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MAXIMALLY GENERATED COHEN-MACAULAY MODULES

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0. Introduction.

This paper examines a class of finitely generated Cohen-Macaulay modules of the maximal possible dimension over a Cohen-Macaulay local or homogeneous ring. The number of generators of such a module is bounded by the multiplicity of the module. The class examined herein consists of those modules for which this bound is attained. These modules are denominated MGMCM modules (*M*aximally *G*enerated *M*aximal *C*ohen-*M*acaulay modules).

The major result of this paper is to show the existence of MGMCM modules for two dimensional homogeneous Cohen-Macaulay domains. This is done by showing that such rings possess a Gorenstein ideal primary to the irrelevant maximal ideal with the maximum possible number of generators, and then constructing a MGMCM module by extending this ideal by the canonical module.

The existence of a MGMCM module is of some interest in that if the residue class field of the ring is infinite such a module is the lifting of a direct sum of copies of the residue class field. The MGMCM modules over a local or homogeneous Cohen-Macaulay ring R are those modules M which are Cohen-Macaulay modules of maximal possible dimension such that $\hat{M}$, the completion of M, has a linear A-resolution where $\hat{R} = A/I$, A a regular local ring. This provides the fact that if a local Cohen-Macaulay ring R admits a MGMCM module, then the associated graded ring of R with respect to the maximal ideal admits a small Cohen-Macaulay module.

The existence of MGMCM module of rank m over a hypersurface domain $R = A/(f)$ with A a regular local or homogeneous ring has been shown by Eisenbud [3] to yield a presentation of f^m as the determinant of a $m[e(R)] \times m[e(R)]$-matrix. Therefore as a consequence of Theorem 4.8, we have for a homogeneous polynomial f of degree e, $f \in k[x, y, z]$, that the polynomial f^2 is the determinant of a $2e \times 2e$-matrix with linear entries.

Section one provides an introduction to notation and properties of

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MGMCM modules. In section two some elementary cases in which MGMCM modules exist are explicitly given. Rings of dimension one, rings of minimal multiplicity and rings associated with the maximal minors of a matrix with linear entries are explicitly shown to possess MGMCM modules. Section three uses a remarkable device of Eisenbud, namely matrices with no generalized zeros to construct Gorenstein ideals in any two dimensional homogeneous domain. A matrix A is said to have no generalized zeros, if A has as entries linear forms from $k[x_1, \ldots, x_m]$ and no nontrivial k-linear combination of the rows of A and the columns of A produces a zero entry. The key result here is Eisenbud's [4] who – among other things – shows that matrices with no generalized zeros have nonzero determinant.

Section four is concerned with the construction and existence of MGMCM modules and shows that the existence of a Cohen-Macaulay ideal I of codimension two with $e(R)[r(R/I)+1]$ generators implies the existence of a MGMCM module of multiplicity $e(R)[r(R/I)+1]$. ($e(R)$ is the multiplicity of R while $r(R/I)$ is the type of R/I.)

Section five deals with the question of what ranks can occur as the rank of a MGMCM module M over a normal homogeneous hypersurface domain R of dimension two, where M has the property that $[\mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda^{\mathrm{rank}\, M} M, R), R)] = 0$ in the class group of R. Such a module is called orientable (see [8]). The semigroup of ranks of such modules is described and shown to depend on the multiplicity of the ring. In particular if the multiplicity of the ring R is odd and greater than one, the ring is shown to have an ideal primary to the irrelevant maximal ideal with three times the multiplicity of the ring generators. This shows the existence of a MGMCM module of rank three.

Section six concludes the paper with four questions we consider to have some interest.

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1. Preliminaries.

A *homogeneous ring* over a field k is a graded ring $\bigoplus_{i \geq 0} R_i$ with $R_0 = k$ and generated by R_1 as a k-algebra. Let $R_+ = \bigoplus_{i \geq 1} R_i$ be the irrelevant maximal ideal of R. By an R-module for a homogeneous ring we will mean a graded R-module. All modules will be finitely generated.

In this paper all rings are Noetherian Cohen–Macaulay commutative rings with unit and either local with maximal ideal $\mathfrak{m}$ or homogeneous with irrelevant maximal ideal $\mathfrak{m}$ and the field $R/\mathfrak{m}$ will be denoted by k.

If R is such a ring, a Cohen-Macaulay R-module M is said to be a *Maximal Cohen-Macaulay module* if $\operatorname{depth}_{\mathfrak{m}} M = \dim R$. For M an R-module, let $\nu(M)$ be the minimal number of generators of M and let $e(M)$ denote the multiplicity of M. The module M is said to have rank m if for every associated prime q of R, $M_q \cong \bigoplus^m R_q$.

PROPOSITION (1.1) (see [14], [15]). *For R as above, if M is a Maximal Cohen-Macaulay R-module, then $\nu(M) \leqq e(M)$.*

PROOF. After a purely transcendental extension of the residue class field, we may assume that $\mathfrak{m}$ has a minimal reduction generated by a maximal M-regular sequence $\mathfrak{x}$. Then $\nu(M) = \dim_k(M/\mathfrak{m}M) \leqq \operatorname{length}(M/\mathfrak{x}M) = e(M)$.

If M has positive rank, then $e(M) = e(R)\operatorname{rank}(M)$ so that the inequality of (1.1) becomes $\nu(M) \leqq e(R)\operatorname{rank}(M)$.

A Maximal Cohen-Macaulay module M will be called a MGMCM *module* (*M*aximally *G*enerated *M*aximal *C*ohen-*M*acaulay) provided that $\nu(M) = e(M)$. Such a module is denominated an Ulrich module in [8]. In [15] the question was asked: Does every Cohen-Macaulay ring R admit a MGMCM module?

If R is zero dimensional the answer is positive and all such modules can be characterized.

PROPOSITION (1.2). *Let R be as above, and zero dimensional with residue class field k, then M is a* MGMCM *R-module if and only if $M \cong \bigoplus^{\nu(M)} k$.*

PROOF. If M is a MGMCM module, then $\dim_k M/\mathfrak{m}M = \nu(M) = e(M) = \operatorname{length} M$. So $\operatorname{ann} M = \mathfrak{m}$ and hence $M \cong \bigoplus^{\nu(M)} k$. The converse is clear.

The question of the existence of a MGMCM module over a given ring R becomes by the following lemma a question of whether some MGMCM module can be lifted from a zero dimensional specialization of R.

LEMMA (1.3). *Let R be as above with k an infinite field and let $\dim M = \dim R$, then M is a* MGMCM *module if and only if there exists a regular sequence $\mathfrak{x}$ on M such that $\mathfrak{x}M = \mathfrak{m}M$.*

In fact the regular sequence in (1.3) can be taken to be general. This allows reduction of many problems to the zero dimensional case. An example of this is:

PROPOSITION (1.4). *R as above, M a* MGMCM *R-module and M'' a Maximal Cohen-Macaulay module with $0 \to M' \to M \to M'' \to 0$ exact. Then M' and M'' are* MGMCM *modules.*

PROOF. After extending the residue class field of R if needed one can select a general maximal regular M-sequence $\mathfrak{x}$ with $\mathfrak{x}M = \mathfrak{m}M$ such that $\mathfrak{x}$

is also a regular sequence on M' and M''. Denoting reduction modulo $\mathfrak{x}$ by two bars one obtains $0 \to \bar{\bar{M}}' \to \bigoplus^{\nu(M)} k \to \bar{\bar{M}}'' \to 0$ is exact. Thus $\bar{\bar{M}}' \cong \bigoplus^{\nu(M')} k$ and $\bar{\bar{M}}'' \cong \bigoplus^{\nu(M'')} k$. Thus by (1.3), M' and M'' are MGMCM modules.

Providing interest and of particular import in the study of MGMCM modules is the relation of these modules to linear resolutions.

Suppose R is a local (respectively homogeneous) ring and S is a regular local (respectively homogeneous) ring with $S \twoheadrightarrow R$ a local (respectively homogeneous) surjection. Denote the maximal (respectively irrelevant maximal) ideal of S by $\mathfrak{n}$. Then an R-module M has a *linear S-resolution*, if for a free minimal S-resolution of M

$$\cdots \xrightarrow{\varphi_2} \bigoplus^{\beta_1(M)} S \xrightarrow{\varphi_1} \bigoplus^{\beta_0(M)} S \xrightarrow{\varphi_0} M \to 0$$

the complex

$$\cdots \xrightarrow{\mathrm{gr}(\varphi_2)} \mathrm{gr}\left(\bigoplus^{\beta_1(M)} S\right) \xrightarrow{\mathrm{gr}(\varphi_1)} \mathrm{gr}\left(\bigoplus^{\beta_0(M)} S\right) \xrightarrow{\mathrm{gr}(\varphi_0)} \mathrm{gr}_{\mathfrak{m}}(M) \to 0,$$

where $\bigoplus^{\beta_i(M)} S$ is filtered by

$$F_j\left(\bigoplus^{\beta_i(M)} S\right) = \begin{cases} \bigoplus^{\beta_i(M)} S & \text{for } j < i \\ \mathfrak{n}^{j-i}\left(\bigoplus^{\beta_i(M)} S\right) & \text{for } j \geqq i \end{cases}$$

is exact. Hence the latter complex is a minimal free $\mathrm{gr}_{\mathfrak{n}}(S)$ resolution of $\mathrm{gr}_{\mathfrak{m}}(M)$. If M is as above and has a linear S-resolution, then $\mathrm{gr}_{\mathfrak{m}}(M)$ has a minimal graded free $\mathrm{gr}_{\mathfrak{n}}(S)$-resolution of the form

$$\cdots \to \bigoplus^{\beta_d(M)} \mathrm{gr}_{\mathfrak{n}}(S)(-d) \to \cdots \bigoplus^{\beta_1(M)} \mathrm{gr}_{\mathfrak{n}}(S)(-1) \to \bigoplus^{\beta_0(M)} \mathrm{gr}_{\mathfrak{n}}(S) \to \mathrm{gr}_{\mathfrak{m}}(M) \to 0.$$

A sequence of elements $\{x_1, \ldots, x_n\}$ in a ring R is a *d-sequence* (see [10]) with respect to the R-module M if:

i) $\{x_1, \ldots, x_n\}$ is a minimal generating set for the ideal $(\mathfrak{x})$ generated by $\{x_1, \ldots, x_n\}$ and
ii) $(x_1, \ldots, x_i)M :_M x_{i+1} \cap (\mathfrak{x})M = (x_1, \ldots, x_i)M$ for $i = 0, \ldots, n-1$.

PROPOSITION (1.5). *Let $(R, \mathfrak{m})$ be a local (respectively homogeneous) Cohen-*

Macaulay ring, and let $(S, \mathfrak{n})$ *be a local (respectively homogeneous) regular ring with a local (respectively homogeneous) surjection* $S \to R$. *And let* $R/\mathfrak{m}$ *be infinite. Let* M *be a Maximal Cohen–Macaulay* R*-module. Then the following are equivalent:*

i) M *is a* MGMCM R*-module.*
ii) *The ideal* $\mathfrak{n}$ *is generated by a* d*-sequence on* M.
iii) M *has a* S*-linear resolution.*

PROOF. i) $\Rightarrow$ ii). Let $\bar{x}_1, \ldots, \bar{x}_d$ be a general maximal regular M-sequence in R such that $\bar{\mathfrak{x}}M = \mathfrak{m}M$. Let x_i be a preimage of $\bar{x}_i$ in S. Extend the set $\{x_1, \ldots, x_d\}$ to $\{x_1, \ldots, x_n\}$ a minimal set of generators of $\mathfrak{n}$. If $i < d$, then

$$(x_1, \ldots, x_i)M :_M x_{i+1} \cap \mathfrak{n}M = (\bar{x}_1, \ldots, \bar{x}_i)M :_M \bar{x}_{i+1} \cap \mathfrak{m}M = (\bar{x}_1, \ldots, \bar{x}_i)M.$$

If $i \geqq d$, then

$$\begin{aligned}(x_1, \ldots, x_i)M :_M x_{i+1} \cap \mathfrak{n}M &= (\bar{x}_1, \ldots, \bar{x}_i)M :_M \bar{x}_{i+1} \cap \mathfrak{m}M = \mathfrak{m}M \\ &= (\bar{x}_1, \ldots, \bar{x}_i)M = (x_1, \ldots, x_i)M.\end{aligned}$$

Hence $\{x_1, \ldots, x_n\}$ is a d-sequence on M.

ii) $\Rightarrow$ iii). This was proved in [10].

iii) $\Rightarrow$ i). Since M has a S-linear resolution there is a resolution

$$\begin{aligned}0 \to \bigoplus^{\beta_p(M)} \mathrm{gr}_{\mathfrak{n}}(S)(-p) \to \cdots \to \bigoplus^{\beta_1(M)} \mathrm{gr}_{\mathfrak{n}}(S)(-1) \\ \to \bigoplus^{\beta_0(M)} \mathrm{gr}_{\mathfrak{n}}(S) \to \mathrm{gr}_{\mathfrak{m}}(M) \to 0.\end{aligned}$$

Applying a result of Herzog and Kühl [7; p. 1632],

$$\begin{aligned}e(M) = e(\mathrm{gr}_{\mathfrak{m}}(M)) &= \nu(\mathrm{gr}_{\mathfrak{m}}(M))\left\{\sum_{i=1}^{p}\left(\prod_{\substack{j=1\\ j\neq i}}^{p}\frac{j}{i-j}\right)\cdot\binom{i}{p}\right\} \\ &= \nu(\mathrm{gr}_{\mathfrak{m}}(M))\left(\prod_{j=1}^{p-1}\frac{j}{p-j}\right) = \nu(\mathrm{gr}_{\mathfrak{m}}(M)) = \nu(M).\end{aligned}$$

Hence M is a MGMCM R-module.

For the next corollary note that even if M is graded, M is not necessarily isomorphic to $\mathrm{gr}_{\mathfrak{m}}(M)$.

COROLLARY 1.6. *Let* $(R, \mathfrak{m})$ *be a local (respectively homogeneous) Cohen-Macaulay ring. Let* M *be a* MGMCM R*-module, then* $\mathrm{gr}_{\mathfrak{m}}(M)$ *is a Maximal Cohen-Macaulay* $\mathrm{gr}_{\mathfrak{m}}(R)$*-module with the same number of generators. Further* *assume that* $\mathrm{gr}_{\mathfrak{m}}(M)$ *has a rank* (e.g. *if* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is a domain*), *then* $\operatorname{rank} M = \operatorname{rank} \mathrm{gr}_{\mathfrak{m}}(M)$, *and if* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay, then* $\mathrm{gr}_{\mathfrak{m}}(M)$ *is a* MGMCM *module.*

PROOF. Let $(S, \mathfrak{n})$ be a regular ring with $\hat{R} \cong S/I$. Then the Betti numbers of $\hat{M}$ as a S-module are the same as those of $\mathrm{gr}_{\mathfrak{m}}(M)$ as $\mathrm{gr}_{\mathfrak{n}}(S)$-module. Thus $\mathrm{gr}_{\mathfrak{m}}(M)$ is Maximal Cohen-Macaulay with the same number of generators as M. If $\mathrm{gr}_{\mathfrak{m}}(M)$ has a rank, then

$$\operatorname{rank} \mathrm{gr}_{\mathfrak{m}}(M) = e(\mathrm{gr}_{\mathfrak{m}}(R))^{-1} e(\mathrm{gr}_{\mathfrak{m}}(M)) = e(R)^{-1} e(M) = \operatorname{rank} M.$$

Therefore $\operatorname{rank} \mathrm{gr}_{\mathfrak{m}}(M) = \operatorname{rank} M$.

2. Elementary results on the existence of Maximally Generated Maximal Cohen-Macaulay modules.

This section provides an account of the folklore on the existence of MGMCM modules. Some of these results were mentioned in [15]. Again $(R, \mathfrak{m})$ is a local (or homogeneous) Cohen-Macaulay ring.

LEMMA (2.1). *If R is a one dimensional ring, then $\mathfrak{m}^{e(R)-1}$ is a* MGMCM *R-module.*

PROOF. Since $\dim R = 1$, $e(R) = \dim_k(\mathfrak{m}^{e(R)-1}/\mathfrak{m}^{e(R)})$ (see [14]; page 36, Theorem 2.31), and we have

$$e(\mathfrak{m}^{e(R)-1}) = e(R) = \dim_k(\mathfrak{m}^{e(R)-1}/\mathfrak{m}^{e(R)}) = \nu(\mathfrak{m}^{e(R)-1}).$$

The fact that a Cohen-Macaulay ring is regular, if and only if it has multiplicity one can be reinterpreted as:

LEMMA (2.2). *R is a* MGMCM *R-module if and only if R is regular.*

A local Cohen-Macaulay ring R satisfies the inequality $e(R) \geqq$ embedding $\text{dimension}(R) - \dim R + 1$ (see [1], [13]). Such a ring R is said to be of *minimal multiplicity*, if equality holds in this relation.

EXAMPLE (2.3). Let k be a field and f a homogeneous polynomial of degree two in $k[x_1, \ldots, x_n]$, then $k[x_1, \ldots, x_n]/(f)$ is a ring of minimal multiplicity.

We can now characterize rings of minimal multiplicity having positive dimension.

LEMMA (2.4). *Let M be an R-module and let x be an M-regular element of R. Let $\bar{M} = M/xM$. Then there exists an exact sequence*

$$0 \to \mathrm{syz}_n(M) \to \mathrm{syz}_n(\bar{M}) \to \mathrm{syz}_{n-1}(M) \to 0$$

for every $n > 0$, *where* $\mathrm{syz}_n(M)$ *is the* nth *syzygy module of the module* M *over* R.

PROOF. Consider P a minimal free R-resolution of M. The map $P^{\cdot} \xrightarrow{\mu_x} P^{\cdot}$ given by multiplication by x yields the short exact sequence of complexes

$$0 \to P^{\cdot} \to C(\mu_x) \to P^{\cdot}[-1] \to 0$$

with $C(\mu_x)$ the mapping cone of the map μ_x, which is a minimal resolution of $\bar{M}$ as an R-module. Truncation yields the required exact sequences for $n > 1$. For $n = 1$ the claim is obvious.

PROPOSITION (2.5). $(R, \mathfrak{m})$ *a* d*-dimensional ring with* $k \cong R/\mathfrak{m}$ *and* $d > 0$. *Then the following conditions are equivalent*:

i) *There exists a* MGMCM R*-module* N *such that* $\mathrm{syz}_1(N)$ *is a* MGMCM R*-module.*
ii) *For some* $i > 0$, $\mathrm{syz}_i(k)$ *is a* MGMCM R*-module.*
iii) *For every* $i \geqq d$, $\mathrm{syz}_i(k)$ *is a* MGMCM R*-module.*
iv) R *has minimal multiplicity.*

PROOF. After an extension of the residue class field if needed we may assume that k is infinite.

ii) $\Rightarrow$ i) By assumption $M = \mathrm{syz}_i(k)$ is a MGMCM R-module. By Lemma (2.4), we have

$$0 \to \mathrm{syz}_i(\bar{M}) \to \mathrm{syz}_i(\bar{\bar{M}}) \to \mathrm{syz}_{i-1}(\bar{M}) \to 0,$$

where one bar denotes reduction by the first $d-1$ elements of a maximal general regular M-sequence and two bars denotes reduction by the entire general regular M-sequence. But since M is a MGMCM module, $\bar{\bar{M}} \cong \bigoplus^{\nu(M)} k$, so

$$\mathrm{syz}_i(\bar{\bar{M}}) = \bigoplus^{\nu(M)} (\mathrm{syz}_i(k)).$$

Therefore $\mathrm{syz}_i(\bar{\bar{M}})$ is a MGMCM module. Moreover $i \geqq d$, since $M = \mathrm{syz}_i(k)$ is Cohen-Macaulay. Hence as $\mathrm{syz}_{i-1}(\bar{M})$ is a Maximal Cohen-Macaulay module, by Proposition (1.4), $\mathrm{syz}_i(\bar{M})$ and $\mathrm{syz}_{i-1}(\bar{M})$ are MGMCM modules.

i) $\Rightarrow$ iv) By hypothesis, we have

$$0 \to \mathrm{syz}_1(N) \to \bigoplus^{\nu(N)} R \to N \to 0.$$

By reduction modulo a general maximal regular N-sequence we obtain

$$0 \to \mathrm{syz}_1(N) \otimes \bar{\bar{R}} \to \bigoplus^{\nu(N)} \bar{\bar{R}} \to \bigoplus^{\nu(N)} k \to 0$$

is exact with $\dim \bar{\bar{R}} = 0$. Therefore

$$\operatorname{syz}_1(N) \otimes \bar{\bar{R}} \cong \bigoplus^{\nu(N)} \mathfrak{m}_{\bar{\bar{R}}}.$$

Since on the other hand, $\operatorname{syz}_1(N)$ is MGMCM, $\operatorname{syz}_1(N) \otimes \bar{\bar{R}}$ is annihilated by $\mathfrak{m}_{\bar{R}}$, and therefore $\bar{\mathfrak{m}}^2 = 0$. Hence $e(R) = e(\bar{\bar{R}}) =$ embedding dimension $\bar{\bar{R}}$ $+1 =$ embedding dimension $R - d + 1$. Therefore R has minimal multiplicity.

iv) $\Rightarrow$ iii) The Betti numbers for rings of minimal multiplicity have been computed by J. Sally [13]. For $i \geqq d$, the ith Betti number $\beta_i(k)$ equals $\sum_{j=0}^{d} [e(R) - 1]^{i-j} \binom{d}{j}$. So

$$\nu(\operatorname{syz}_i(k)) = \beta_i(k) = [e(R) - 1]^{i-d} e(R)^d, \quad \text{for } i \geqq d.$$

One has $e(\operatorname{syz}_{i+1}(k)) = e(R)\beta_i(k) - e(\operatorname{syz}_i(k))$. If $\operatorname{syz}_i(k)$ is a MGMCM module, then

$$\begin{aligned} e(\operatorname{syz}_{i+1}(k)) &= e(R)^{d+1}[e(R)-1]^{i-d} - e(R)^d[e(R)-1]^{i-d} \\ &= e(R)^d[e(R)-1]^{i+1-d} = \nu(\operatorname{syz}_{i+1}(k)). \end{aligned}$$

Therefore $\operatorname{syz}_{i+1}(k)$ is a MGMCM module. So it is enough to show that $\operatorname{syz}_d(k)$ is a MGMCM module.

Since $\beta_d(k) = e(R)^d$, we have to show that $e(\operatorname{syz}_d(k)) = e(R)^d$. Again by the results of [13] we have for $i \leqq d$ that

$$\beta_i(k) = \sum_{j=0}^{i} [e(R)-1]^{i-j} \binom{d}{j}.$$

Set $Z = [e(R) - 1]$. Then

$$\begin{aligned} e(\operatorname{syz}_d(k)) &= \left| \sum_{i=0}^{d-1} (-1)^i \beta_i(k) e(R) \right| = \left| \sum_{i=0}^{d-1} (-1)^i e(R) \sum_{j=0}^{i} \binom{d}{j} Z^{i-j} \right| \\ &= \frac{e(R)}{Z^d} \left| \sum_{j=0}^{d-1} \left(\sum_{i=j}^{d-1} (-1)^i Z^i \right) \binom{d}{j} Z^{d-j} \right| \\ &= \frac{1}{Z^d} \left| \sum_{j=0}^{d-1} \{(-1)^j Z^j - (-1)^d Z^d\} \binom{d}{j} Z^{d-j} \right| \\ &= \left| \sum_{j=0}^{d-1} (-1)^j \binom{d}{j} + (-1)^{d+1} \sum_{j=0}^{d-1} \binom{d}{j} Z^{d-j} \right| \\ &= |1 + (e(R)^d - 1)| = e(R)^d = \nu(\operatorname{syz}_d(k)). \end{aligned}$$

iii) $\Rightarrow$ ii) Clear.

The condition that the ring have positive dimension is necessary as can be seen by the following example.

EXAMPLE (2.6). Let k be a field and $n \geqq 2$ an integer and let $R = k[x]/(x^n)$. Then $0 \to k \to R \to R \to k \to 0$ is exact, so $\text{syz}_2(k)$ is a MGMCM module, while R has minimal multiplicity, if and only if $n = 2$.

This example is the only one, if the ring is Gorenstein and contains k.

PROPOSITION (2.7). *If R is a local zero dimensional Gorenstein ring with residue class field k, and* $\text{syz}_i(k)$ *is a* MGMCM *module for some $i > 0$, then R is a hypersurface ring.*

PROOF. Consider the minimal complex

$$0 \to \text{syz}_i(k) \to \cdots \to R \to k \to 0.$$

As R is zero dimensional by (1.2), $\text{syz}_i(k) \cong \bigoplus^N k$. Dualizing this complex, with respect to R, since R is self-injective we obtain

$$0 \to k \to R \to \cdots \to \bigoplus^N R \to \bigoplus^N k \to 0$$

and therefore $N = 1$. The Betti numbers of k are therefore bounded and the conclusion follows from [3, Corollary 6.2].

Quotient rings associated with maximal minors of a matrix admit an MGMCM module.

PROPOSITION (2.8). *Let r, s be nonnegative integers with $s \geqq r$. Let $A = k[x_1, \ldots, x_n]$ and let C be a $r \times s$-matrix whose entries are linear forms in A with $I = I_r(C)$ the ideal of A generated by maximal minors of C having grade $(s-r+1)$. Then the ring $R = A/I$ admits a* MGMCM *module M. If I is a prime ideal, then M can be taken to have rank one.*

PROOF. If I is prime, let Y be the matrix obtained from the matrix C by deletion of a row. Then $I_{r-1}(Y) \supsetneqq I$, hence

$$\text{ht}(I_{r-1}(Y)) \geqq \text{ht}(I) + 1 = s-(r-1)+1.$$

Since on the other hand $\text{ht}(I_{r-1}(Y)) \leqq s-(r-1)+1$, it follows that

$$\text{ht}(I_{r-1}(Y)) = s-(r-1)+1 = \text{ht}(I)+1.$$

Therefore as $I_{r-1}(Y)$ is a Cohen-Macaulay ideal, $I_{r-1}(Y)/I$ is a height one Cohen-Macaulay ideal and hence $I_{r-1}(Y)/I$ is a Maximal Cohen-Macaulay R-module [9, 4.13]. Moreover

$$v(I_{r-1}(Y)/I) = v(I_{r-1}(Y)) = \binom{s}{r-1} = e(R)$$

(see [11]). Hence R admits a rank one MGMCM module.

Otherwise, let X be a generic $r \times s$-matrix and $S = A[X]/I_r(X)$. Then $R \cong S/(y_1, \ldots, y_{rs})$, where $y_1, \ldots, y_{rs}$ is a regular sequence consisting of linear forms. Hence $e(S) = e(R)$. By the previous paragraph, S admits a MGMCM module M. The module $M \otimes_S R$ is a MGMCM R-module.

This result can be generalized somewhat.

PROPOSITION (2.9). *Let $A = k[x_1, \ldots, x_n]$ and let I be a homogeneous Cohen-Macaulay ideal of grade g with linear resolution having generators in degree d, and let J be a homogeneous Cohen-Macaulay ideal of grade $g+1$ with linear resolution, which contains I and has its generators in degree d or $d-1$. Then A/I has a* MGMCM *module of rank one.*

For A/I to admit a MGMCM of rank one, it does not suffice that I has a linear resolution.

EXAMPLE (2.10). Let n be a positive integer greater than two, let X be a generic symmetric $n \times n$-matrix, let $P = k[X]$, and let $I = I_{n-1}(X)$, and set $R = P/I$. The ideal I has a linear resolution (see [6], [12]), and $e(R) = \binom{n+1}{3}$ (see [11]), while the class group of R is cyclic of order two with nontrivial element given by an ideal with n generators (see [5]). Thus R has no rank one MGMCM module.

3. Construction of Gorenstein ideals with a large number of generators.

For this section, let $R = \bigoplus_{i \geq 0} R_i$ be a homogeneous domain and $R_0 = k$ an infinite field.

Let A be a square matrix whose entries are linear forms in $k[x_1, \ldots, x_m]$. The matrix A is said to have *no generalized zeros* if no nontrivial k-linear combination of the rows and columns of A has a component which is zero.

The key result on matrices with no generalized zeros is:

THEOREM (3.1) (Eisenbud). *Let A be a $n \times n$-matrix of linear forms in $k[x_1, \ldots, x_m]$ with no generalized zeros and $z_1, \ldots, z_{n-1}$ linear forms in $k[x_1, \ldots, x_m]$. Then* $\det A \not\equiv 0$ *modulo $(z_1, \ldots, z_{n-1})$. In particular* $\det A \neq 0$.

PROOF. See [4].

We now proceed to construct a matrix with no generalized zeros.

Let U be a subspace of R_i and V a subspace of R_j with $\dim_k U = \dim_k V$. Let $\{u_1, \ldots, u_n\}$ be a basis of U and $\{v_1, \ldots, v_n\}$ be a basis of V and let $\{c_1, \ldots, c_m\}$ be a basis of R_{i+j}. Then for some $y_{ij}^t \in k$ one has

$$u_i v_j = \sum_{t=1}^{m} y_{ij}^t c_t$$

with not all y_{ij}^t, $t = 1, \ldots, m$, equal to zero. Set

$$a_{ij} = \sum_{t=1}^{m} y_{ij}^t x_t \in k[x_1, \ldots, x_m]$$

and let A to be the $n \times n$-matrix with (i,j)th entry a_{ij}.

LEMMA (3.2). *The matrix A has no generalized zeros.*

PROOF. The application of row and column operations to the matrix A is equivalent to the selection of different bases of U and V as vector spaces over k. Since R is a domain, the product of any nonzero element of U with any nonzero element of V is nonzero. Hence it follows from the construction of A given above that A has no generalized zeros.

PROPOSITION (3.3). *Let s be a nonnegative integer and let U_i and V_i be subspaces of R_i for $i = 0, \ldots, s$ such that $\dim_k U_i \leqq \dim_k V_{s-i}$. Then there exists a nonempty open subset W of $R_s^* = \mathrm{Hom}_k(R_s, k)$ such that for all $\varphi \in W$, all $i = 0, \ldots, s$, and for all nonzero $u \in U_i$, one has $\varphi(u \cdot V_{s-i}) \neq 0$ (i.e. there exists a $v \in V_{s-i}$ such that $\varphi(uv) \neq 0$).*

PROOF. It suffices to prove this in the case $\dim_k U_i = \dim_k V_{s-i}$ for all $i = 0, \ldots, s$. Fix i and let $\{u_1, \ldots, u_n\}$ be a basis of U_i and let $\{v_1, \ldots, v_n\}$ be a basis of V_{s-i} and $\{c_1, \ldots, c_m\}$ be a basis of R_s. With $u_p v_j = \sum_{t=1}^m y_{pj}^t c_t$, let $A_i(\mathfrak{x})$ be the $n \times n$-matrix, whose (p, j)th entry is the linear form $\sum_{t=1}^m y_{pj}^t x_t$ with $\mathfrak{x} = (x_1, \ldots, x_m)$.

Let c_t^* be the element of R_s^* with $c_t^*(c_j) = \delta_{tj}$. Then $\{c_t^*, \ldots, c_m^*\}$ form a basis of R_s^*. Set

$$W_i = \left\{ \varphi \in R_s^* \;\middle|\; \varphi = \sum_{t=1}^{m} z_t c_t^*, \quad \det A_i(\mathfrak{z}) \neq 0, \quad z_t \in k \right\}.$$

For $\varphi = \sum_{t=1}^m z_t c_t^* \in R_s^*$, the matrix $A_i(\mathfrak{z})$ coincides with the matrix $(\varphi(u_p v_j))$. If in addition $\varphi \in W_i$, then $\det A_i(\mathfrak{z}) \neq 0$, and hence φ defines a nondegenerate bilinear form $U_i \times V_{s-i} \to k$. Therefore $\varphi(u V_{s-i}) \neq 0$ for all $0 \neq u \in U$.

Since by Lemma (3.2), $A_i(\mathfrak{x})$ is a matrix with no generalized zeros, it follows from Theorem (3.1) that the open set W_i is nonempty. Now set $W = \bigcap_{i=0}^{s} W_i$. Since k is infinite, W is a nonempty open set satisfying the conditions of the proposition.

For $\varphi \in R_s^*$, $\varphi \neq 0$, set $I(\varphi)_i = \{a \in R_i \mid \varphi(aR_{s-i}) = 0\}$ and set $I(\varphi) = \bigoplus_{i \geq 0} I(\varphi)_i$. Let $R(\varphi) = R/I(\varphi)$.

PROPOSITION (3.4). *For $\varphi \in R_s^*$, $\varphi \neq 0$, $I(\varphi)$ is a homogeneous Gorenstein ideal primary to the irrelevant maximal ideal. The artinian ring $R(\varphi) = R/I(\varphi)$ has its socle lying in degree s.*

PROOF. $I(\varphi)$ is obviously a homogeneous ideal. If $i > s$, then $R_{s-i} = 0$, thus $I(\varphi)_i = R_i$. So $I(\varphi)$ is primary to the irrelevant maximal ideal $\mathfrak{m}$. The socle of $R(\varphi)$ is the image in $R(\varphi)$ of the set $X = \{x \in R \mid x\mathfrak{m} \in I(\varphi)\}$. Since $I(\varphi)$ is homogeneous, every element of X is the sum of homogeneous elements of X. If $x \in R_i \cap X$, then $\varphi(xR_1R_{s-i-1}) = 0$. Thus if $i \neq s$, then $x \in I(\varphi)$. Therefore the socle of $R(\varphi)$ lies in degree s. Since $\dim_k \ker \varphi = \dim_k R_s - 1$, the dimension of the socle of $R(\varphi)$ equals one. So the ideal $I(\varphi)$ is Gorenstein.

The following special case of Proposition (3.3) is our principal application of that result.

PROPOSITION (3.5). *Let s be a nonnegative integer and let U_i, $i = 0, \ldots, s$ be subspaces of R_i such that $\dim_k U_i \leqq \dim_k R_{s-i}$ for $i = 0, \ldots, s$. Then there exists a nonempty open set $W \subset R_s^*$, such that for $\varphi \in W$ one has $U_i \cap I(\varphi) = 0$.*

PROOF. Applying Proposition (3.3) to the case $V_i = R_i$, $i = 0, \ldots, s$, yields the required result.

The existence of Gorenstein ideals with many generators can now be shown in dimension two.

COROLLARY (3.6). *In addition to the assumptions from the beginning of this section, suppose that R is 2-dimensional, then there exists a Gorenstein ideal of R primary to $\mathfrak{m}$ with at least $2e(R)$ generators.*

PROOF. Since $\dim R = 2$ for $i \gg 0$ one has

$$\dim_k R_i - \dim_k R_{i-1} = e(R). \tag{3.7}$$

Let t be a sufficiently large integer such that (3.7) holds for all $i \geqq t$. Set $s = 2t$. By Proposition (3.5) with $U_i = 0$ for $i \neq t$ and $U_t = R_t$, there exists $\varphi \in R_s^*$ with $I(\varphi) \cap R_t = 0$. Hence as R is a homogeneous domain, $I(\varphi) \cap R_i = 0$ if $i \leqq t$. Thus $v(I(\varphi)) \geqq \dim_k I(\varphi)_{t+1}$. Since $R(\varphi)$ is Gorenstein by Proposition 3.4, and hence has symmetric Hilbert function, we obtain

$$\dim_k I(\varphi)_{t+1} = \dim_k R_{t+1} - \dim_k R(\varphi)_{t+1} = \dim_k R_{t+1} - \dim_k R(\varphi)_{t-1}.$$

But $R(\varphi)_{t-1} \cong R_{t-1}$, since $I(\varphi)_{t-1} = 0$. Thus

$$\dim_k I(\varphi)_{t+1} = \dim_k R_{t+1} - \dim_k R_{t-1} = 2e(R).$$

The last equality is satisfied since (3.7) holds for t and $t+1$. Therefore $\nu(I(\varphi)) \geqq 2e(R)$.

4. The existence theorem.

In this section R is a Cohen-Macaulay ring admitting a canonical module ω_R and I is a codimension two Cohen-Macaulay ideal of R. Let $r(R/I) = \nu(\mathrm{Ext}_R^2(R/I, \omega_R))$, the type of R/I.

The following extension of the theory of Bourbaki sequences [2; VII, §4.9], [8] enables us to construct MGMCM modules.

THEOREM (4.1). *For R and I as above, there exists a Maximal Cohen-Macaulay R-module M with at least $\nu(I)$ generators and multiplicity equal to $e(R)[r(R/I)+I]$.*

PROOF. The short exact sequence $0 \to I \to R \to R/I \to 0$ yields isomorphisms

$$\mathrm{Hom}_R(I, \omega_R) \cong \omega_R, \quad \mathrm{Ext}_R^1(I, \omega_R) \cong \mathrm{Ext}_R^2(R/I, \omega_R) \cong \omega_{R/I}$$

(see [9; Satz 5.12]) and

$$\mathrm{Ext}_R^i(I, \omega_R) = 0 \quad \text{for } i \geqq 2$$

(see [9; Satz 6.1]). Further ω_R is a Maximal Cohen-Macaulay module with $\mathrm{Hom}_R(\omega_R, \omega_R) = R$ (see [9; Satz 6.1]) and $e(\omega_R) = e(R)$.

Let $\xi_1, \ldots, \xi_{r(R/I)}$ be a set of minimal generators of $\mathrm{Ext}_R^1(I, \omega_R)$ and let M be the extension of I by $\bigoplus^{r(R/I)} \omega_R$ corresponding to the element $(\xi_1, \ldots, \xi_{r(R/I)})$ of

$$\mathrm{Ext}_R^1\left(I, \bigoplus^{r(R/I)} \omega_R\right) \cong \bigoplus^{r(R/I)} \mathrm{Ext}_R^1(I, \omega_R).$$

Applying $\mathrm{Hom}_R(-, \omega_R)$ to the short exact sequence

$$0 \to \bigoplus^{r(R/I)} \omega_R \to M \to I \to 0,$$

one obtains the long exact sequence

$$(4.2)\qquad 0 \to \omega_R \to \operatorname{Hom}_R(M, \omega_R) \to \bigoplus^{r(R/I)} R \xrightarrow{\delta} \omega_{R/I} \to \operatorname{Ext}^1_R(M, \omega_R) \to 0$$

and $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for $i \geqq 2$. By choice of M, the map δ is surjective. Therefore $\operatorname{Ext}^1_R(M, \omega_R) = 0$. Hence M is a Maximal Cohen-Macaulay module.

Since

$$0 \to \bigoplus^{r(R/I)} \omega_R \to M \to I \to 0$$

is exact, $v(M) \geqq v(I)$ and

$$e(M) = e(I) + r(R/I)e(\omega_R) = e(R)[r(R/I) + 1].$$

An immediate consequence of this result is:

Corollary (4.3). *Let R and I be as above with $v(I) \geqq e(R)[r(R/I)+1]$, then R admits a* MGMCM *module with $e(R)[r(R/I)+1]$ generators.*

For an R-module M, we say that M is self-dual with respect to ω_R, if $M \cong \operatorname{Hom}_R(M, \omega_R)$.

Corollary (4.4). *Let R and I be as above and let I be a Gorenstein ideal of R, then R admits a Maximal Cohen-Macaulay module M self-dual with respect to ω_R with at least $v(I)$ generators, and multiplicity $2e(R)$.*

Proof. Take M as constructed in (4.1). Since R/I is Gorenstein, the complex (4.2) becomes

$$0 \to \omega_R \to \operatorname{Hom}_R(M, \omega_R) \to R \xrightarrow{\delta} R/I \to 0.$$

Hence the kernel of δ is I. So $\operatorname{Hom}_R(M, \omega_R)$ is an extension of I by ω_R. $\operatorname{Hom}_R(M, \omega_R)$ is a Maximal Cohen-Macaulay module. Hence as

$$\operatorname{Ext}^1_R(I, \omega_R) \cong \omega_{R/I} \cong R/I,$$

and as M corresponds to ξ_M and $\operatorname{Hom}_R(M, \omega_R)$ corresponds to $\xi_{\operatorname{Hom}_R(M, \omega_R)}$ with ξ_M and $\xi_{\operatorname{Hom}_R(M, \omega_R)}$ generators of R/I, it follows that $\xi_M = u\xi_{\operatorname{Hom}_R(M, \omega_R)}$ for $u \in R$, u a unit. But then $M \cong \operatorname{Hom}_R(M, \omega_R)$.

The ring R is called *generically Gorenstein,* if for every associated prime q of R, the ring R_q is Gorenstein. In this case ω_R has rank one.

COROLLARY (4.5). *Let R and I be as above and in addition let R be generically Gorenstein, then R admits a Maximal Cohen-Macaulay module of rank $[r(R/I)+1]$ and with at least $v(I)$ generators.*

For a normal ring R and M a R-module of rank m, the *determinant of M*, $\det M$, is the class of $\mathrm{Hom}_R(\mathrm{Hom}_R(\Lambda^m M, R), R)$ in the class group of R (see [2; VII, §4.7]). Such an R-module is called *orientable* (see [8]) if $\det M = 0$. Two facts on the determinant are:

i) if $0 \to N' \to N \to N'' \to 0$ is exact, then $\det N = \det N' + \det N''$, and,
ii) N is a rank one torsion-free orientable module, if and only if N is free or isomorphic to an ideal of codimension at least two.

Applying these facts to the defining sequence of the module M in (4.1) yields:

COROLLARY (4.6). *Let R and I be as above, and in addition let R be Gorenstein and normal, then R admits an orientable Maximal Cohen-Macaulay module M with rank $[r(R/I)+1]$ and at least $v(I)$ generators.*

REMARK (4.7). Note that the preceeding results imply that for R and I as in (4.1)

$$v(I) \leqq e(R)[r(R/I)+1].$$

Consequently the ideal of (3.6) has exactly $2e(R)$ generators and is generated by the elements in degree $t+1$.

Applying Corollary (3.6) and the results of this section we obtain our main result.

THEOREM (4.8). *If R is a homogeneous 2-dimensional Cohen-Macaulay domain with infinite residue class field, then R admits a* MGMCM *module self-dual with respect to ω_R rank 2.*

5. Admissible ranks of orientable MGMCM modules.

The theory of Bourbaki sequences of [2], [8] provides a means by which the structure of orientable MGMCM modules over a normal Gorenstein domain can be investigated.

An exact sequence of R-modules $0 \to F \to M \to I \to 0$ with F free, M a Maximal Cohen-Macaulay module, and I a codimension two Cohen-Macaulay ideal or $I \cong R$, is called a *Bourbaki sequence*.

A Bourbaki sequence $0 \to F \to M \to I \to 0$ is said to be *tight* if $v(M) = v(I)+v(F)$ (see [8]).

LEMMA (5.1). *Let R be a local (respectively homogeneous) normal Gorenstein domain with infinite residue class field. Then M is an orientable Maximal Cohen-Macaulay module, if and only if there exists a tight Bourbaki sequence*

$$0 \to \bigoplus^{N} R \to M \to I \to 0$$

(respectively in the homogeneous case if all generators of M lie in the same degree: with homogeneous maps and all generators of I lie in the same degree).

PROOF. See [8; Propositions 1.8, 1.9, and the proof of 2.1].

REMARK (5.2). If R is a local (respectively homogeneous) normal Cohen-Macaulay ring with infinite residue class field that admits a rank one orientable MGMCM module M, then R is regular and $M \cong R$.

PROOF. Clearly $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \cong R$.

Given that 2-dimensional homogeneous Gorenstein normal domains with infinite residue class fields admit a rank two orientable MGMCM module and the proscriptions against admitting a rank one orientable MGMCM module, it is natural to ask for what ranks does there exist a MGMCM module. Theorem (5.9) answers this question for homogeneous hypersurface domains of dimension two. As might be expected the answer depends on the multiplicity of the ring.

For the remainder of this section, let A be $k[x_1, x_2, x_3]$ a polynomial ring over an infinite field. Let $f \in A$, f a homogeneous form of degree $e > 1$, and let $R = A/(f)$ be normal. Clearly $e(R) = e$.

LEMMA (5.3). *Let M be an orientable rank m* MGMCM *R-module with all generators in the same degree, and let*

$$0 \to \bigoplus^{m-1} R \to M \to I \to 0$$

be a tight homogeneous Bourbaki sequence with the generators of I lying in degree a. Then $m(e-1) = 2a$.

PROOF. By hypothesis $R = A/(f)$. Thus we have by Proposition (1.5):

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
 & \overset{m-1}{\bigoplus} A(-e) & \longrightarrow & \overset{em}{\bigoplus} A(-1) & \\
 & \mu_f \downarrow & & \downarrow & \\
 & \overset{m-1}{\bigoplus} A & \overset{i}{\longrightarrow} & \overset{em}{\bigoplus} A & \\
 & \downarrow & & \downarrow & \\
0 \to & \overset{m-1}{\bigoplus} R & \longrightarrow & M \to I \to 0 & \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

Here i is split injective, since the Bourbaki sequence is tight. Taking the mapping cone, one obtains an exact sequence

$$(5.4) \qquad 0 \to \overset{m-1}{\bigoplus} A(-e-a) \to \overset{em}{\bigoplus} A(-1-a) \to \overset{em-m+1}{\bigoplus} A(-a) \to I \to 0$$

which is a minimal A-resolution of I.

Letting J be the preimage of I in A, then $A/J = R/I$, so that J is a codimension three Cohen-Macaulay ideal of A.

CASE 1). f is a minimal generator of J.

Then

$$0 \to \overset{m-1}{\bigoplus} A(-e-a) \to \overset{em}{\bigoplus} A(-1-a) \to \overset{em-m+1}{\bigoplus} A(-a) \oplus A(-e) \to J \to 0$$

is an A-resolution of J.

Computing the Hilbert series of A/J one obtains

$$(5.5) \qquad H(A/J, \lambda) = \frac{1-\lambda^e-(em-m+1)\lambda^a+em\lambda^{a+1}-(m-1)\lambda^{a+e}}{(1-\lambda)^3}.$$

But since A/J is a zero dimensional ring, $H(A/J, \lambda)$ is a polynomial. Therefore, if the numerator of (5.5) is $P(\lambda)$, $(1-\lambda)^3$ divides $P(\lambda)$, and hence the first and second formal derivatives of $P(\lambda)$ evaluated at 1 are zero. So

$$\begin{aligned}
0 &= P^{(2)}(1) \\
&= -e(e-1)-a(a-1)(em-m+1)+ema(a+1)-(m-1)(q+e)(a+e-1) \\
&= m(e-1)-2a.
\end{aligned}$$

Therefore $m(e-1) = 2a$.

CASE 2). f is not a minimal generator of J.

If $0 \to F_3 \to F_2 \to F_1 \to J \to 0$ is a minimal free A-resolution of J one obtains

$$0 \to F_3 \to F_2 \oplus A(-e) \to F_1 \to I \to 0$$

is a minimal free A-resolution of I. By comparison with (5.4) one obtains $a = e-1$ and

$$0 \to \bigoplus^{m-1} A(1-2e) \to \bigoplus^{em-1} A(-e) \to \bigoplus^{em-m+1} A(-e+1) \to J \to 0$$

is a minimal A-resolution of J.

By the Betti number formula of [7] one obtains:

$$m-1 = \frac{e(e-1)}{[2e-1-(e)][2e-1-(e-1)]} = 1.$$

Thus $m = 2$ and $2(e-1) = 2a$.

COROLLARY (5.6). *Let R be as above and $e(R) = e \equiv 0 \bmod 2$ and M an orientable* MGMCM *R-module of rank m with all generators lying in the same degree. Then $m \equiv 0 \bmod 2$.*

PROOF. By (5.3), $m(e-1) \equiv 0 \bmod 2$. But $e \equiv 0 \bmod 2$, hence $m \equiv 0 \bmod 2$.

The following result allows us to reduce the general case to that dealt with above. Note that even if M is graded, it is not necessarily true that all its generators lie in the same degree, whereas $\mathrm{gr}_{\mathfrak{m}}(M)$ always has that property.

LEMMA (5.7). *Let S be a local (or homogeneous) Cohen-Macaulay ring with (irrelevant) maximal ideal $\mathfrak{m}$, $S/\mathfrak{m}$ infinite, and $gr_{\mathfrak{m}}(S)$ normal. Let M be a* MGMCM *orientable S-module, then $\mathrm{gr}_{\mathfrak{m}}(M)$ is an orientable Maximal Cohen-Macaulay $\mathrm{gr}_{\mathfrak{m}}(S)$-module. In particular, if $\mathrm{gr}_{\mathfrak{m}}(S)$ is Cohen-Macaulay, then $\mathrm{gr}_{\mathfrak{m}}(M)$ is an orientable* MGMCM $\mathrm{gr}_{\mathfrak{m}}(S)$*-module.*

PROOF. By (1.6), it suffices to show that $\mathrm{gr}_{\mathfrak{m}}(M)$ is orientable. As $\mathrm{gr}_{\mathfrak{m}}(M)$ is a torsion free $\mathrm{gr}_{\mathfrak{m}}(S)$-module, by Bourbaki's Theorem [2; VII, §4.9, Theorem 6], there exists an ideal I of $\mathrm{gr}_{\mathfrak{m}}(S)$ such that

$$0 \to \bigoplus^{m-1} \mathrm{gr}_{\mathfrak{m}}(S) \xrightarrow{i} \mathrm{gr}_{\mathfrak{m}}(M) \to I \to 0$$

is exact. Pick a set of generators $\{e_1, \ldots, e_{m-1}\}$ of $\bigoplus^{m-1} \mathrm{gr}_{\mathfrak{m}}(S)$ and let $u_j \in M$ be such that $\bar{u}_j = i(e_j) \in \mathrm{gr}_{\mathfrak{m}}(M)$. Let U be the submodule of M

generated by $\{u_1, .., u_{m-1}\}$. Let J be an S-module such that

$$(*) \qquad 0 \to U \to M \to J \to 0$$

is exact. Then $(*)$ induces a complex

$$\bigoplus^{m-1} \mathrm{gr}_{\mathfrak{m}}(S) \xrightarrow{i} \mathrm{gr}_{\mathfrak{m}}(M) \to \mathrm{gr}_{\mathfrak{m}}(J) \to 0$$

which is exact on the right. Hence there is a surjection $\varphi: I \to \mathrm{gr}_{\mathfrak{m}}(J)$.

Since $\nu(M) = \nu(\mathrm{gr}_{\mathfrak{m}}(M)) > m-1 \geqq \nu(U)$, it follows that $J \neq 0$, and hence $\mathrm{gr}_{\mathfrak{m}}(J) \neq 0$. Because

$$\mathrm{rank}(\mathrm{gr}_{\mathfrak{m}}(J)) \geqq 1 = \mathrm{rank}(I),$$

and I is torsion free, it is clear that $I \cong \mathrm{gr}_{\mathfrak{m}}(J)$.

As

$$\mathrm{rank}(J) = (e(S))^{-1}e(J) = (e(\mathrm{gr}_{\mathfrak{m}}(S)))^{-1}e(\mathrm{gr}_{\mathfrak{m}}(J)) = \mathrm{rank}(\mathrm{gr}_{\mathfrak{m}}(J)) = 1,$$

and since $\mathrm{gr}_{\mathfrak{m}}(J)$ and hence J is torsion free, it follows that J is an S-ideal. Therefore,

$$\mathrm{rank}(U) = \mathrm{rank}(M) - \mathrm{rank}(J) = \mathrm{rank}(\mathrm{gr}_{\mathfrak{m}}(M)) - 1 = m-1 \geqq \nu(U),$$

and hence U is a free S-module. Therefore and because M is orientable, $(*)$ now implies that J is isomorphic to an S-ideal $\tilde{J}$ with $\mathrm{ht}(\tilde{J}) \geqq 2$ (including the case $\tilde{J} = S$).

Let $l \geqq 0$ be the integer with $\tilde{J} \subseteqq \mathfrak{m}^l$, $\tilde{J} \nsubseteq \mathfrak{m}^{l+1}$. On $\tilde{J}$ we consider the following filtration

$$F: F_i(\tilde{J}) = \tilde{J} \quad \text{for} \quad i \leqq l,$$

and

$$F_i(\tilde{J}) = \mathfrak{m}^{i-l}\tilde{J} \quad \text{for} \quad i \geqq l.$$

On S we consider the $\mathfrak{m}$-adic filtration. Then $0 \to \tilde{J} \to S \to S/\tilde{J} \to 0$ induces a complex

$$(**) \qquad 0 \to \mathrm{gr}_F(\tilde{J}) \xrightarrow{\psi} \mathrm{gr}_{\mathfrak{m}}(S) \to \mathrm{gr}_{\mathfrak{m}}(S/\tilde{J}) \to 0.$$

By definition of l, $\psi \neq 0$. Moreover,

$$\mathrm{gr}_F(\tilde{J}) \cong (\mathrm{gr}_{\mathfrak{m}}(\tilde{J}))(-l) \cong \mathrm{gr}_{\mathfrak{m}}(J)(-l)$$

is torsion free of rank one. Hence ψ is injective. But then it is an easy exercise to show that $(**)$ is exact. Since

$$\dim \operatorname{gr}_{\mathfrak{m}}(S/\tilde{J}) \leqq \dim \operatorname{gr}_{\mathfrak{m}}(S)-2,$$

it is then clear that $\operatorname{gr}_F(\tilde{J})$ and hence I is isomorphic to an ideal of height at least 2. Then the exact sequence

$$0 \to \bigoplus^{m} \operatorname{gr}_{\mathfrak{m}}(S) \to \operatorname{gr}_{\mathfrak{m}}(M) \to I \to 0$$

implies that $\operatorname{gr}_{\mathfrak{m}}(M)$ is orientable.

For a normal Cohen-Macaulay ring S, let $Y(S)$ be the semigroup of ranks of orientable MGMCM S-modules.

THEOREM (5.8). *Let (R, m) be a 2-dimensional hypersurface ring with infinite residue class field and $\operatorname{gr}_{\mathfrak{m}}(R)$ normal. If $e(R) \equiv 0 \bmod 2$, then $Y(R) \subseteqq 2\mathsf{N}$.*

PROOF. By (5.7), $Y(R) \subset Y(\operatorname{gr}_{\mathfrak{m}}(R))$. By (5.6) and (5.7), $Y(\operatorname{gr}_{\mathfrak{m}}(R)) \subseteqq 2\mathsf{N}$.

A more complete classification can be obtained, if the ring is a homogeneous hypersurface domain.

THEOREM (5.9). *Let R be a 2-dimensional normal homogeneous hypersurface domain with infinite residue class field. Then*

i) *if $e(R) = 1$, then $Y(R) = \mathsf{N}$,*
ii) *if $e(R) \equiv 0 \bmod 2$, then $Y(R) = 2\mathsf{N}$,*
iii) *if $e(R) \equiv 1 \bmod 2$ and $e(R) \neq 1$, then*
$Y(R) = 2\mathsf{N}+3\mathsf{N} = \mathsf{N}\setminus\{1\}$.

PROOF. Recall $e = e(R)$. If $e = 1$, i) follows from Lemma (2.2). By (4.8), $Y(R) \supseteqq 2\mathsf{N}$. If $e \equiv 0 \bmod 2$, then by (5.8), $2\mathsf{N} \supseteqq Y(R)$ and hence $Y(R) = 2\mathsf{N}$. Hence it suffices to show that if $e \neq 1$, $e \equiv 1 \bmod 2$, then R admits a rank 3 orientable MGMCM module.

By (4.6) it suffices to show that R has a codimension two Cohen-Macaulay ideal I with 3e generators and $r(R/I) = 2$.

The Hilbert function of R is

$$h(t) = \begin{cases} \binom{t+2}{2} & t < e \\ \binom{t+2}{2} - \binom{t-e+2}{2} = et - \dfrac{e(e-3)}{2} & t \geqq e. \end{cases}$$

Set $a = \frac{1}{2}(3e+1)$, $s = \frac{1}{2}(5e-3)$, $x = [s/2]$. Then $x \leqq s-x$. Since $h(t)$ is an increasing function of t, $h(x) \leqq h(s-x)$. By (3.5) there exists a nonempty open subset W_1 of R_s^* such that for $\varphi_1 \in W_1$, $R_x \cap I(\varphi_1) = 0$. Since R is a homogeneous domain, $I(\varphi_1) \cap R_j = 0$ for $j \leqq s/2$.

Now as $s-a+1 = e-1 < s/2$ and as $R(\varphi_1) = R/I(\varphi_1)$ is Gorenstein with socle degree s, one obtains

$$\begin{aligned}\dim_k I(\varphi_1)_{a-1} &= \dim_k R_{a-1} - \dim_k R(\varphi_1)_{a-1} = \dim_k R_{a-1} - \dim_k R(\varphi_1)_{s-a+1}\\ &= \dim_k R_{a-1} - \dim_k R_{e-1} = h(a-1) - h(e-1) = \tfrac{1}{2}e(e+1)\\ &= \dim_k R_{e-1}.\end{aligned}$$

Hence by (3.5) there exists a nonempty open set W_2 of R_s^* such that for $\varphi_2 \in W_2$, $I(\varphi_1)_{a-1} \cap I(\varphi_2)_{a-1} = 0$.

For any $\varphi_2 \in W_2$, φ_1 and φ_2 are k-linearly independent. Set $I = I(\varphi_1) \cap I(\varphi_2)$. I is a codimension two Cohen-Macaulay ideal. Moreover, since $I(\varphi_1)$ and $I(\varphi_2)$ are Gorenstein and since φ_1 and φ_2 are k-linearly independent, we obtain $r(R/I) = 2$. Since $I_{a-1} = 0$, and R *is* a homogeneous domain, it follows that $I_j = 0$ for all $j \leqq a-1$.

Thus $v(I) \geqq \dim_k I_a$, while

$$\begin{aligned}\dim_k I_a &= \dim_k I(\varphi_1)_a + \dim_k I(\varphi_2)_a - \dim_k [I(\varphi_1) + I(\varphi_2)]_a\\ &\geqq \dim_k I(\varphi_1)_a + \dim_k I(\varphi_2)_a - h(a).\end{aligned}$$

As $R(\varphi_i)$, $i = 1, 2$, is Gorenstein with socle degree s,

$$\dim_k I(\varphi_i)_a = h(a) - \dim_k R(\varphi_i)_a = h(a) - \dim_k R(\varphi_i)_{s-a} = h(a) - h(s-a).$$

Therefore $v(I) \geqq h(a) - 2h(s-a) = 3e$. Thus by (4.7), $v(I) = 3e$ and by (4.6), R admits a rank 3 orientable MGMCM module.

6. Questions.

For the sake of completeness we list here some open questions.

QUESTION (6.1). If R is a two dimensional local Cohen-Macaulay ring, does R admit a MGMCM module? If R admits a canonical module, by (4.3) it would suffice to show that R possesses an ideal I of codimension two such that $v(I) = e(R)[r(R/I)+1]$.

QUESTION (6.2). If R is a two dimensional normal homogeneous hypersurface domain, what are the possible ranks of indecomposable orientable MGMCM modules?

For I a codimension two Cohen-Macaulay ideal, I is said to be *linked* to the ideal J if there is a regular sequence x_1, x_2 in I such that $(x_1, x_2):I = J$. The ideals I and J are said to be *evenly linked* provided that there exists a sequence of ideals $I_0, \ldots, I_n$ with $I = I_0$, $J = I_n$, I_i is linked to I_{i+1} for $i = 0, \ldots, n-1$ and n is even. The *even linkage class* of I is the equivalence class induced by the relation evenly linked. The even linkage class of I is said to be *indecomposable*, if no ideal in the class has the form $x''I' + x'I''$, where I' and I'' are codimension two Cohen-Macaulay ideals and x', x'' is a regular sequence with $x' \in I'$ and $x'' \in I''$. By [8, Remark 2.6] the question (6.2) is equivalent to determining the indecomposable even linkage classes of ideals I occurring in a Bourbaki sequence $0 \to F \to M \to I \to 0$ with M a MGMCM module.

QUESTION (6.3). If R is a d-dimensional homogeneous hypersurface ring, does R admit a MGMCM module? In particular it is not known whether $k[x, y, z, w]/(f)$ admits a MGMCM module, where f is a homogeneous polynomial of degree three.

QUESTION (6.4). If R is a zero dimensional local ring, when can the ith syzygy module of the residue class field of R be a MGMCM R-module (a direct sum of copies of the residue class field by Proposition (1.2))? Proposition (2.7) asserts that if R is Gorenstein and contains k, then $R \cong k[x]/(x^n)$. If R is arbitrary with ith syzygy module of k isomorphic to $\bigoplus^N k$, then the Poincaré series of R is of the form

$$P(\lambda) = \frac{1 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{i-1}\lambda^{i-1}}{(1 - N\lambda^i)}.$$

NOTE ADDED IN PROOF. Meanwhile, Question 6.3 has been answered affirmatively by J. Backelin and J. Herzog.

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