MATH. SCAND. 61 (1987), 204–212

ALGEBRAIC AND GEOMETRIC APPROACH TO THE CLASSIFICATION OF SEMISPACES

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Let L be a linear space over an ordered field K and let M be a subspace of L. Maximal convex subsets of $L \setminus M$ are called *semispaces* of L at M (comp. Hammer [2]). Klee [4] gave a classification of semispaces in the case of $K = \mathsf{R}$ and $M = 0$. Those semispaces correspond to linear orderings of L (see [3] and [6]). What is more, the conditions

(1) S is a semispace at M in L,

(2) S is a (strong) preordering on L with $S_0 = M$,

(see Section 1 for definitions) are equivalent ([6, Proposition 2.2]).

Consideration of those equivalent algebraic objects gives some adventages. Firstly, instead of semispaces at M in L, one can consider semispaces at 0 in L/M. Secondly, the classical results about ordered groups (see [1]) can be adopted in a natural way to linear orderings. Since the field of reals is the only complete ordered field, indecomposable real orderings are one-dimensional. This fact enables one to obtain Klee's classification purely algebraically.

Thanks to Theorem 1.5 or more general Theorem 2.5, a classification of semispaces of a linear space over an Archimedean field (and a classification of ordered abelian groups) can be reduced to the case of the field of reals, that is, to the classification of Klee.

It seems that the algebraic method of the proof of Theorem 1.5 cannot be applied to the proof of the more general Theorem 2.5 and to the proof of Theorem 2.4. The last theorem (announced in the introduction of [6]) and the description [5] of convex half-spaces (i.e. convex sets with convex complements) in R^n give a description of convex half-spaces in any finite-dimensional space over an Archimedean field.

The first part of the paper is written in terms of orderings and the second independent one in terms of convexity.

Received August 11, 1985; in revised form July 2, 1986.

1. Description of linear preorderings.

Let K be an ordered field with the set K^+ of positive elements, and let L be a linear space over K. A set $S \subset L$ is called a (strong) K-*preordering* on L, if it satisfies the following conditions

(i) $S+S \subset S,\ K^+S \subset S,$

(ii) $L = S \dot{\cup} S_0 \dot{\cup} (-S)$ for some linear subspace S_0 of L,

where the symbol $\dot{\cup}$ stands for the disjoint union. Then L becomes a partially ordered linear space with the relation $<$ defined as follows: $a < b \Leftrightarrow b-a \in S$. If $a < b$ for every $a \in A$, $b \in B$, we write $A < B$. The space L is fully ordered by the relation $<$ if $S_0 = 0$. In this case S is called a (strong) K-*ordering* on L. It is easy to see that the above definitions coincide with those of [6]. We leave to the reader the proof of the following result.

LEMMA 1.1. *Let M be a subspace of L and let $v: L \to L/M$ denote the natural homomorphism. There is a natural one-to-one correspondence between K-preorderings on L satisfying $S_0 = M$ and K-orderings $\bar{S}$ on L/M, which is given by $\bar{S} = v(S)$ and $S = v^{-1}(\bar{S})$.*

The above Lemma reduces the investigation of K-preorderings to the study of more familiar K-orderings. Many properties of them are analogous to those of [1], stated for ordered groups (see Propositions 1.2–1.4 below). Therefore some proofs are sketched only.

Let $\mathscr{V}$ be a chain of subspaces of L. For $V \in \mathscr{V}$ denote by $\tilde{V}$ the union of all $W \in \mathscr{V}$ properly contained in V. We say that the chain $\mathscr{V}$ is *admissible*, if for every $x \in L$ there exists a smallest subspace $V_x \in \mathscr{V}$ containing x. In other words, $x \in V_x \setminus \tilde{V}_x$ and, as a consequence, $V = \tilde{V}$ if and only if $V \neq V_x$ for every $x \in L$. For an admissible chain $\mathscr{V}$ on L we have

$$L = \bigcup_{x \in L} (V_x \setminus \tilde{V}_x) = \dot{\bigcup_{V \in \mathscr{V}}} (V \setminus \tilde{V}),$$

and it is clear that we can reject every $V \in \mathscr{V}$ equal to $\tilde{V}$, obtaining another admissible chain with the same operator $\tilde{}$. Any chain closed under intersections is admissible. Each admissible chain $\mathscr{V}$ contains a smallest subspace $V_0 \in \mathscr{V}$. Evidently $\tilde{V}_0 = \emptyset$, and except for this case, every $\tilde{V}$ is a subspace of V.

PROPOSITION 1.2. *Let $\mathscr{V}$ be an admissible chain of subspaces of L with the smallest subspace 0, and let $\mathscr{W} = \mathscr{V} \setminus \{0\}$. There is a natural one-to-one correspondence between the set of K-orderings S on L satisfying $\underline{S \cap V} < S \setminus V$ for all $V \in \mathscr{W}$ and the set of systems $(\bar{S}_V;\ V \in \mathscr{W})$ of K-orderings $\bar{S}_V$ on $V/\tilde{V}$.*

PROOF. Let S be a K-ordering on L. Then $L \setminus 0 = \dot{\cup}_{V \in \mathscr{W}}(V \setminus \tilde{V})$ and $S = \dot{\cup}_{V \in \mathscr{W}} S_V$ where $S_V = S \cap (V \setminus \tilde{V})$. The assumption $S \cap V < S \setminus V$ means that $S_V < S_W$ for $V \subsetneq W$. Obviously, S_V is a K-preordering on V with $(S_V)_0 = \tilde{V}$, and this gives us a K-ordering $\overline{S_V}$ on $V/\tilde{V}$, for all $V \in \mathscr{W}$. Conversely, the inverse images S_V of $\overline{S_V}$ are K-preorderings on V (for all $V \in \mathscr{W}$), satisfying $S_V \pm S_W \subset S_V$ for $W \subsetneq V$. Thus we get a K-ordering $S = \cup_{V \in \mathscr{W}} S_V$ on L satisfying the desired condition.

We say that S in question (Proposition 1.2) is the *ordinal sum* of K-orderings $\overline{S_V}$. We call a K-ordering S *decomposable* if it is a proper ordinal sum, that is, if there exists a proper non-zero subspace V of L satisfying $S \cap V < S \setminus V$. Otherwise S is called *indecomposable*. The next proposition gives us the unique decomposition of a K-ordering S into an ordinal sum of indecomposable K-orderings. The family of these indecomposable K-orderings is called the *skeleton* of S.

PROPOSITION 1.3. *If S is a K-ordering on L, then the family $\mathscr{V}$ of all subspaces $V \subset L$ satisfying $S \cap V < S \setminus V$ is an admissible chain with the smallest subspace 0 and it is closed under intersections and unions. The corresponding K-orderings $\overline{S_V}$ are indecomposable.*

PROOF. Let $V, W \in \mathscr{V}$ and suppose that there exist $x \in V \setminus W$ and $y \in W \setminus V$. We can assume that $x, y \in S$. Then $x \in S \cap V$, $y \in S \setminus V$ and hence $x < y$. By symmetry we get $y < x$, contradiction. The rest of the proof is immediate.

Recall [1] that a group or a group ordering S is called *Archimedean* if for every $x, y \in S$ we have $\mathbf{N}x \not< y$ (that is, there exists an integer $n > 0$ such that $nx > y$). Let us assume in the sequel that the field K (i.e., the ordering K^+) is Archimedean. A well-known theorem of Hölder (see [1, pp. 45 and 126]) states that each Archimedean group (respectively field) is an ordered subgroup (respectively subfield) of the group (respectively field) of real numbers $\mathbf{R}$. Consequently, we can assume that $K \subset \mathbf{R}$ and $K^+ = \mathbf{R}^+ \cap K$.

PROPOSITION 1.4. *If L is a linear space over an Archimedean field K and S is a K-ordering on L, then the following conditions are equivalent*:

(1) *S is indecomposable,*

(2) *S is Archimedean,*

(3) *$S = f^{-1}(\mathbf{R}^+)$ for some K-linear imbedding $f: L \hookrightarrow \mathbf{R}$.*

In particular, indecomposable real ordered spaces are at most one-dimensional.

PROOF. (1) $\Leftrightarrow$ (2). We show a more general equivalence for K being an arbitrary ordered field: S is decomposable if and only if $K^+x < y$ for some

$x,y \in S$. The condition $0 < K^+x < y$ gives us a proper non-zero subspace

$$V = \{z \in L; |z| \leqq \lambda x \text{ for some } \lambda \in K^+\}$$

satisfying $S \cap V < S \setminus V$. Conversely, for any such subspace V, if $x \in S \cap V$ and $y \in S \setminus V$, then $K^+x < y$.

(2)$\Leftrightarrow$(3). This is a version of Hölder's theorem and the proof is similar. For a fixed $x \in S$ and $y \in L$ the real number $f(y)$ is defined by the section $(L(y)|U(y))$, where

$$L(y) = \{\lambda \in K; \lambda x \leqq y\}$$

and

$$U(y) = \{\lambda \in K; \lambda x > y\}.$$

The mapping f is K-linear since $L(y+z) = L(y)+L(z)$ and $L(\mu y) = \mu L(y)$ for $y,z \in L$ and $\mu \in K^+$.

The above propositions allow us to prove two extension theorems. Note that the first of them is true also in the more general situation when K and F are arbitrary ordered fields, not necessarily Archimedean (see Theorem 2.5). However, the proof of the general case uses methods of convexity.

THEOREM 1.5. *Let K be a subfield of an Archimedean field F such that $K^+ = F^+ \cap K$. For any K-preordering S on L there exists an F-preordering T on $FL = F \otimes_K L$ such that $S = T \cap L$ and $T_0 = FS_0$. In particular, an ordering on L extends to an ordering on FL.*

PROOF. (a) Reduction. Assume that our theorem holds for orderings. Let $S_0 = M$. Consider the following commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\ \varphi\ } & FL \\ \nu\downarrow & & \downarrow\mu \\ L/M & \xrightarrow{\ \psi\ } & F(L/M) = FL/FM \end{array}$$

It follows from Lemma 1.1 that $S = \nu^{-1}(\bar{S})$ where $\bar{S}$ is a K-ordering on L/M. Since $\bar{S} = \psi^{-1}(\bar{T})$ for an F-ordering $\bar{T}$ on $F(L/M)$, we see that

$$S = \nu^{-1}\psi^{-1}(\bar{T}) = \varphi^{-1}\mu^{-1}(\bar{T}) = T \cap L,$$

where $T = \mu^{-1}(\bar{T})$ is an F-preordering on FL satisfying $T_0 = FM$.

(b) The case of orderings. Let S be a K-ordering. Since $L \subset FL \subset \mathbb{R}L$ it can be assumed that $F = \mathbb{R}$. If S is indecomposable, then, in virtue of

Proposition 1.4, we have the following commutative diagram:

$$\begin{array}{ccc} L & \overset{f}{\hookrightarrow} & \mathsf{R} \\ \cap & \nearrow_{g} & \\ \mathsf{R}L & & \end{array}$$

where $S = f^{-1}(\mathsf{R}^+)$ and g is R-linear. Take some (for example, lexicographic) R-ordering t on $\mathrm{Ker}(g)$. Proposition 1.2 gives an R-ordering $T = t \,\dot{\cup}\, g^{-1}(\mathsf{R}^+)$ on $\mathsf{R}L$ satisfying $T \cap L = S$. For an arbitrary K-ordering S consider the skeleton $(\overline{S_V}\,; V \in \mathscr{W})$ over K. As above, we obtain a family $(\overline{T}_V\,; V \in \mathscr{W})$, where $\overline{T}_V$ is an R-ordering on $\mathsf{R}(V/\tilde{V}) = \mathsf{R}V/\mathsf{R}\tilde{V}$ satisfying $\overline{T_V} \cap (V/\tilde{V}) = \overline{S_V}$. Proposition 1.2 gives us an R-ordering $T = \dot{\cup}_{V \in \mathscr{W}} T_V$ on $\mathsf{R}L$. As in (a), $T_V \cap V = S_V$, and finally $T \cap L = S$.

THEOREM 1.5'. *Let S be a strong ordering on an abelian group A. Then $S = T \cap A$ for some R-ordering T on $\mathsf{R}A = \mathsf{R} \otimes_{\mathsf{Z}} A$.*

PROOF. By Levi Theorem (see [1, p. 36]), the group A is torsion-free. Hence it is contained in $\mathsf{R}A = \mathsf{R} \otimes_{\mathsf{Q}} A_{(0)}$, where $A_{(0)}$ denotes the linear space over the field Q of rationals, being the localization of A. The ordering S can be extended to a Q-ordering $S_{(0)} = \{s/t\,; s \in S, t \in \mathsf{N}\}$ on $A_{(0)}$, and we are in position to apply Theorem 1.5. Since R is a flat Z-module, it is possible to give also a direct proof similar to the part (b) of the proof of Theorem 1.5; it is based on original results from [1] instead of Propositions 1.2–1.4.

The above theorems reduce our investigation to the real field R. Propositions 1.3 and 1.4 give us the following description of this case:

PROPOSITION 1.6. *Every R-preordering has the form*

$$S = \dot{\bigcup_{V \in \mathscr{W}}} (\mathsf{R}^+ e_V + \tilde{V}),$$

where $\mathscr{W} = \mathscr{V} \setminus \{V_0\}$ for some admissible chain $\mathscr{V}$, $\dim(V/\tilde{V}) = 1$ and $e_V \in V \setminus \tilde{V}$ for each $V \in \mathscr{W}$. In this case, $S_0 = V_0$.

PROOF. Lemma 1.1 reduces the proof to the case when S is an R-ordering. In virtue of Proposition 1.4, the skeleton of S consists of one-dimensional orderings, each of the form $\mathsf{R}^+ e$. The rest is immediate.

A more familiar description of R-orderings (or, equivalently, R-semispaces at 0) is known as the so-called "Klee representation" (see [3], [4]). Jamison [3] pointed out that this representation is possible only over the field R.

Here we give a relative generalization for K-preorderings over an Archimedean field K.

Let K denote a subring of R, for example the ring of integers or an Archimedean field. A subset Φ of $\mathrm{Hom}_K(L, \mathsf{R})$ totally ordered by $\prec$ is called *admissible with respect to a subspace* V_0 of L, if for every $x \in L \setminus V_0$ there exists a smallest member $\varphi_x \in \Phi$ with $\varphi_x(x) \neq 0$. It is called simply *admissible* if $V_0 = 0$. If Φ is admissible with respect to V_0 then

$$S = S(\Phi, \prec) = \{x \in L \setminus V_0 \,; \varphi_x(x) > 0\}$$

is a K-preordering on L with $S_0 = V_0$. Conversely, we have

THEOREM 1.7. *If K is an Archimedean field, then any K-preordering S has the form $S = S(\Phi, \prec)$, where Φ is admissible with respect to S_0. Every abelian group ordering has the form $S(\Phi, \prec)$ for some admissible Φ.*

PROOF. By Theorem 1.5 or 1.5′ we can assume that $K = \mathsf{R}$. In this case, the description of S presented in Proposition 1.6 gives us a family $\{\psi_V : V \to \mathsf{R}\,; V \in \mathscr{W}\}$ of linear functionals defined by the conditions $\psi_V(e_V) = 1$ and $\psi_V(\tilde{V}) = 0$. Let φ_V denote an extension of ψ_V on L. The family $\Phi = \{\varphi_V\,; V \in \mathscr{W}\}$ is admissible with respect to $V_0 = S_0$ if the ordering $\prec$ is defined as follows:

$$\varphi_W \prec \varphi_V \Leftrightarrow W \supsetneq V.$$

In fact, if $x \in L \setminus V_0$, then $x \in V_x \setminus \tilde{V}_x$ with $V_x \in \mathscr{W}$, and therefore $\varphi_x = \varphi_{V_x} \in \Phi$. Since $\varphi_x(x) > 0$ if and only if $x \in \mathsf{R}^+ e_{V_x} + \tilde{V}_x$, we obtain $S = S(\Phi, \prec)$.

2. Convexity under extension of the space.

Let K_2 be an ordered field and let K_1 be a subfield of K_2. Consider a linear space L_2 over K_2 and a subspace L_1 of L_2 over K_1 such that every subset of L_1 linearly independent over K_1 is also linearly independent over K_2. For instance, if L_1 is a linear space over K_1, then in the part of L_2 we can take the space $K_2 \otimes_{K_1} L_1$ over K_2. The spaces Q^n and R^n are simple examples of such L_1 and L_2.

The symbols $\mathrm{aff}_i A$, $\mathrm{conv}_i A$ mean the affine and the convex hulls of a set A of the space L_i over K_i, $i = 1,2$.

LEMMA 2.1. *For every $A \subset L_1$ we have*

$$\mathrm{conv}_1 A = L_1 \cap \mathrm{conv}_2 A.$$

PROOF. Let $a \in L_1 \cap \mathrm{conv}_2 A$. There exists a finite minimal set $\{a_0, \ldots, a_n\} \subset A$ for which $a \in \mathrm{conv}_2\{a_0, \ldots, a_n\}$. Hence $a = \alpha_0 a_0 + \ldots + \alpha_n a_n$ for some $\alpha_i \in K_2$

such that $0 \leqq \alpha_i \leqq 1$, $i = 0,\ldots,n$, and $\alpha_0+\ldots+\alpha_n = 1$. We can assume that $a_0 = 0$. Since $a_1,\ldots,a_n$ are linearly independent and $a = \alpha_1 a_1+\ldots+\alpha_n a_n$ over K_2 it follows that a is a linear combination of $a_1,\ldots,a_n$ over K_1, that is, $\alpha_1,\ldots,\alpha_n \in K_1$. From $\alpha_0+\ldots+\alpha_n = 1$ we get $\alpha_0 \in K_1$. Consequently,

$$a \in \mathrm{conv}_1\{a_0,\ldots,a_n\} \subset \mathrm{conv}_1 A.$$

We see that $L_1 \cap \mathrm{conv}_2 A \subset \mathrm{conv}_1 A$. The inverse inclusion is obvious.

From the above Lemma we obtain that *a subset of L_1 is convex if and only if it is the intersection of L_1 with a convex subset of L_2*. More exactly: $A = L_1 \cap \mathrm{conv}_2 A$ for every convex subset A of L_1.

LEMMA 2.2. *If A_i, $i = 1,\ldots,n$, are convex subsets of L_1 and $\bigcap_{i=1}^n A_i = \emptyset$, then $\bigcap_{i=1}^n \mathrm{conv}_2 A_i = \emptyset$.*

PROOF. Suppose that $\bigcap_{i=1}^n \mathrm{conv}_2 A_i \neq \emptyset$. Thus $\bigcap_{i=1}^n \mathrm{conv}_2 F_i \neq \emptyset$ for some finite $F_i \subset A_i$, $i = 1,\ldots,n$. Consequently, there exist minimal finite subsets G_i of A_i, $i = 1,\ldots,n$, such that $\bigcap_{i=1}^n \mathrm{conv}_2 G_i \neq \emptyset$. Let w be a point of this set.

Suppose that there exists a point $z \in \bigcap_{i=1}^n \mathrm{aff}_2 G_i$ different from w. Let $G_1 = \{c_1,\ldots,c_m\}$. Obviously, w and z have the forms

$$w = \alpha_1 c_1+\ldots+\alpha_m c_m, \quad z = \beta_1 c_1+\ldots+\beta_m c_m,$$

where $\alpha_1+\ldots+\alpha_m = 1$, $\beta_1+\ldots+\beta_m = 1$, $\alpha_j \geqq 0$ and $\alpha_j, \beta_j \in K_2$ for $j = 1,\ldots,m$. Since $w \neq z$ and $(\alpha_1-\beta_1)+\ldots+(\alpha_m-\beta_m) = 0$, at least one of the scalars

$$\lambda_j = \alpha_j/(\alpha_j-\beta_j), \quad j = 1,\ldots,m,$$

is defined and non-negative. Let $\gamma_1 = \lambda_{j_0}$ be the smallest non-negative one. We have $\alpha_j+\lambda_{j_0}(\beta_j-\alpha_j) \geqq 0$ for $j = 1,..,m$ with the equality for $j = j_0$. Moreover,

$$\sum_{j=1}^m [\alpha_j+\lambda_{j_0}(\beta_j-\alpha_j)] = \lambda_{j_0} \sum_{j=1}^m \beta_j+(1-\lambda_{j_0}) \sum_{j=1}^m \alpha_j = 1.$$

Therefore the point

$$g_1 = \sum_{j=1}^m [\alpha_j+\lambda_{j_0}(\beta_j-\alpha_j)]c_j = w+\gamma_1(z-w)$$

of the half-line with the end-point w through z belongs to the convex hull of a proper subset of G_1. Generally, a point

$$g_i = w+\gamma_i(z-w), \quad \text{where} \quad \gamma_i \geqq 0,$$

of this half-line belongs to the convex hull of a proper subset of G_i for every $i = 1, \ldots, n$. Let $\gamma_{i_0} = \min\{\gamma_1, \ldots, \gamma_m\}$. Obviously, $g_{i_0} \in \mathrm{conv}_2 G_i$ for $i = 1, \ldots, n$. Since g_{i_0} belongs to the convex hull of a proper subset of G_{i_0}, we obtain a contradiction with the minimality of G_{i_0}. Consequently, $\bigcap_{i=1}^m \mathrm{aff}_2 G_i = \{w\}$. This means that w is the only solution over K_2 of a system of linear equations with coefficients from the field K_1. So $w \in L_1$. Since $w \in \mathrm{conv}_2 G_i$, from Lemma 2.1 we obtain $w \in \mathrm{conv}_1 G_i$, $i = 1, \ldots, n$. Hence $w \in \bigcap_{i=1}^n A_i$. A contradiction with the assumption.

The following consequence of Lemma 2.2 is of independent interest:

PROPOSITION 2.3. *For arbitrary convex sets A_i of L_1, $i = 1, \ldots, n$, we have*

$$\mathrm{conv}_2 \bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \mathrm{conv}_2 A_i.$$

PROOF. It is sufficient to consider the case $n = 2$.

Let $x \in \mathrm{conv}_2 A_1 \cap \mathrm{conv}_2 A_2$. For some finite $F_1 \subset A_1$ and $F_2 \subset A_2$ we have $x \in \mathrm{conv}_2 F_1 \cap \mathrm{conv}_2 F_2$. Consider an extreme point y of $\mathrm{conv}_2 F_1 \cap \mathrm{conv}_2 F_2$. We can find minimal sets $M_1 \subset F_1$ and $M_2 \subset F_2$ such that $y \in \mathrm{conv}_2 M_1 \cap \mathrm{conv}_2 M_2$. Since y is extreme, $\mathrm{conv}_2 M_1 \cap \mathrm{conv}_2 M_2 = \{y\}$. From Lemma 2.2 we get $y \in \mathrm{conv}_2(M_1 \cap M_2)$. The arbitrariness of the extreme point y implies $\mathrm{conv}_2 F_1 \cap \mathrm{conv}_2 F_2 \subset \mathrm{conv}_2(M_1 \cap M_2)$. Consequently, $x \in \mathrm{conv}_2(M_1 \cap M_2) \subset \mathrm{conv}_2(A_1 \cap A_2)$, which proves the inclusion $\supset$. The inverse inclusion is obvious.

The example of intervals with end-points $a_i, b_i \in \mathbb{Q}$ such that $a_i < \sqrt{2} < b_i$, $i = 1, 2, \ldots$ and $\lim a_i = \sqrt{2} = \lim b_i$ shows that the above equality and Lemma 2.2 do not hold for infinite intersections.

Let us observe that Lemma 2.2 and Proposition 2.3 enable simple transfers of some theorems on intersection of convex sets (e.g. of Helly-type theorems) from linear spaces over the field of reals into linear spaces over subfields of reals, and consequently, over Archimedean fields.

THEOREM 2.4. *A subset of L_1 is a convex half-space of L_1 if and only if it is the intersection of L_1 with a convex half-space of L_2.*

PROOF. Obviously, the intersection of L_1 with a convex half-space of L_2 is a convex half-space of L_1.

Let G be a convex half-space of L_1. From Lemma 2.2 we obtain $\mathrm{conv}_2 G \cap \mathrm{conv}_2(L_1 \setminus G) = \emptyset$. By a known result of Kakutani (comp. Theorem 2.3 in [7]: the proof is correct also in the general situation of linear spaces over ordered fields) there exists a convex half-space H of L_2 such that $\mathrm{conv}_2 G \subset H$ and $H \subset \mathrm{conv}_2(L_1 \setminus G) = \emptyset$. So $G \subset H$ and $H \cap (L_1 \setminus G) = \emptyset$. Since $G \subset L_1$, we get $G = H \cap L_1$.

THEOREM 2.5. *A subset of L_1 is a semispace of L_1 at a subspace M of L_1 if and only if it is the intersection of L_1 with a semispace of L_2 at the subspace* $\mathrm{conv}_2 M = \mathrm{aff}_2 M$.

PROOF. It follows from Lemma 2.1 that $M = L_1 \cap \mathrm{conv}_2 M$. Then the "if" part of our Theorem is evident in the language of preorderings. However, it can be shown without preorderings using the following characterization of semispaces (see [6, Proposition 2.2]): a subset S of a linear space L over an ordered field is a semispace of L at a subspace M if and only if S is convex and $L = S \dot{\cup} M \dot{\cup} (-S)$.

Conversely, let S be a semispace of L_1 at M. Thanks to Lemma 2.2 the set $\mathrm{conv}_2 S$ is disjoint with the subspace $\mathrm{conv}_2 M$. By Zorn's Lemma there exists a semispace T of L_2 at $\mathrm{conv}_2 M$ containing $\mathrm{conv}_2 S$. From the "if" part it follows that $T \cap L_1$ is a semispace of L_1 at M. Since $S \subset T \cap L_1$ is also a semispace of L_1 at M, we have $S = T \cap L_1$.

As we pointed out in the introduction, Theorem 2.5 together with Klee classification enables a classification of semispaces in any linear space over an Archimedean field. Similarly, Theorem 2.4 together with part 1 of Theorem 1 of [5] gives a description of convex half-spaces in a finite-dimensional linear space over an Archimedean field. The analogous description does not concern infinite-dimensional spaces because of the existence of convex half-spaces which are not translates of semispaces (see Remark 2.5 in [6]).

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