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M-STRUCTURE IN TENSOR PRODUCTS OF BANACH SPACES

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Abstract.

We investigate M-summands and M-ideals in tensor products of Banach spaces, normed by the injective tensor norm or its right-sided projective hull. Spaces of compact operators are considered, too. The main result is that $Z \subset X \hat{\otimes}_\varepsilon Y$ is an M-ideal if and only if there is an M-ideal J in Y with $Z = X \hat{\otimes}_\varepsilon J$ provided $\{0\}$ and X are the only M-ideals in X.

1. Introduction and notation.

The objective of this paper is to study certain subspaces of a tensor product of two Banach spaces, the so-called M-ideals and M-summands. A closed subspace J of a Banach space X is defined to be an M-ideal if there is an l^1-direct complement of J°, the polar of J, in X':

$$X' = J^\circ \oplus_1 (J^\circ)^\perp.$$

It can easily be shown (see [4, Lemma 1.2]) that the orthogonal space $(J^\circ)^\perp$ is uniquely determined in this case. It depends on whether or not it is weak*-closed if there is an l^∞-direct complement of J in X. Provided this is true, i.e.

$$X = J \oplus_\infty J^\perp,$$

we call J an M-summand. The according projections are called M-projections. Similarly one defines L-summands and L-projections so that $J \subset X$ is an M-ideal if and only if $J^\circ \subset X'$ is an L-summand. These concepts have been introduced in [3] in the case of real Banach spaces, later the definitions have been extended to complex spaces in [17]. Basic properties of M-ideals are discussed in [4], for more recent results cf. [6].

It can be shown (see [27]) that the M-ideals of a C*-algebra coincide with the closed two-sided ideals. In particular, $J \subset C(K)$ is an M-ideal if and only if $J = \{f \in C(K): f|_D = 0\}$ for some closed $D \subset K$. (Of course, there is an elementary proof of this assertion, e.g. based on Lemma 1.1 below.) The same characterization holds for spaces $C(K, X) = X \hat{\otimes}_\varepsilon C(K)$ of vector-valued continuous functions if X has no M-ideal apart from $\{0\}$ and X itself [4, Proposition 10.1].

In the second section we shall prove variants of this result for an injective tensor product where $C(K)$ is replaced by an arbitrary Banach space Y and for spaces of compact operators. Particularly we solve a problem raised in [5].

In the third section we shall consider the right-sided projective hull of the injective tensor norm. In several instances we are able to characterize the M-summands of the thus normed tensor products. Dual versions lead to the classification of L-summands in spaces of absolutely summing operators.

We shall study M-ideals in a Banach space by means of the extreme points of the dual unit ball. The closed unit ball of a Banach space X is denoted by B_X, S_X stands for the unit sphere. We shall use the following elementary lemma repeatedly.

1.1 Lemma. *For* $X = U \oplus_1 V$ *we have* $\operatorname{ex} B_X = \operatorname{ex} B_U \cup \operatorname{ex} B_V$.

In order to apply this lemma in the context of section 2, we need information on the extreme functionals on an injective tensor product. We state the fundamental result for the sake of easy reference.

1.2 Theorem. *If* H *is a closed subspace of* $K_{w^*}(X', Y)$, *the space of compact operators from* X' *into* Y *which are weak*-weakly continuous, containing* $X \otimes Y$, *then* $\operatorname{ex} B_{H'} = \operatorname{ex} B_{X'} \otimes \operatorname{ex} B_{Y'}$.

For the proof of this theorem the reader is referred to [24] (real case) and [21] (complex case), a unified approach can be found in [23]. The special case $H = X \hat{\otimes}_\varepsilon Y$ has been treated in [29].

Finally, let us fix some notation. We use the term tensor norm in the sense of Grothendieck [12]. $X \otimes_\alpha Y$ denotes the algebraic tensor product of X and Y equipped with the norm α and $X \hat{\otimes}_\alpha Y$ its completion. ε stands for the injective and π for the projective tensor norm, that is

$$\|\Sigma x_i \otimes y_i\|_\varepsilon = \sup\{|\Sigma x'(x_i)y'(y_i)| : x' \in B_{X'}, y' \in B_{Y'}\},$$

$$\|u\|_\pi = \inf\{\Sigma \|x_i\| \cdot \|y_i\| : u = \Sigma x_i \otimes y_i\}.$$

The dual of $X \hat{\otimes}_\alpha Y$ may be identified with a space of bilinear forms or, equivalently, with a space of linear operators from X into Y'. They are called α'-integral operators in [12], we write $(X \hat{\otimes}_\alpha Y)' = L^{\alpha'}(X, Y')$. In the case $\alpha = \varepsilon$, the notion integral operator is used. Often the duality between some space and its dual is denoted by $\langle \cdot, \cdot \rangle$. Background material on tensor products of Banach spaces can be found in e.g. [7, chapter VIII] and [10].

Our results refer to real and complex spaces, unless stated otherwise.

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2. The injective tensor norm.

We start with an elementary observation.

2.1 Proposition. *If P is an* M*-projection on Y, then* Id $\otimes P$ *is an* M*-projection on $X \hat{\otimes}_\varepsilon Y$.*

Proof. Straightforward verification.

A restatement of Proposition 2.1 is: If J is an M-summand in Y, then $X \hat{\otimes}_\varepsilon J$ is an M-summand in $X \hat{\otimes}_\varepsilon Y$. We are going to establish a corresponding statement on M-ideals. First we present a general result on tensor norms.

2.2 Proposition. *Let α be a tensor norm having the following property for all pairs of Banach spaces X and Y:*

If P is an M*-projection on Y, then* Id $\otimes P$ *is an* M*-projection on $X \hat{\otimes}_\alpha Y$.*

Then α also satisfies:

If J is an M*-ideal in Y, then the closure (with respect to α) of $X \otimes J$ is an* M*-ideal in $X \hat{\otimes}_\alpha Y$.*

Proof. The adjoint of Id $\otimes P$ maps $T \in L^{\alpha'}(X, Y') = (X \hat{\otimes}_\alpha Y)'$ to $P' \circ T$. Hence, if P is an M-projection, we have by assumption on α

$$\|T\|_{\alpha'} = \|P' \circ T\|_{\alpha'} + \|T - P' \circ T\|_{\alpha'}. \tag{1}$$

Now recall from [12, p. 14] that $T \in L^{\alpha'}(X, Y')$ if and only if $T'' \in L^{\alpha'}(X'', Y''')$ with equality of α'-norms. Let E denote the L-projection from Y' onto J°. It follows for $T \in L^{\alpha'}(X, Y')$

$$\begin{aligned}
\|T\|_{\alpha'} &= \|T''\|_{\alpha'} \\
&= \|E'' \circ T''\|_{\alpha'} + \|T'' - E'' \circ T''\|_{\alpha'} \quad \text{(apply (1) with } P = E') \\
&= \|E \circ T\|_{\alpha'} + \|T - E \circ T\|_{\alpha'}.
\end{aligned}$$

Moreover, $T = E \circ T$ if and only if $T(X) \subset E(Y') = J^\circ$, that is $T \in (X \otimes J)^\circ$. This means that $(X \otimes J)^{-\alpha}$ is an M-ideal in $X \hat{\otimes}_\alpha Y$.

Of course, a tensor norm which preserves M-projections of the left factor preserves M-ideals of the left factor, too.

2.3 Corollary. *If J is an* M*-ideal in Y, then $X \hat{\otimes}_\varepsilon J$ is an* M*-ideal in $X \hat{\otimes}_\varepsilon Y$.*

Proof. Apply Propositions 2.1 and 2.2 and note that the ε-closure of $X \otimes J$ is $X \hat{\otimes}_\varepsilon J$.

Remarks. 1) The "left-sided" analogues of 2.1 and 2.3 are valid, too, since the ε-norm is symmetric.

2) The importance of M-ideals in approximation theory stems from the fact that they are proximinal subspaces (see [3, Corollary 5.6], [4, Proposition 6.5]). It should be pointed out that the analogous version of Corollary 2.3 for proximinal subspaces is false (see [18, Theorem 2.7]).

3) A proof of Corollary 2.3 which uses special features of the ε-norm rather than general properties of tensor norms may be given as follows. Again, let E denote the L-projection from Y' onto J°. For $x \in X$, $y \in Y$ and $\varphi \in (X \hat{\otimes}_\varepsilon Y)'$, i.e. an integral bilinear form, put $\varphi_x(y) = \varphi(x, y)$ and $Q(\varphi)(x, y) = \langle E(\varphi_x), y\rangle$. To prove that $Q(\varphi)$ is an integral bilinear form, too, and that Q is an L-projection onto $(X \otimes J)^{\circ}$ one needs a second representation of $Q(\varphi)$. In fact, if μ is a positive Radon measure on $S = B_{X'} \times B_{Y'}$ such that

$$\varphi(x, y) = \int_S \langle x', x\rangle\langle y', y\rangle d\mu(x', y')$$

for all $x \in X$, $y \in Y$, then also

$$Q(\varphi)(x, y) = \int_S \langle x', x\rangle\langle E(y'), y\rangle d\mu(x', y'). \tag{2}$$

In the proof of (2) it is crucial that $E'(y)$ considered as a function on $B_{Y'}$ satisfies the barycentric calculus (see [3, Corollary 4.2] in the real case, [2, p. 240] in the complex case). Using (2) it is easy to complete the proof.

4) Finally, let us rephrase Corollary 2.3 in terms of operators:

2.4 Corollary. *Let J be an* M-*ideal in Y and suppose V' or Y to have the approximation property. Then the space of J-valued compact operators is an* M-*ideal in $K(V, Y)$.*

Proof. If V' or Y has the approximation property, then $K(V, J)$ (respectively $K(V, Y)$) and $V' \hat{\otimes}_\varepsilon J$ (respectively $V' \hat{\otimes}_\varepsilon Y$) are isometrically isomorphic, cf. e.g. [7, p. 242].

We have tacitly employed the fact that the approximation property is inherited by M-ideals. To see this, let J be an M-ideal in Y, a space with the approximation property. Let $K \subset J$ be compact and let $\varepsilon > 0$. Since $J^{\circ\circ} (= J'')$ is norm one complemented in Y'', it follows from the approximation property of Y that there is a finite rank operator $S\colon J \to J''$ with $\|Sx - x\| \leqq \varepsilon$ for $x \in K$. Now choose a finite δ-net $\{x_1, \ldots, x_n\}$ for K with $\delta < \varepsilon/(1 + \|S\| \cdot (1 + \varepsilon))$ and use the principle of local reflexivity (see [22, 28.1.3]) to obtain an operator

$$T\colon \mathrm{lin}(\{x_1, \ldots, x_n\} \cup S(J)) \to J$$

with $\|T\| \leqq 1+\varepsilon$ and $Tx_i = x_i$ for $i = 1, \ldots, n$. We conclude $\|TSx - x\| \leqq \varepsilon(2+\varepsilon)$ for $x \in K$. By the way, the same argument shows that the metric and the bounded approximation property are inherited by M-ideals.

REMARK. We did not succeed in removing the approximation assumption from Corollary 2.4. It would be most interesting to consider $V = Y = H^\infty$ here, since it is not known if H^∞ has the approximation property. On the other hand, the space of compact operators must not be replaced by the whole operator space in 2.4, that is, $L(V, J)$ is in general not an M-ideal in $L(V, Y)$ even though J is an M-ideal in Y. This phenomenon is discussed in [9] and [31] for $V = Y = C_{\mathbf{C}}(K)$.

Our next topic is the investigation of the converses of 2.1 and 2.3. First we are going to consider M-summands in $K_{w^*}(X', Y)$, the space of compact operators from X' into Y which are weak*-weakly continuous.

2.5 THEOREM. *Suppose X has no non-trivial* (i.e. *different from* $\{0\}$ *and* X) M-*summand. Let Z be an* M-*summand in $K_{w^*}(X', Y)$. Then there is an* M-*summand J in Y such that*

$$Z = \{T \in K_{w^*}(X', Y) : T(X') \subset J\} = K_{w^*}(X', J).$$

PROOF. Let us abbreviate $K_{w^*}(X', Y)$ by H. By assumption on Z there is a decomposition $H' = V_1 \oplus_1 V_2$ with weak*-closed subspaces V_1 and V_2, say $V_1 = Z^\circ$. It follows from 1.1 and 1.2 that

$$\operatorname{ex} B_{H'} = \operatorname{ex} B_{V_1} \cup \operatorname{ex} B_{V_2} \tag{3}$$

and

$$\operatorname{ex} B_{H'} = \operatorname{ex} B_{X'} \otimes \operatorname{ex} B_{Y'}. \tag{4}$$

Now, fix $q \in \operatorname{ex} B_{Y'}$. Since the map $x' \to x' \otimes q$ from X' into H' is weak*-continuous, the spaces

$$N_{q,i} := \{x' : x' \otimes q \in V_i\}$$

are weak*-closed ($i = 1, 2$). It follows that $N_{q,1}$ and $N_{q,2}$ are L-orthogonal, and $N_{q,1} \oplus_1 N_{q,2}$ is a weak*-closed subspace of X' (by virtue of the Krein-Smulyan theorem) containing $\operatorname{ex} B_{X'}$ (according to (3) and (4)). Hence $X' = N_{q,1} \oplus_1 N_{q,2}$ by the Krein-Milman theorem. By assumption on X, however, we must have

$$N_{q,1} = X' \quad \text{or} \quad N_{q,1} = \{0\}. \tag{5}$$

In an analogous manner, fix $p \in \operatorname{ex} B_{X'}$ and define weak*-closed subspaces of Y' by

$$M_{p,i} := \{y' : p \otimes y' \in V_i\}.$$

Again we deduce a decomposition $Y' = M_{p,1} \oplus_1 M_{p,2}$. By (5), all the spaces $M_{p,1}$ coincide, as their unit balls have the same extreme points. Thus, we may write

$$Y' = M_1 \oplus_1 M_2$$

with $M_i = \{y' : p \otimes y' \in V_i \text{ for some (hence all) } p \in \operatorname{ex} B_{X'}\}$ being weak*-closed.

Finally, let J be the closed subspace of Y such that $J^\circ = M_1$. By construction, J is an M-summand in Y. It remains to show that

$$V_1 = \{T \in H : T(X') \subset J\}^\circ \tag{6}$$

which implies $Z = K_{w^*}(X', J)$.

In fact, both spaces occurring in (6) are weak*-closed L-summands of H' (cf. [19, Proposition 6.1]). Therefore it is enough to show that their unit balls have the same extreme points, which are necessarily of the form $p \otimes q$. To finish the proof just note $p \otimes q \in V_1$ iff $q \in M_1 = J^\circ$ iff $p \otimes q$ annihilates $K_{w^*}(X', J)$.

REMARKS. 1) Theorem 2.5 generalizes previous results in [16] and [21].

2) The proof applies to operator spaces $H \subset K_{w^*}(X', Y)$ containing the finite rank operators, which are invariant under $T \mapsto P \circ T$ for M-projections P, e.g. $H = X \hat{\otimes}_\varepsilon Y$, as well.

3) The isometric isomorphism between $K(V, Y)$ and $K_{w^*}(V'', Y)$ yields the following result:

2.6 COROLLARY. *If V has no non-trivial* L-*summand, then every* M-*summand in $K(V, Y)$ has the form $K(V, J)$ with some* M-*summand J in Y.*

PROOF. Apply Theorem 2.5 with $X = V'$ and note that M-summands of a dual space are necessarily weak*-closed (see [4, Theorem 5.6]).

We remark in passing that the same method as in the proof of 2.5 yields a dual result for L-summands of the projective tensor product if the extreme point structure of the spaces involved is sufficiently rich. The following result can be proved in analogy to 2.5, one only has to take into account (see [25], [30])

$$\operatorname{dent} B_{X \hat{\otimes}_\pi Y} = \operatorname{dent} B_X \otimes \operatorname{dent} B_Y,$$

where dent C denotes the set of denting points of a subset C of a Banach space.

2.7 PROPOSITION. *If* $B_X = \overline{\text{co}}\,\text{dent}\,B_X$ *and* $B_Y = \overline{\text{co}}\,\text{dent}\,B_Y$ *and if* X *has no non-trivial* L-*summand, then every* L-*summand in* $X \hat{\otimes}_\pi Y$ *has the form* $X \otimes_\pi J$ *with some* L-*summand* J *in* Y.

This proposition applies in particular to spaces with the Radon-Nikodým property (see [7, Theorem VII.3.3]).

Next, we shall consider M-ideals.

2.8 THEOREM. *Suppose* X *has no non-trivial* M-*ideal. Then every* M-*ideal* Z *in* $X \hat{\otimes}_\varepsilon Y$ *has the form* $Z = X \hat{\otimes}_\varepsilon J$ *with some* M-*ideal* J *in* Y.

To prepare the proof we present two results which are of independent interest.

2.9 LEMMA. *Let* $p \in \text{ex}\, B_{X'}$, $y \in S_Y$. *Suppose* μ *is a Radon probability on* $S = B_{X'} \times B_{Y'}$ *such that*

$$p(x) = \int_S x \otimes y \, d\mu$$

for all $x \in X$. *Then* $\text{supp}(\mu)$, *the support of* μ, *is contained in*

$$\Gamma \cdot \{p\} \times \{y' \in B_{Y'} : |\langle y', y\rangle| = 1\},$$

where Γ *is the set of scalars of modulus one.*

PROOF. Consider the measurable mapping $U : S \to B_{X'}$, $U(x', y') = \langle y', y\rangle \cdot x'$. Then we have for the image measure $\nu = U(\mu)$ and $x \in X$

$$\int_{B_{X'}} x\, d\nu = \int_S x \circ U \, d\mu$$
$$= \int_S x \otimes y \, d\mu = p(x).$$

Since p is extreme, it follows $\nu = \delta_p$ [1, Corollary I.2.4], hence

$$1 = \nu(\{p\}) = \mu(U^{-1}(\{p\}))$$
$$= \mu(\{(x', y') \in S : \langle y', y\rangle \cdot x' = p\}$$
$$\leqq \mu(\Gamma \cdot \{p\} \times \{y' \in B_{Y'} : |\langle y', y\rangle| = 1\}).$$

2.10 PROPOSITION. *Let* $p \in \text{ex}\, B_{X'}$ *and suppose* Z *is an* M-*ideal in* $X \hat{\otimes}_\varepsilon Y$. *Then the closure of* $(p \otimes \text{Id})(Z)$ *is an* M-*ideal in* Y.

Proof. Let E denote the L-projection from $(X \hat{\otimes}_\varepsilon Y)'$ onto Z°. Given $y'_0 \in S_{Y'}$, consider $E(p \otimes y'_0)$. We shall prove the existence of a (uniquely determined) functional $P(y'_0) \in Y'$ such that

$$E(p \otimes y'_0) = p \otimes P(y'_0). \tag{7}$$

To this end, represent the integral bilinear form $E(p \otimes y'_0)$ by a positive Radon measure μ_1 on $S := B_{X'} \times B_{Y'}$ such that $\|E(p \otimes y'_0)\| = \mu_1(S)$. In the same manner, let $p \otimes y'_0 - E(p \otimes y'_0)$ be represented by μ_2. Since E is an L-projection, $\mu := \mu_1 + \mu_2$ is a probability measure for which

$$\langle p \otimes y'_0, x \otimes y \rangle = \int_S x \otimes y \, d\mu$$

for all $x \in X$, $y \in Y$.

Assume for the moment that y'_0 attains its norm on B_Y. In this case

$$p(x) = \int_S x \otimes y_0 \, d\mu$$

for a suitable $y_0 \in S_Y$ and all $x \in X$. Then we have by Lemma 2.9

$$\operatorname{supp}(\mu_i) \subset \operatorname{supp}(\mu) \subset \Gamma \cdot \{p\} \times B_{Y'}$$

so that there are $\lambda_1 \in \Gamma$, $y'_1 \in B_{Y'}$ with

$$\begin{aligned} \langle E(p \otimes y'_0), x \otimes y \rangle &= \int_S x \otimes y \, d\mu_1 \\ &= \langle \lambda_1 p \otimes y'_1, x \otimes y \rangle \\ &= \langle p \otimes \lambda_1 y'_1, x \otimes y \rangle \end{aligned}$$

for all $x \in X$, $y \in Y$. Thus (7) is valid in case y'_0 attains its norm.

In the general case, the Bishop-Phelps theorem (see [7, Theorem VII.1.4]) yields a sequence of norm attaining functionals $y'_n \in S_{Y'}$ converging to y'_0 in norm. The validity of (7) for y'_n gives

$$E(p \otimes y'_0) = \lim E(p \otimes y'_n) = \lim p \otimes P(y'_n)$$

so that $P(y'_0) = \lim P(y'_n)$ exists, and (7) is proved in the general case, too.

Now define a mapping P from Y' into itself by (7). Obviously, P is an L-projection. It is left to prove

$$P(Y') = (p \otimes \mathrm{Id})(Z)^\circ.$$

"$\subset$" is immediate from the definitions of E and P. Conversely, suppose $y' \in Y'$ satisfies

$$0 = \langle y', (p \otimes \mathrm{Id})(u) \rangle = \langle p \otimes y', u \rangle$$

for all $u \in Z$. Then $p \otimes y' \in Z^\circ$ so that

$$p \otimes Py' = E(p \otimes y') = p \otimes y'.$$

Hence $Py' = y'$, and Proposition 2.10 is completely proved.

PROOF OF THEOREM 2.8. Again, we denote the L-projection onto Z° by E. By Proposition 2.10 we may partition the set $\mathrm{ex}\, B_{Y'}$ into the two subsets

$$C_1 = \{q \in \mathrm{ex}\, B_{Y'} : E(x' \otimes q) = x' \otimes q \quad \text{for all } x' \in X'\}$$
$$C_2 = \{q \in \mathrm{ex}\, B_{Y'} : E(x' \otimes q) = 0 \qquad \text{for all } x' \in X'\},$$

because X was assumed to have no non-trivial M-ideals. Next, we apply Proposition 2.10 once more to obtain a family of L-projections P_p on Y', indexed by the extreme functionals $p \in \mathrm{ex}\, B_{X'}$, with weak*-closed ranges which satisfy

$$E(p \otimes y') = p \otimes P_p(y')$$

for all $y' \in Y'$. Furthermore,

$$P_p(Y') = \{y' : p \otimes y' \in Z^\circ\} =: M_p.$$

Obviously, $C_1 \subset M_p$ for all $p \in \mathrm{ex}\, B_{X'}$. On the other hand, let $q \in \mathrm{ex}\, B_{M_p}$. Then $q \in \mathrm{ex}\, B_{Y'}$ (since M_p is an L-summand) and $p \otimes q \in Z^\circ$, that is $q \in C_1$. M_p is weak*-closed and thus

$$M_p = \overline{\mathrm{lin}}^{w*} C_1$$

independently of p. In other words,

$$J := \{y \in Y : q(y) = 0 \quad \text{for all } q \in C_1\}$$

is an M-ideal in Y. To prove $X \hat{\otimes}_\varepsilon J = Z$ it is enough to show $(X \hat{\otimes}_\varepsilon J)^\circ = Z^\circ$. Both spaces are weak*-closed L-summands (Corollary 2.3), therefore it is enough to check the coincidence of the extreme points of the unit ball, which must have the form $p \otimes q$ with p and q extremal (cf. 1.1 and 1.2). In fact, $p \otimes q \in Z^\circ$ iff $q \in C_1$ iff $q \in J^\circ$ iff $p \otimes q \in (X \hat{\otimes}_\varepsilon J)^\circ$.

2.11 Corollary. *Let X and Y be Banach spaces without proper* M-*ideals* (i.e. *every* M-*ideal is an* M-*summand*). *Then $X \hat{\otimes}_\varepsilon Y$ fails to have proper* M-*ideals.*

Proof. It is known that a Banach space without proper M-ideals is isometrically isomorphic to a c_0-sum of Banach spaces without non-trivial M-ideals, (see [15, Proposition I.2.11]). If $X = (\oplus_i X_i)_{c_0}$ and $Y = (\oplus_j Y_j)_{c_0}$ are represented in this way, then

$$X \hat{\otimes}_\varepsilon Y = (\oplus_{i,j} X_i \hat{\otimes}_\varepsilon Y_j)_{c_0},$$

and $X_i \hat{\otimes}_\varepsilon Y_j$ has no non-trivial M-ideal by Theorem 2.8. Now an application of [4, Proposition 4.9] completes the proof.

Remarks. 1) Classes of Banach spaces without proper M-ideals are spaces which are M-ideals in their bidual (see [16]) and spaces which do not contain a copy of c_0 (see [16]). Neither of these classes is stable with respect to forming injective tensor products. For example, consider $L^p = L^p[0,1]$ for $1 < p < \infty$, $p \neq 2$. By Khintchin's inequality, l^2 is isomorphic to a subspace of L^p (respectively L^q, where $1/p + 1/q = 1$).

Thus,

$$c_0 \hookrightarrow l^2 \hat{\otimes}_\varepsilon l^2 \hookrightarrow L^q \hat{\otimes}_\varepsilon L^p = K(L^p),$$

moreover $K(L^p)$ is not an M-ideal in its bidual $L(L^p)$ for p as above (see [20]).
2) The operator version of Theorem 2.8 is:

2.12. Corollary. *If V' has the approximation property and if V' fails to contain non-trivial* M-*ideals, then every* M-*ideal in $K(V, Y)$ has the form $K(V, J)$ where J is an* M-ideal in Y.

In particular, this corollary applies to $V = L^p(\mu)$, $1 < p \leqq \infty$, in the case of real scalars one has to exclude $V = l^\infty(2)$. Thus, we recover the result from [28] that $K(l^p)$ has no non-trivial M-ideal for $1 < p < \infty$.
Without the approximation assumption in Corollary 2.12 we have the following weaker result.

2.13 Proposition. *If X and Y have no non-trivial* M-*ideals, then neither has $K_{w^*}(X', Y)$.*

Proof. The proof of Proposition 2.10 shows that the closure of $\{T(p) : T \in Z\}$ is an M-ideal in Y if $p \in \operatorname{ex} B_{X'}$ and Z is an M-ideal in $H := K_{w^*}(X', Y)$. As a matter of fact, the essential property to be used is that H may be embedded isometrically into $C(B_{X'} \times B_{Y'})$. Analogously, $\{T'(q) : T \in Z\}^-$ is an M-ideal in X for $q \in \operatorname{ex} B_{Y'}$. One can, therefore, partition $\operatorname{ex} B_{Y'} = C_1 \cup C_2$ in the same way as in the proof of Theorem 2.8. Parallelling this proof one arrives at the conclusion that

$$J = \{y : q(y) = 0 \quad \text{for all } q \in C_1\}$$

is an M-ideal in Y. Hence, $J = \{0\}$ or $J = Y$. In the first case it follows that $\operatorname{ex} B_{Y'} = C_1$ and $Z = \{0\}$. (The L-summand which is complementary to Z° is isometrically isomorphic to Z', but has no extreme points because of $C_2 = \emptyset$.) In the second case $\operatorname{ex} B_{Y'} = C_2$ holds and thus $Z = H$.

2.14 COROLLARY. *If X' and Y have no non-trivial* M*-ideals, then neither has* $K(X, Y)$.

This conclusion has been established in [16] and [21] for reflexive spaces. The results achieved so far support the point of view that M-structure properties are very well reflected by the injective tensor norm. So it may come as a surprise that there is an injective tensor product with non-trivial L-summands.

2.15 PROPOSITION.

$$l^2(2, \mathsf{R}) \hat{\otimes}_\varepsilon l^2(2, \mathsf{R}) \cong l^2(2, \mathsf{R}) \oplus_1 l^2(2, \mathsf{R}).$$

PROOF. Let $X = l^2(2, \mathsf{R}) \hat{\otimes}_\varepsilon l^2(2, \mathsf{R} = L(l^2(2, \mathsf{R}))$. It is well-known that $\operatorname{ex} B_X = \mathrm{O}(2, \mathsf{R})$, the group of orthogonal 2×2 matrices. Let E (*respectively* $\hat{E}$) denote the linear span of all orthogonal matrices with determinant $+1$ (respectively -1). Then $X = E \oplus \hat{E}$ and $E \cong \hat{E} \cong l^2(2, \mathsf{R})$. If P denotes the projection from X onto E, then $P(\operatorname{ex} B_X) \subset \{0, 1\} \cdot \operatorname{ex} B_X$ so that P is an L-projection by [19, Theorem 4.6].

The exceptional character of Proposition 2.15 is underlined by the following

2.16 PROPOSITION. *If H is a complex Hilbert space or a real Hilbert space of dimension $\geqq 3$, then $H \hat{\otimes}_\varepsilon H$ contains no non-trivial* L*-summand.*

PROOF. If H is infinite-dimensional, then $H \hat{\otimes}_\varepsilon H = K(H)$ is an M-ideal in its bidual $L(H)$ (see [8] or [27]) and hence cannot contain a non-trivial L-summand (see [4, Corollary 1.14]).

In the rest of the proof assume $\dim(H) < \infty$. We shall make use of the following observation. If an L-summand J of X contains an extreme point x_0 of B_X, then the whole connected component of x_0 (with respect to the norm topology) is contained in J. This remark proves 2.16 in the case of complex scalars, since $\operatorname{ex} B_{H \hat{\otimes}_\varepsilon H}$, the group of unitary complex matrices, is connected.

Next consider Euclidean spaces of odd dimension n. In this case we have

$$\operatorname{ex} B_{H \otimes_\varepsilon H} = \mathrm{O}(n, \mathrm{R}) = \mathrm{SO}(n, \mathrm{R}) \cup -\mathrm{SO}(n, \mathrm{R}).$$

The above remark shows $\pm \mathrm{SO}(n) \subset J$ for every L-summand $J \neq \{0\}$ in $H \otimes_\varepsilon H$, hence $J = H \otimes_\varepsilon H$.

Now let $H = H_n$ be a Euclidean space of even dimension $n \geqq 4$. As in the proof of 2.15 write E_n ($\hat{E}_n$) for the linear span of the orthogonal matrices of determinant $+1$ (-1), of course, $E_n \cong \hat{E}_n$ and $H_n \otimes_\varepsilon H_n = E_n + \hat{E}_n$. The connectedness of $\mathrm{SO}(n)$ shows that E_n and $\hat{E}_n$ are the only candidates for non-trivial L-summands, again according to our observation. But it is quickly verified that $R \oplus 0 \in E_n \cap \hat{E}_n$ for $n \geqq 4$, where R denotes the rotation operator with angle $\pi/4$ on H_2 and 0 is the zero operator on H_{n-2}. Hence the sum $E_n + \hat{E}_n$ is not direct.

Using the same method one can show that the L-summands in $K(l^p(2))$ must be trivial, this time the proof depends on the extreme point characterization in [13], here $1 < p < \infty$.

3. The projective hulls of the injective tensor norm.

Let α be a tensor norm. Then there is a tensor norm $\alpha/$ with the property: If X, Y, and Z are Banach spaces and if $Q: Z \to Y$ is a quotient map, then

$$\mathrm{Id} \otimes Q: X \otimes_{\alpha/} Z \to X \otimes_{\alpha/} Y$$

is a quotient map, and $\alpha/$ is the smallest tensor norm dominating α with respect to this property. It can be defined by considering any L^1-space Z and quotient map $Q: Z \to Y$ and transporting the α-norm on $X \otimes Z$ to $X \otimes Y$ via $\mathrm{Id} \otimes Q$ (cf. [12, p. 30, Corollary 4]). $\alpha/$ is called the right-sided projective hull of α; the modifications for the left-sided version are obvious.

We are interested in the M-structure properties of $\alpha/$ in comparison with those of α. First of all, it is routine to establish the following:

3.1 Proposition. *Let J be an* M*-summand in X and $J \hat{\otimes}_\alpha Y$ be an* M*-summand in $X \hat{\otimes}_\alpha Y$. Then $J \hat{\otimes}_{\alpha/} Y$ is an* M*-summand in $X \hat{\otimes}_{\alpha/} Y$.*

3.2 Corollary. *If J is an* M*-summand in X, then $J \hat{\otimes}_{\varepsilon/} Y$ is an* M*-summand in $X \hat{\otimes}_{\varepsilon/} Y$.*

Proof. Immediate from Proposition 2.1.

3.3 Corollary. If J is an M-ideal in X, then the $\varepsilon/$-closure of $J \otimes Y$ is an M-ideal in $X \hat{\otimes}_{\varepsilon/} Y$.

PROOF. Immediate from Corollary 3.2 and Proposition 2.2 (and the comment following it, to be sure).

REMARKS. 1) Since an M-summand J is (by its very definition) the range of a contractive projection, the α-closure of $J \otimes Y$ coincides with $J \hat{\otimes}_\alpha Y$ for all tensor norms α.

2) If J is an M-summand in Y, then $X \hat{\otimes}_{\varepsilon/} J$ need not be an M-summand in $X \hat{\otimes}_{\varepsilon/} Y$, see below.

3) It is well-known that $X \hat{\otimes}_\varepsilon Y = X \hat{\otimes}_{\varepsilon/} Y$ whenever X is an L^1-predual, cf. e.g. [14].

In the remainder of this section we shall be concerned with the converse of Corollary 3.2. To this end, we need some information on the extreme points of the unit ball of $(X \hat{\otimes}_{\varepsilon/} Y)' = L^{(\varepsilon/)'}(X, Y')$. It is known that the space of $(\varepsilon/)'$-integral operators coincides isometrically with the space of absolutely summing operators, this is, of course, implicit in [12] and is stated explicity in [26, Theorem 3.2]. Recall that an operator T between Banach spaces X and Y is called absolutely summing, if there is a number $c \geqq 0$ such that

$$\Sigma\|Tx_i\| \leqq c \cdot \sup\{\Sigma|x'(x_i)| : x' \in B_{X'}\}$$

for every finite family $\{x_1, \ldots, x_n\} \subset X$. The smallest such number c is denoted by $\pi_1(T)$, π_1 defines a complete norm on the space $\Pi^1(X, Y)$ of absolutely summing operators.

The following results are implicitly contained in [29]. We prefer, however, to give a simple proof based on Lemma 2.9.

3.4 LEMMA. *Suppose $T \in \Pi^1(X, Y)$ with $\pi_1(T) = 1$ and $T'(q) \in \operatorname{ex} B_{X'}$ for some $q \in S_{Y'}$. Then $T'(Y') = \operatorname{lin}\{T'(q)\}$, in particular, T is a one-dimensional operator.*

PROOF. Considering jT instead of T (with $j: Y \to l^\infty(B_{Y'})$ the canonical embedding) we may assume that T is integral with integral norm one (see [22, Theorem 19.2.7]). Consequently, there exists a Radon probability μ on $S := B_{X'} \times B_{Y''}$ such that

$$\langle Tx, y' \rangle = \int_S x \otimes y' d\mu$$

for all $x \in X$, $y' \in Y'$. Particularly we have, letting $p := T'(q) \in \operatorname{ex} B_{X'}$,

$$p(x) = \int_S x \otimes q d\mu$$

for $x \in X$. From Lemma 2.9 we infer $\operatorname{supp}(\mu) \subset \Gamma \cdot \{p\} \times B_{Y''}$ so that there is a measure ν on $\Gamma \times B_{Y''}$ with

$$\langle Tx, y' \rangle = \int_{\Gamma \times B_{Y''}} \langle \lambda y'', y' \rangle d\nu(\lambda, y'') \cdot p(x).$$

Hence $T'y' \in \operatorname{lin}\{p\}$.

3.5 PROPOSITION. *For $p \in \operatorname{ex} B_{X'}$ and $y_0 \in \operatorname{ex} B_Y$, $p \otimes y_0 \in \operatorname{ex} B_{\Pi^1(X,Y)}$.*

PROOF. (Cf. [29, Proposition 2]:) Suppose $p \otimes y_0 = (T_1 + T_2)/2$ with $\pi_1(T_i) \leqq 1$. Choose $q \in S_{Y'}$ with $q(y_0) = 1$. Hence $p = (T_1'(q) + T_2'(q))/2$. Since $\|T_i'(q)\| \leqq \|T_i\| \leqq \pi_1(T_i) \leqq 1$ and since p is extreme, $T_1'(q) = T_2'(q) = p$. By Lemma 3.4, $T_i = p \otimes y_i$ for some $y_i \in Y$, and the result follows.

We now pass to the main result of this section.

3.6 THEOREM. *Let X or Y have the approximation property. In case Y has no non-trivial* M*-summand, every* M*-summand in $X \otimes_{\varepsilon /} Y$ has the form $J \otimes_{\varepsilon /} Y$ with some* M*-summand J in X.*

PROOF. Suppose there is a decomposition $H := X \hat{\otimes}_{\varepsilon /} Y = Z_1 \oplus_\infty Z_2$. Then there are weak*-closed L-summand $V_i = Z_i^\circ$ such that $H' = V_1 \oplus_1 V_2$. As in the proof of Theorem 2.5 we obtain a decomposition $X' = N_1 \oplus_1 N_2$ where

$$\begin{aligned} N_i &= \{x' \in X' : x' \otimes q \in V_i \quad \text{for some } q \in \operatorname{ex} B_{Y'}\} \\ &= \{x' \in X' : x' \otimes y' \in V_i \quad \text{for all } \quad y' \in Y'\} \end{aligned}$$

is weak*-closed. (To see this it is enough to know $\operatorname{ex} B_{X'} \otimes \operatorname{ex} B_{Y'} \subset \operatorname{ex} B_{H'}$, this follows from Proposition 3.5 since $H' = \Pi^1(X, Y')$.)

Now let J_i be the closed subspace of X with $J_i^\circ = N_i$. J_1 and J_2 are complementary M-summands by construction. It is left to prove $Z_i = J_i \hat{\otimes}_{\varepsilon /} Y$. Using Corollary 3.2 we see that it suffices to show $Z_1 \subset J_1 \hat{\otimes}_{\varepsilon /} Y$. In fact, if $u \in Z_1$, then $\langle u, x' \otimes y' \rangle = 0$ for $x' \in N_1$, $y' \in Y'$. Denote the M-projection from X onto J_2 by P_2. It follows

$$0 = \langle u, P_2'(x') \otimes y' \rangle = \langle (P_2 \otimes \mathrm{Id})(u), x' \otimes y' \rangle$$

for $x' \in X', y' \in Y'$. Hence $\Phi((P_2 \otimes \mathrm{Id})(u)) = 0$, where $\Phi : X \hat{\otimes}_{\varepsilon /} Y \to X \hat{\otimes}_\varepsilon Y$ denotes the natural operator. It is a consequence of the approximation assumption that Φ is injective (see [12, p. 15]). We conclude $(P_2 \otimes \mathrm{Id})(u) = 0$, that is $u \in J_1 \hat{\otimes}_{\varepsilon /} Y$.

3.7 COROLLARY. *Under the assumption of Theorem* 3.6, *and* L*-projection E*

on $\Pi^1(X, Y')$ which is continuous with respect to the weak-operator topology has the form $E(T) = T \circ P$, where P is an* M*-projection on X.*

PROOF. E is the adjoint of an M-projection on $X \otimes_{\varepsilon/} Y$.

REMARKS. 1) In the special case of reflexive spaces X and Y no additional continuity assumptions on an L-projection E need be made in 3.7, if we specify X to have the approximation property. The reason is that in this situation $\Pi^1(X, Y)$ is reflexive (see [11, Theorem 4.2]).

2) Pietsch's factorization theorem (see [22, Theorem 17.3.2 and 17.3.3]) can be employed to prove that $T \to T \circ P$ is an L-projection on $\Pi^1(X, Y)$ for M-projections P without recourse to tensor products.

3) The non-symmetry of the $\varepsilon/$-norm is stressed by the following example: Let $X = l^1(3, \mathbb{R})$, $Y = l^\infty(3, \mathbb{R})$, and let P denote the M-projection from Y onto the first two coordinates. Then $\mathrm{Id} \otimes P$ is not an M-projection on $X \otimes_{\varepsilon/} Y$, equivalently $T \to P' \circ T$ is not an L-projection on $\Pi^1(X)$. In fact, it is quickly established by means of elementary calculations that $\pi_1(\mathrm{Id} - P') = 1$, $\pi_1(P') = \pi_1(\mathrm{Id}) = 2$.

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