{h_1}] = \lambda_2[\omega_2, u_{h_2}]$, $\lambda_i \in \boldsymbol{T}$, $\omega_i \in \hat{A}_H$, $h_i \in H^{\perp}$, $i = 1,2$. Let s be the source map of $G(B_\beta, A_H)$. Then

$$\omega_1 = s(\lambda_1[\omega_1, u_{h_1}]) = s(\lambda_2[\omega_2, u_{h_2}]) = \omega_2.$$

In the same way,

$$\omega_1 \circ \mathrm{Ad}\, u_{h_1}^* = r(\lambda_1[\omega_1, u_{h_1}]) = r(\lambda_2[\omega_2, u_{h_2}]) = \omega_2 \circ \mathrm{Ad}\, u_{h_2}^*.$$

Thus

$$\omega_1 \circ \mathrm{Ad}\, u_{h_1}^* u_{h_2} = \omega_1.$$

It follows from Assertion C and the multiplication law (1) that $\operatorname{Ad} u_{h_1}^* u_{h_2} = \operatorname{Ad} u_{h_2 - h_1}$. By Assertion D we must therefore have $h_1 - h_2 \in H$, that is $h_1 = h_2$ since $H^\perp \cap H = \{0\}$. Now $\lambda_1 = \lambda_2$ follows immediately.

Observe that the map $(\lambda, \mu) \to \lambda[\omega, u_h]$ from $\boldsymbol{T} \times \hat{A}_H$ into $G(B_\beta, A_H)$ is continuous. Since $\hat{A}_H$ is homeomorphic to $\hat{H}$ and since $\hat{H}$ is connected (H is torsion-free), it follows from Lemma 2 that the sets

$$\{\lambda[\omega, u_h] \mid \lambda \in \boldsymbol{T},\ \omega \in \hat{A}_H\}, \quad h \in H^\perp,$$

form a partition of $G(B_\beta, A_H)$ into connected compact open subsets. We can now prove our first major result.

PROPOSITION 3. *Let b_1 be a bicharacter on $\mathbf{Z}^n$, $m \in \mathrm{N}$ and assume that $H_1 \subseteqq \mathbf{Z}^m$ is a complemented maximal kernel subgroup for β_1.*

If $\alpha: B_\beta \to B_{\beta_1}$ is a $*$-isomorphism such that $\alpha(A_H) = A_{H_1}$, there exists a group isomorphism

$$\varphi: H^\perp \to H_1^\perp$$

such that

$$\alpha(u_h a u_h^*) = u_{\varphi(h)} \alpha(a) u_{\varphi(h)}^*, \quad a \in A_H,\ h \in H^\perp.$$

PROOF. Observe that the dual map $\alpha^*: B_{\beta_1}^* \to B_\beta^*$ induces a topological groupoid isomorphism of $G(B_{\beta_1}, A_{H_1})$ onto $G(B_\beta, A_H)$. The proof can now proceed as the proof of [16, Theorem 9]. We leave it to the reader to make the necessary small changes.

REMARK 4. We shall need the observation that the isomorphism φ of Proposition 3 is determined by the requirement

$$[\omega, u_h] \circ \alpha^{-1} \in \boldsymbol{T}[\omega \circ \alpha^{-1}, u_{\varphi(h)}], \quad \omega \in \hat{A}_H,\ h \in H^\perp.$$

To prove our main result, we turn to group cohomology. With the aid of $\lambda_H: H^\perp \to \hat{H}$ we can make $C(\hat{H}, \boldsymbol{T})$ into a $H^\perp$-module by defining the $H^\perp$-action as follows

$$g \cdot f(t) = \alpha_g(f)(t) = f(\lambda_H(g)t), \quad f \in C(\hat{H}, \boldsymbol{T}),\ t \in \hat{H},\ g \in H^\perp.$$

We consider $\boldsymbol{T}$ as a $H^\perp$-module with trivial $H^\perp$-action and observe that the map

$$i_H: \boldsymbol{T} \to C(\hat{H}, \boldsymbol{T})$$

which identifies $\lambda \in \boldsymbol{T}$ with the corresponding constant function, defines a $H^\perp$-module morphism. Thus i_H induces a group homomorphism

$$i_H^*: H^2(H^\perp, \boldsymbol{T}) \to H^2(H^\perp, C(\hat{H}, \boldsymbol{T})).$$

This map will be of crucial importance in what follows.

Before we state our next result, we observe that a bicharacter b on $\mathbf{Z}^n$, $n \in \mathbf{N}$, defines a 2-cocycle, i.e. an element in $Z^2(\mathbf{Z}^n, \boldsymbol{T})$ (trivial $\mathbf{Z}^n$-action on $\boldsymbol{T}$). Thus each bicharacter b defines an element $[b]$ in $H^2(\mathbf{Z}^n, \boldsymbol{T}) = Z^2(\mathbf{Z}^n, \boldsymbol{T})/B^2(\mathbf{Z}^n, \boldsymbol{T})$. Only the proof of the next lemma will be used later. It is well-known.

LEMMA 5. *For every antisymmetric bicharacter β on $\mathbf{Z}^n$ there exists a bicharacter b on $\mathbf{Z}^n$ such that $b(g,h)b(h,g)^{-1} = \beta(g,h)$, $g,h \in \mathbf{Z}^n$. As a consequence, every element in $H^2(\mathbf{Z}^n, \boldsymbol{T})$ is represented by a bicharacter in $Z^2(\mathbf{Z}^n, \boldsymbol{T})$.*

PROOF. There is an antisymmetric real $n \times n$-matrix B such that

$$\beta(g,h) = e^{2\pi i \langle g, Bh \rangle}, \quad g,h \in \mathbf{Z}^n.$$

Here $\langle \cdot,\cdot \rangle$ denotes the inner product in $\mathbf{R}^n$. Let $A = \frac{1}{2}B$. Then $A - A^t = B$. Define

$$b(g,h) = e^{2\pi i \langle g, Ah \rangle}, \quad g,h \in \mathbf{Z}^n.$$

Then b has the desired property.

The last statement is now an easy consequence of, for example, [10, Proposition 3.2].

The main step toward our main result is

PROPOSITION 6. *Let b_1 be a bicharacter on $\mathbf{Z}^{n_1}$, $n_1 \in \mathbf{N}$, and let $H_1 \subseteqq \mathbf{Z}^{n_1}$ be a complemented maximal kernel subgroup for the nondegenerate antisymmetric bicharacter β_1. Then the following conditions are equivalent:*

a) *there is a $*$-isomorphism $\alpha: B_\beta \to B_{\beta_1}$ such that $\alpha(A_H) = A_{H_1}$;*

b) *there is a group isomorphism $\varphi: \mathbf{Z}^n \to \mathbf{Z}^{n_1}$ such that*
   i) $\varphi(H) = H_1$, $\varphi(H^\perp) = H_1^\perp$,
   ii) $\beta(g,h) = \beta_1(\varphi(g), \varphi(h))$, $g \in H^\perp$, $h \in H$,
   iii) *the bicharacter $b(\cdot,\cdot)\overline{b_1(\varphi(\cdot), \varphi(\cdot))}$ on $H^\perp$ represents an element in* $\ker i_H^*$.

PROOF. a) $\Rightarrow$ b): By Proposition 3 there exists a group isomorphism $\varphi: H^\perp \to H_1^\perp$ such that

$$\alpha(u_h a u_h^*) = u_{\varphi(h)} \alpha(a) u_{\varphi(h)}^*, \quad a \in A_H, \ h \in H^\perp.$$

Using Assertions A and B, this gives us a $*$-isomorphism $\pi: C(\hat{H}) \to C(\hat{H}_1)$ such that

$$\pi \circ \alpha_g = \alpha_{\varphi(g)} \circ \pi, \quad g \in H^\perp.$$

The dual map gives a homeomorphism $j:\hat{H}_1 \to \hat{H}$ satisfying

$$\lambda_H(g)j(\cdot) = j(\lambda_{H_1}(\varphi(g))\cdot), \quad g \in H^\perp.$$

Set $\Phi(\cdot) = j(\cdot)j(0)^{-1}$. Then

$$(*) \qquad \lambda_H(g) = \Phi(\lambda_{H_1}(\varphi(g))), \quad g \in H^\perp.$$

Since Φ is a homeomorphism and $\lambda_{H_1}(H_1^\perp)$ is dense in $\hat{H}_1$ (because β_1 is nondegenerate), it follows from (*) that Φ is a topological group isomorphism. Let $\Phi_*: H \to H_1$ be the dual group isomorphism. Then $\Phi_* \oplus \varphi$: $H \oplus H^\perp \to H_1 \oplus H_1^\perp$ is an isomorphism extending φ, and we denote this by φ again. By construction, φ satisfies i). We check ii):

$$\begin{aligned}\beta(g,h) &= \lambda_H(g)(h) = \Phi(\lambda_{H_1}(\varphi(g)))(h) = \lambda_{H_1}(\varphi(g))(\Phi_*(h))\\ &= \lambda_{H_1}(\varphi(g))(\varphi(h)) = \beta_1(\varphi(g),\varphi(h)), \quad g \in H^\perp,\ h \in H.\end{aligned}$$

To check iii) we define $t_1: G(B_\beta, A_H) \to \hat{A}_H \times H^\perp$ by

$$t_1(\lambda[\omega, u_g]) = (\omega, g), \quad \lambda \in \boldsymbol{T},\ \omega \in \hat{A}_H,\ g \in H^\perp,$$

and $t_2: G(B_{\beta_1}, A_{H_1}) \to \hat{A}_H \times H^\perp$ by

$$t_2(\lambda[\omega, u_g]) = (\omega \circ \alpha, \varphi^{-1}(g)), \ \omega \in \hat{A}_{H_1},\ g \in H_1^\perp,\ \lambda \in \boldsymbol{T}.$$

Using Remark 4, we see that the following diagram commutes:

$$\begin{array}{ccc} G(B_\beta, A_H) & \xrightarrow{\ t_1\ } & \hat{A}_H \times H^\perp \\ {\scriptstyle (\alpha^{-1})^*}\big\downarrow & \nearrow {\scriptstyle t_2} & \\ G(B_{\beta_1}, A_{H_1}) & & \end{array}$$

Thus the functions

$$s_1, s_2: \hat{A}_H \times H^\perp \to G(B_\beta, A_H)$$

given by $s_1(\omega, g) = [\omega, u_g]$ and

$$s_2(\omega, g) = \alpha^*([\omega \circ \alpha^{-1}, u_{\varphi(g)}]), \quad \omega \in \hat{A}_H,\ g \in H^\perp,$$

both define continuous sections of the $\boldsymbol{T}$-bundle

$$G(B_\beta, A_H) \xrightarrow{\ t_1\ } \hat{A}_H \times H^\perp.$$

It follows that we can define a continuous function $f: \hat{A}_H \times H^\perp \to \boldsymbol{T}$ by

$$f(\omega, g)s_1(\omega, g) = s_2(\omega, g), \quad \omega \in \hat{A}_H,\ g \in H^\perp.$$

Now we use the groupoid structure of $G(B_\beta, A_H)$ to calculate

$$\begin{aligned} s_1(\omega,h)s_1(\omega\circ \operatorname{Ad} u_g, g) &= [\omega, u_h][\omega\circ\operatorname{Ad} u_g, u_g] = [\omega\circ\operatorname{Ad} u_g, u_h u_g] \\ &= b(h,g)[\omega\circ\operatorname{Ad} u_g, u_{h+g}] = b(h,g)s_1(\omega\circ\operatorname{Ad} u_g, h+g) \end{aligned}$$

and

$$\begin{aligned} s_2(\omega,h)s_2(\omega\circ\operatorname{Ad} u_g, g) &= \alpha^*([\omega\circ\alpha^{-1}, u_{\varphi(h)}])\alpha^*([\omega\circ\operatorname{Ad} u_g\circ\alpha^{-1}, u_{\varphi(g)}]) \\ &= \alpha^*([\omega\circ\operatorname{Ad} u_g\circ\alpha^{-1}, u_{\varphi(h)}u_{\varphi(g)}]) \\ &= b_1(\varphi(h),\varphi(g))\alpha^*([\omega\circ\operatorname{Ad} u_g\circ\alpha^{-1}, u_{\varphi(h+g)}]) \\ &= b_1(\varphi(h),\varphi(g))s_2(\omega\circ\operatorname{Ad} u_g, h+g), \quad \omega\in\hat{A}_H,\ h,g\in H^\perp. \end{aligned}$$

Inserting $fs_1 = s_2$, one finds

$$\begin{aligned} &f(\omega,h)f(\omega\circ\operatorname{Ad} u_g, g)b(h,g)s_1(\omega\circ\operatorname{Ad} u_g, g+h) \\ &\quad = b_1(\varphi(h),\varphi(g))f(\omega\circ\operatorname{Ad} u_g, g+h)s_1(\omega\circ\operatorname{Ad} u_g, g+h). \end{aligned}$$

On substituting $\omega\circ\operatorname{Ad} u_g^*$ for ω, it follows that

$$(**)\qquad f(\omega\circ\operatorname{Ad} u_g^*, h)f(\omega,g)f(\omega,g+h)^{-1}b(h,g) = b_1(\varphi(h),\varphi(g))$$

for $\omega\in\hat{A}_H$, $g,h\in H^\perp$.

Let $F: H^\perp \to C(\hat{H})$ be defined by

$$F_h(t) = f(\pi^{*-1}(t), -h)\quad t\in\hat{H},\ h\in H^\perp.$$

Here $\pi: C(\hat{H})\to A_H$ is the $*$-isomorphism considered in Assertion A. Using Assertion B, one may rewrite (**) as

$$g\cdot F_hF_gF_{g+h}^{-1}b(h,g) = b_1(\varphi(h),\varphi(g)),\quad h,g\in H^\perp,$$

from which iii) follows.

b) ⇒ a): Since β and β_1 are trivial on H and H_1, respectively, the proof of Lemma 5 gives that we can assume that so are b and b_1. This can be done without violating iii) since this condition depends only on $\overline{\beta(\cdot,\cdot)}\beta_1(\varphi(\cdot),\varphi(\cdot))$.

Let $\varphi^*: \hat{H}_1\to\hat{H}$ be the group isomorphism dual to $\varphi|_H$. Then there is a $*$-isomorphism $\tilde{\varphi}: C(\hat{H})\to C(\hat{H}_1)$ such that $\tilde{\varphi}(h) = \varphi(h)$, $h\in H\subseteqq C(\hat{H})$. Let $\pi: C(\hat{H})\to A_H$ and $\pi_1: C(\hat{H}_1)\to A_{H_1}$ be the $*$-isomorphisms given by Assertion A, and set $\psi = \pi_1\circ\tilde{\varphi}\circ\pi^{-1}: A_H\to A_{H_1}$. Since $u = w$, we find

$$\begin{aligned} \psi(u_gu_hu_g^*) &= \beta(g,h)\psi(u_h) = \beta(g,h)u_{\varphi(h)} = \beta_1(\varphi(g),\varphi(h))u_{\varphi(h)} \\ &= u_{\varphi(g)}u_{\varphi(h)}u_{\varphi(g)}^* = u_{\varphi(g)}\psi(u_h)u_{\varphi(g)}^*\quad h\in H,\ g\in H^\perp, \end{aligned}$$

where we have used ii).

Using Assertion B, the condition iii) translates into the existence of a function $f : H^{\perp} \to A_H$ taking unitary values such that

$$(***) \qquad \operatorname{Ad} u_g(f(h))f(g)f(g+h)^{-1}b_1(\varphi(g), \varphi(h)) = b(g,h)$$

for $g,h \in H^{\perp}$.

Set

$$C_0 = \operatorname{span}\{au_h \mid a \in A_H,\ h \in H^{\perp}\}$$

and

$$C_0^1 = \operatorname{span}\{au_h \mid a \in A_{H_1},\ h \in H_1^{\perp}\}.$$

It follows from Lemma 1 that every element c in C_0 admits a unique decomposition

$$c = \sum_h a_h u_h, \quad a_h \in A_H,\ h \in H^{\perp} \text{ (finite sum)}.$$

Define $\alpha(c) = \sum_h \psi(a_h f(h))u_{\varphi(h)}$. Then α is clearly a linear map of C_0 onto C_0^1. Using (***), we find

$$\begin{aligned} \psi(af(g))u_{\varphi(g)}\psi(bf(h))u_{\varphi(h)} &= \psi(af(g))u_{\varphi(g)}\psi(bf(h))u^*_{\varphi(g)}u_{\varphi(g)}u_{\varphi(h)} \\ &= \psi(af(g)\operatorname{Ad} u_g(b)\operatorname{Ad} u_g(f(h)))b_1(\varphi(g), \varphi(h))u_{\varphi(g+h)} \\ &= \psi(a\operatorname{Ad} u_g(b)f(g+h))b(g,h)u_{\varphi(g+h)} = \alpha(au_g bu_h), \quad g,h \in H^{\perp},\ a,b \in A_H. \end{aligned}$$

Thus α is a homomorphism.

Inserting $g = h = 0$ in (***) gives $f(0) = \mathbf{1}$. If we insert $g = -g$ and $h = g$, we get

$$\operatorname{Ad} u_{-g}(f(g))f(-g)\overline{b_1(\varphi(g), \varphi(g))} = \overline{b(g,g)}, \quad g \in H^{\perp}.$$

A direct check confirms that this equality implies that α is a $*$-homomorphism.

Since C_0 is dense in B_β and P_H is faithful by Lemma 1, we have

$$\|z\| = \sup\{\|P_H(y^*z^*zy)\|^{1/2} \mid y \in B_0, P_H(y^*y) \leqq \mathbf{1}\}$$

for all $z \in B_\beta$, and the same in B_{β_1}. Since

$$\begin{aligned} P_{H_1}(y^*\alpha(z)^*\alpha(z)y_1) &= P_{H_1} \circ \alpha(\alpha^{-1}(y_1^*)z^*z\alpha^{-1}(y_1)) \\ &= \psi \circ P_H(\alpha^{-1}(y_1^*)z^*z\alpha^{-1}(y_1)) \end{aligned}$$

and

$$P_H(\alpha^{-1}(y_1^*)\alpha^{-1}(y_1)) = P_H \circ \alpha^{-1}(y_1^*y_1) = \psi^{-1} \circ P_{H_1}(y_1^*y_1)$$

for all $z \in C_0$, $y_1 \in C_0^1$, we find that α is isometric on C_0 and therefore extends to a $*$-isomorphism of B_β onto B_{β_1}.

In view of Proposition 6, it becomes interesting to obtain a description of $\ker i_H^*$. This is the aim of the following lemma. In particular, it follows from this lemma and [10, Proposition 3.2], that $\ker i_H^* \neq 0$ whenever $H^2(H^\perp, \boldsymbol{T}) \neq 0$, i.e. whenever Rank $H^\perp > 1$. Hence we cannot conclude directly from Proposition 6 that β and β_1 are isomorphic and have to investigate $\ker i_H^*$ more carefully.

LEMMA 7. *Every element in* $\ker i_H^*$ *is represented by a 2-cocycle*

$$(g, h) \to j(g)(\lambda_H(h)), \quad g, h \in H^\perp,$$

where $j \in \operatorname{Hom}(H^\perp, H)$.

Conversely, every such 2-cocycle represents an element in $\ker i_H^*$.

PROOF. For the proof, we make the identifications: $H = \mathsf{Z}^k$, $H^\perp = \mathsf{Z}^m$ and consider the short exact sequence

$$(*) \qquad 0 \to \boldsymbol{T} \xrightarrow{i_H} C(\boldsymbol{T}^k, \boldsymbol{T}) \xrightarrow{q} Q \to 0$$

where Q is the quotient $C(\boldsymbol{T}^k, \boldsymbol{T})/i_H(\boldsymbol{T})$ and q the quotient map. We also need the real analogue of (*),

$$(**) \qquad 0 \to \mathsf{R} \to C(\boldsymbol{T}^k, \mathsf{R}) \xrightarrow{q_\mathsf{R}} Q_\mathsf{R} \to 0.$$

Here $C(\boldsymbol{T}^k, \mathsf{R})$ is viewed as a Z^m-module with the action of Z^m given by λ_H. In order to connect these sequences, we consider the map $e: C(\boldsymbol{T}^k, \mathsf{R}) \to C(\boldsymbol{T}^k, \boldsymbol{T})$ given by

$$e(f)(t) = e^{2\pi i f(t)}, \quad f \in C(\boldsymbol{T}^k, \mathsf{R}),\ t \in \boldsymbol{T}^k.$$

Then e is a Z^m-module map and induces a Z^m-module map $\tilde{e}: Q_\mathsf{R} \to Q$ in the obvious way.

Observe that every $f \in C(\boldsymbol{T}^k, \boldsymbol{T})$ defines an element in $\operatorname{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))$ by the formula

$$f_*[\gamma] = [f \circ \gamma]$$

for all loops $\gamma: \boldsymbol{T} \to \boldsymbol{T}^k$. Since the constant functions lie in the kernel of $f \to f_*$, we can define a map

$$t: Q \to \operatorname{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))$$

by

$$t(q(f)) = f_*, \quad f \in C(\boldsymbol{T}^k, \boldsymbol{T}).$$

If we give $\mathrm{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))$ the trivial $\mathbf{Z}^m$-module structure, the map t becomes a module map. By a well-known lifting criterion, we have an exact sequence

$$0 \to Q_{\mathbf{R}} \xrightarrow{\tilde{e}} Q \xrightarrow{t} \mathrm{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T})) \to 0,$$

where the injectivity of $\tilde{e}$ follows from the connectedness of $\boldsymbol{T}^k$.

Combining all the maps and using the functorial properties of $H^*(\mathbf{Z}^m, \cdot\,)$, we get a commutative diagram

$$
\begin{array}{ccccccc}
 & & H^1(\mathbf{Z}^m, \mathrm{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))) & & & & \\
 & & t^* \uparrow & & & & \\
H^1(\mathbf{Z}^m, C(T^k, \boldsymbol{T})) & \xrightarrow{q^*} & H^1(\mathbf{Z}^m, Q) & \xrightarrow{\delta} & H^2(\mathbf{Z}^m, \boldsymbol{T}) & \xrightarrow{i_H^*} & H^2(\mathbf{Z}^m, C(\boldsymbol{T}^k, \boldsymbol{T})) \\
e^* \uparrow & & \uparrow \tilde{e}^* & & & & \\
H^1(\mathbf{Z}^m, C(\boldsymbol{T}^k, \mathbf{R})) & \xrightarrow[q_{\mathbf{R}}^*]{} & H^1(\mathbf{Z}^m, Q_{\mathbf{R}}) & & & &
\end{array}
$$

which is exact in the vertical and horizontal directions.

We want to consider yet another map, a right inverse s for t. To define it, observe that every element x of $\mathrm{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))$ is represented by an element $(z_1, z_2, \ldots, z_k)$ in $\mathbf{Z}^k$ such that

$$x[\gamma] = [\gamma_1^{z_1} \gamma_2^{z_2} \cdots \gamma_k^{z_k}]$$

for all loops $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)\colon \boldsymbol{T} \to \boldsymbol{T}^k$. Let f_x be the character in $C(\boldsymbol{T}^k, \boldsymbol{T})$ defined

$$f_x(t_1, t_2, \ldots, t_k) = t_1^{z_1} t_2^{z_2} \cdots t_k^{z_k}, \quad (t_1, t_2, \ldots, t_k) \in \boldsymbol{T}^k$$

and let $s(x) = q(f_x)$. Then s is a $\mathbf{Z}^m$-module map and $t \circ s = \mathrm{id}$.

Since

$$H^1(\mathbf{Z}^m, \mathrm{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))) = \mathrm{Hom}(\mathbf{Z}^m, \mathrm{Hom}(\pi_1(\boldsymbol{T}^k), \pi_1(\boldsymbol{T}))),$$

the definition of δ gives that the statement of the lemma follows from

$$\ker i_H^* = \operatorname{ran} \delta = \delta(\operatorname{ran} s^*).$$

Observing that integration with respect to Haar measure produces a splitting in (**) so that $q_{\mathbb{R}}^*$ is surjective, the desired conclusion now follows from a diagram chase in the above diagram.

We can now state and prove our main result.

THEOREM 8. *Let β and β_1 be nondegenerate, anti-symmetric bicharacters on $\mathbf{Z}^n$ and $\mathbf{Z}^{n_1}$, respectively, and assume there are complemented maximal kernel subgroups, H for β and H_1 for β_1, and a *-isomorphism $\alpha: B_\beta \to B_{\beta_1}$ such that $\alpha(A_H) = A_{H_1}$.*

Then there is group isomorphism $\varphi: \mathbf{Z}^n \to \mathbf{Z}^{n_1}$ such that $\varphi(H) = H_1$ and $\beta_1(\varphi(\cdot), \varphi(\cdot)) = \beta(\cdot, \cdot)$.

PROOF. By Proposition 6 we may assume that $n = n_1$, $H = H_1$, and that

a) $\beta(g,h) = \beta_1(g,h), \quad g \in H^\perp,\ h \in H,$

b) $b(\cdot,\cdot)\overline{b_1(\cdot,\cdot)}_{|H^\perp}$ represents an element in $\ker i_H^*$.

We identify $H = \mathbf{Z}^d$, $H^\perp = \mathbf{Z}^{n-d}$. By a) there are a $d \times (n-d)$ real matrix A and $(n-d) \times (n-d)$ anti-symmetric real matrices X, Y such that

$$\beta(\cdot,\cdot) = e^{2\pi i \langle \cdot, B \cdot \rangle} \quad \text{and} \quad \beta_1(\cdot,\cdot) = e^{2\pi i \langle \cdot, B_1 \cdot \rangle},$$

$$B = \begin{pmatrix} 0 & A \\ -A^t & X \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 & A \\ -A^t & Y \end{pmatrix}.$$

By Lemma 7, b) implies the existence of a $d \times (n-d)$ integral matrix C such that

$$b(\cdot,\cdot)\overline{b_1(\cdot,\cdot)} = e^{2\pi i \langle \cdot, C^t A \cdot \rangle}$$

on $H^\perp$ (modulo coboundaries). It follows that $X = Y + C^t A - A^t C \pmod{M_{n-d}(\mathbf{Z})}$. Define

$$D = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}.$$

Then $D \in \mathrm{Gl}_n(\mathbf{Z})$ and $D^t B_1 D = B \pmod{M_n(\mathbf{Z})}$.

Thus $D: \mathbf{Z}^n \to \mathbf{Z}^n$ is an automorphism φ such that $\beta_1(\varphi(\cdot), \varphi(\cdot)) = \beta(\cdot,\cdot)$ and $\varphi(H) = H$. This completes the proof.

REMARK 9. An easy application of Proposition 6 gives that the converse of Theorem 8 also holds: If φ exists, then α exists.

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