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JOINS AND HIGHER SECANT VARIETIES

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Introduction.

The join of two varieties X and Y in $\boldsymbol{P} = \boldsymbol{P}^N$ is the closure of the union of the lines of form xy, $x \in X$, $y \in Y$. The r'th secant variety of $X \subset \boldsymbol{P}$ is the closure of the union of the $(r-1)$-planes spanned by r points in X.

In section 1 we observe that the set $\mathrm{Var}(\boldsymbol{P})$ of closed subvarieties in $\boldsymbol{P}$ becomes a monoid under the operation join, and that the higher secant varieties of X are the powers of X in this monoid.

Using this monoid we prove among other things that the higher secant varieties of a curve always have the expected dimensions (if they are non-linear). Although this was known to Palatini [15, footnote p. 635] it was considered an open problem by Atiyah [3, p. 424]. From this fact Atiyah's arguments give a proof of Nagata's theorem about the minimal section of a ruled surface. Recently, Lange [14] and Zak [19] has proved that these secant varieties have the right dimensions. Lange also gave the application to Nagata's theorem.

We also consider Terracini's lemma. This is a useful tool for the differential study of the secant variety, as shown by works of Zak [18], Fujita-Roberts [6], Dale [4] and others. We give a simple proof of Terracini's lemma for joins.

Together the monoid and Terracini's lemma give a framework for further studies of higher secant varieties. For instance, we have been able to classify varieties with a maximal number of degenerate higher secant varieties (see [1]).

In section 2 we prove a join-defect formula generalising the embedding-obstruction/double point formula from secant varieties to joins. This formula expresses the dimension of the join XY of X and Y in terms of the Segre class of $X \cap Y$ in $X \times Y$.

As a corollary we observe that a special case gives a "refined Bezout theorem" connected with results of Fulton [7, 12.3].

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1. General theory and tangential properties of joints.

1.1. DEFINITIONS, NOTATIONS AND OBSERVATIONS. Let k be an algebraically closed field and V a finite dimensional linear space. Let $\boldsymbol{A} = \boldsymbol{V}(V) = \operatorname{Spec} S(V)$, where S denotes symmetric algebra.

By a variety we will mean a reduced and irreducible algebraic k-scheme. If X and Y are subvarieties of $\boldsymbol{A}$, we define their *sum* $X+Y$ as the closure of the image of $X \times Y$ under the addition morphism $\boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$. Clearly $\dim(X+Y) \leqq \dim X + \dim Y$.

Let $\boldsymbol{P} = \boldsymbol{P}(V) = \operatorname{Proj} S(V)$. For a subvariety $Z \subset \boldsymbol{P}$ let $\hat{Z}$ denote the affine cone in $\boldsymbol{A}$. Let X and Y be subvarieties in $\boldsymbol{P}$, then there is a subvariety $XY \subset \boldsymbol{P}$ the *join* of X and Y such that $(XY)\hat{} = \hat{X} + \hat{Y}$. We have $\dim XY \leqq \dim X + \dim Y + 1$. The set $\operatorname{Var}(\boldsymbol{P})$ of projective subvarieties of $\boldsymbol{P}$ becomes an ordered abelian monoid under join, where the ordering is given by inclusion and the empty variety is the unit element. The empty variety has always dimension -1.

If $x_1, \ldots, x_r$ are points in $\boldsymbol{P}$, then the join $x_1 \ldots x_r$ equals their linear span. With this notation, $X_1 \ldots X_r$ is the closure of the union of the $x_1 \ldots x_r$'s with $x_i \in X_i$, $i = 1, \ldots, r$. If $\dim x_1 \ldots x_r = r-1$, $x_i \in X_i$, we call $x_1 \ldots x_r$ a joining $(r-1)$-plane of $X_1, \ldots, X_r$.

The higher secant varieties of a given $X \in \operatorname{Var}(\boldsymbol{P})$ are the powers of X in this monoid. We use this notation writing X^2 for $\operatorname{Sec}(X)$ and so on. In this situation we sometimes change our terminology and talk about r-secant $(r-1)$-planes or just r-secants if no confusion is likely. With these notations we see that $X \in \operatorname{Var}(\boldsymbol{P})$ is idempotent if and only if it is a linear subspace. We make the following observation generalising [8, Lemma 7.10]. Palatini [16] gave another proof.

OBSERVATION 1.2. *Let $X \in \operatorname{Var}(\boldsymbol{P})$. If $X^i = X^{i+1}$, then X^i is linear.*

PROOF. $X^i = X^{i+1}$ implies X^i idempotent.

If $X \in \operatorname{Var}(\boldsymbol{P})$ and P is a point, we say that P is a *vertex* of X if $XP = X$. The set of vertices of X is denoted $\operatorname{Vert}(X)$. It is easy to vertify that $\operatorname{Vert}(X)$ is a linear subvariety of X and that codim $(\operatorname{Vert}(X), X) \geqq 2$ for X nonlinear, see also [5].

PROPOSITION 1.3. *Let $X, Y \in \operatorname{Var}(\boldsymbol{P})$, then*

i) $XY = X$ *if and only if* $Y \subset \operatorname{Vert}(X)$,

ii) $\dim XY = \dim X + 1$ *implies* $Y \subset \operatorname{Vert}(XY)$.

PROOF. The first statement is trivial. For the second suppose $\dim X = n$, $\dim XY = n+1$. By i) we cannot have $Y \subset \mathrm{Vert}(X)$. Take $y \in Y$, $y \notin \mathrm{Vert}(X)$. Then $Xy \neq X$ giving $Xy = XY$ therefore $XYy = Xy^2 = Xy = XY$ giving $y \in \mathrm{Vert}(XY)$.

As a corallary we obtain the following result of Palatini [16, 3, Teorema I] giving a stronger version of Observation 1.2.

COROLLARY 1.4. *If* $\dim X^{i+1} \leqq \dim X^i + 1$, *then* X^{i+1} *is linear.*

PROOF. By the proposition $X \subset \mathrm{Vert}(X^{i+1})$. But $\mathrm{Vert}(X^{i+1})$ is linear, giving $X^{i+1} \subset \mathrm{Vert}(X^{i+1})$ and X^{i+1} becomes linear.

COROLLARY 1.5. *Suppose* $X_1, \ldots, X_r \subset \boldsymbol{P}$ *are curves. Then* $\dim X_1 \ldots X_r < 2r-1$ *if and only if there is a sequence* $1 \leqq i_1 < \ldots < i_s \leqq r$ *such that* $X_{i_1} \ldots X_{i_s}$ *is linear of dimension* $< 2s-1$.

PROOF. Suppose no such sequence exist. We prove $\dim X_1 \ldots X_r = 2r-1$ by induction. It is certainly true for $r = 1$. Suppose $r \geqq 2$ and let $Z_i = X_1 \ldots \check{X}_i \ldots X_r$ and $X = X_1 \ldots X_r = X_i Z_i$. By induction $\dim Z_i = 2r-3$. Suppose $\dim X < 2r-1$. Then $\dim X \leqq \dim Z_i + 1$ and since $X = Z_i X_i$, we get $X_i \subset \mathrm{Vert}(X)$. This holds for $1 \leqq i \leqq r$ and implies that $X \subset \mathrm{Vert}(X)$. Thus X is linear of dimension $< 2r-1$, giving a contradiction.

The converse is a trivial dimension count.

REMARK 1.6. Each of the corollaries implies that the higher secant-varieties of a curve has the "right" dimensions if they are nonlinear.

We will now take a look at the tangential properties of joins.

For a variety X we let $t_{X,x}$ denote the Zariski tangent space. If $X \subset \boldsymbol{A}$ then naturally $t_{X,x} \subset T_{\boldsymbol{A},x} = V^*$ and we identify $t_{X,x}$ with the corresponding linear subvariety of $\boldsymbol{A}$.

If $X \subset \boldsymbol{P}$ then $t_{X,x} \subset t_{\boldsymbol{P},x}$ and we let $T_{X,x}$ denote the embedded tangent space, that is the linear subvariety L of $\boldsymbol{P}$ such that $t_{L,x} = t_{X,x}$ in $t_{\boldsymbol{P},x}$. Clearly $(T_{X,x})\hat{} = t_{\hat{X},x'}$ for x' in $\boldsymbol{A}$ lying over x.

PROPOSITION 1.7. *Suppose* $X, Y, Z \in \mathrm{Var}(\boldsymbol{P})$.

(1) *If* $x \in X$, *then* $T_{XY,x} \supset x\langle Y\rangle$ *where* $\langle Y\rangle$ *is the linear span of* Y.

(2) *If* XYZ *is nonlinear, then* $(X \cap Y) \subset \mathrm{Sing}(XYZ)$.

PROOF. (1) $T_{XY,x} \supset T_{xY,x} = x\langle Y\rangle$.

(2) Take $w \in X \cap Y$, then $T_{XYZ,w} \supset w\langle Y\rangle \supset YZ$ by (1). Interchanging X and Y gives $T_{XYZ,w} \supset XZ$, thus $T_{XYZ,w} \supset XYZ$. If XYZ is nonlinear this implies that $w \in \mathrm{Sing}(XYZ)$.

As a corollary we get another strengthening of Observation 1.2.

COROLLARY 1.8. *If* $X \in \mathrm{Var}(\boldsymbol{P})$ *with* X^{i+1} *nonlinear, then*

$$X^i \subset \mathrm{Sing}(X^{i+1}).$$

PROOF. Take $z \in X^i$, then $T_{X^{i+1},z} \supset z\langle X\rangle \supset X^{i+1}$.

PROPOSITION 1.9. *Let* X *and* Y *be subvarieties of* $\boldsymbol{A}$. *If* $x \in X$ *and* $y \in Y$, *the image of the differential at* (x, y) *of the addition morphism* $+ : X \times Y \to X + Y$ *is* $t_{X,x} + t_{X,y}$.

PROOF. $+ : \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$ is linear giving $d+ \; = \; +$, and we get

$$+ : t_{X,x} \times t_{X,y} \to t_{X,x} + t_{X,y}.$$

We obtain the following affine version of Terracini's lemma.

COROLLARY 1.10. *Let* X, Y *be as above, then*

(1) $t_{X,x} + t_{Y,y} \subset t_{X+Y,x+y}$ for $x \in X$, $y \in Y$.

(2) *If* $\mathrm{char}(k) = 0$ *then there is a dense open subset* U *in* $X + Y$ *such that*

$$t_{X,x} + t_{Y,y} = t_{X+Y,z}$$

for all $z \in U$, $x \in X$, $y \in Y$ *with* $z = x + y$.

PROOF. (1) Follows immediately from the proposition.

(2) Follows from the theorem of "generic submersiveness", see [9, Proof of III, 10.7].

By passing to affine cones we get the following version of Terracini's lemma.

COROLLARY 1.11. *If* $X, Y \in \mathrm{Var}(\boldsymbol{P})$, *then*

(1) $T_{X,x} T_{Y,y} \subset T_{XY,z}$ *for* $x \in X$, $y \in Y$, *and* $z \in xy$.

(2) *If* $\mathrm{char}(k) = 0$ *then there is a dense open subset* U *in* XY *such that*

$$T_{X,x} T_{Y,y} = T_{XY,z}$$

for all $z \in U$, $x \in X$, $y \in Y$ *with* $z \in xy$.

REMARK. It is now an easy task to formulate and prove more "multiple" versions of Terracini.

2. A join-defect formula.

Let $(\boldsymbol{P})^r$ denote $\boldsymbol{P} \times \ldots \times \boldsymbol{P}$ (r times). The tautological quotient $V_{\boldsymbol{P}} \twoheadrightarrow O_{\boldsymbol{P}}(1)$ induces a homomorphism

$$V_{(P)^r} \to \bigoplus_{i=1}^{r} p_i^* O_P(1)$$

where $p_i : (P)^r \to P$ is the ith projection. Let F denote the cokernel of this homomorphism. For $r \leqq N+1 = \dim V$, let

$$U_r = \{(x_1, \ldots, x_r) | \dim x_1 \ldots x_r = r-1\}.$$

The support of F is the complement $(P)^r - U_r$, and the Fitting ideal $F^0(F)$ induces the reduced scheme structure since the homomorphism above corresponds to the generic $r \times (N+1)$ matrix.

Corresponding to the quotient

$$V_{U_r} \twoheadrightarrow \bigoplus_{i=1}^{r} p_i^* O_P(1)\Big|_{U_r}$$

we get a morphism

$$\varphi : U_r \to G = G(r-1, N) = \mathrm{Grass}_r(V)$$

sending an r-tuple $(x_1, \ldots, x_r)$ to the $(r-1)$-plane $x_1 \ldots x_r$.

Let $F \subset P \times G$ be the incidence correspondence with morphisms $\alpha : F \to P$, $\beta : F \to G$ induced by the projections. Furthermore let $(F)^r$ denote $F \times_G \ldots \times_G F$ (r times), the multiincidence correspondence.

Proposition 2.1. *The morphism $\pi : (F)^r \to (P)^r$ induced by α is the blowing up of $(P)^r$ in $F^0(F)$.*

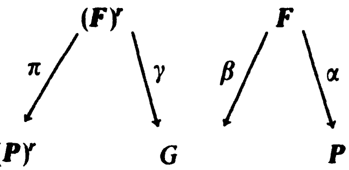
Proof. We will show that the universal property of blowing-up is fulfilled. Let $f : Y \to (P)^r$ be a morphism. By [17, Lemma 1.1] we know that $F^0(f^*F)$ is invertible if and only if the image of $V_Y \to \bigoplus_{i=1}^{r} f^* q_i^* O_P(1)$ is locally free of rank r.

On $(F)^r$ this image is the pullback of the tautological quotient on G.

Suppose the image E of $V_Y \to \bigoplus_{i=1}^{r} f^* q_i^* O_P(1)$ is an r-bundle. This gives an Y-point in G. The induced morphisms $E \to f^* P_i^* O_P(1)$ are surjective, giving Y-points in F for $i = 1, \ldots, r$. We get a morphism $\bar{f} : Y \to (F)^r$ with $\pi \circ \bar{f} = f$.

Remark. If $r = 2$, then $F^0(F)$ is the ideal of the diagonal in $P \times P$ and we recover essentially the diagram in Holme's article [10]. The isomorphism of $(P \times P)^{\sim}$ with $P \times_G F$ was realised by Johnson [11].

We have the following diagram



For $X_i \in \mathrm{Var}(\boldsymbol{P})$ $i = 1, \ldots, n$, let $n_i = \dim(X_i)$ and $n = \sum_{i=1}^{r} n_i$. Let Π denote $X_1 \times \ldots \times X_r$. If $\Pi \cap U_r \neq \phi$, we let B denote the strict transform of Π by π. We see that the join $J = X_1 \ldots X_r$ equals $\beta\alpha^{-1}(S)$, where $S = \gamma(B)$.

PROPOSITION 2.2. *If $\Pi \cap U_r \neq \phi$, then*

(i) *for* $r-1 \leqq m \leqq n+r-2$, *we have* $\dim J \leqq m$ *iff* $\pi_*([B].s_{m+2-r}(\gamma^*Q)) = 0$, *where s denotes the Segre class,*
(ii) *if* $\dim J = n+r-1$, *then*

$$p \cdot q \cdot \deg J = \deg \pi_*([B].s_n(\gamma^*Q))$$

where $p = \deg(\gamma|_B)$ *and* $q = \deg(\alpha|_{\beta^{-1}(S)})$.

PROOF. For $r-1 \leqq m \leqq \min\{N-1, n+r-2\}$ let A be a general linear subspace in $\boldsymbol{P}$ of codimenson $m+1$. Then we have $\dim J \leqq m$ if and only if $J \cap A = \phi$. Put $\Sigma = \beta(\alpha^{-1}(A))$, the Schubert variety of $(r-1)$-planes meeting A. We have $\dim J \leqq m$, iff $S \cap \Sigma = \phi$. Pulling this back by γ the condition becomes $B \cap \gamma^{-1}(S) = \phi$.

In the Chow ring of $(\boldsymbol{F})^r$ we have $[B \cap \gamma^{-1}(\Sigma)] = [B].\gamma^*[\Sigma]$. Pushing this down by π (which is birational on all components of $B \cap \gamma^{-1}(\Sigma)$) we get $\dim J \leqq m$, iff $\pi_*([B].\gamma^*[\Sigma]) = 0$. By Porteous' formula, we have $[\Sigma] = s_{m+2-r}(Q)$. To finish the proof of (i), observe that $s_{m+2-r}(Q) = 0$ for $m \geqq N$.

If $\dim J = n+r-1$, then zero-cycle $\pi_*([B].s_n(\gamma^*(Q))) \neq 0$. We have $p \cdot [S] = \gamma_*[B]$ and $[\beta^{-1}S] = \beta^*[S]$ giving

$$p \cdot q \cdot [J] = \alpha_*\beta^*\gamma_*[B] \quad \text{in} \quad A(\boldsymbol{P}).$$

Using the projection formula and the fact that $s_n(Q) = [\Sigma] = \beta_*\alpha^*[A]$ for A a general plane of codimension $n+r-1$ we get

$$\gamma_*([B].\gamma^*[\Sigma]) = \beta_*(\beta^*\gamma_*[B].\alpha^*[A]).$$

Taking degrees we get

$$\deg(\pi_*([B].s_n(Q))) = \deg([B].\gamma^*[\Sigma]) = \deg(\beta^*\gamma_*[B].\alpha^*[A]).$$

Now,

$$\alpha_*(\beta^*(\gamma_*[B]).\alpha^*[A]) = \alpha_*\beta^*\gamma_*[B].[A] = pq[J].[A]$$

by the projection formula, and taking degrees we get

$$\deg(\pi_*([B].s_n(Q)) = p\cdot q\cdot \deg J.$$

COROLLARY 2.3. *Suppose* $\Pi = X_1 \times \ldots \times X_r \subset U_r$, *then* $\dim X_1 \ldots X_r = n+r-1$ *and*

$$\deg X_1 \ldots X_r = p^{-1}q^{-1} \sum_{i=1}^{r} d_i,$$

where $d_i = \deg(X_i)$.

In particular, if $X_i \subset L_i$, L_i *is a linear subspace such that* $L_i \cap L_1 \ldots \check{L}_i \ldots L_r = \phi$, $i = 1,\ldots,r$, *then* $\deg X_1 \ldots X_r = \prod_{i=1}^r d_i$.

PROOF. We have $B \cong \Pi$ and $\gamma^* \cong \bigoplus_{i=1}^r p_i^* O_{\mathbf{P}}(1)$.

Let $t_i = c_1(p_i^* O_{\mathbf{P}}(1))$, $i = 1,\ldots,r$ and $d = d_1 \ldots d_r$, then

$$[\Pi] = dt_1^{N-n_1} \ldots t_r^{N-n_r}$$

and

$$s(\gamma^*Q) = \prod_{i=1}^{r} \frac{1}{1-t_i} = \prod_{i=1}^{r} (1+t_i+t_i^2+ \ldots +t_i^N).$$

Thus $s(\gamma^*Q)$ is the sum of all monomials of degree n in $t_1,\ldots,t_r$. We get

$$[\Pi].s_n(\gamma^*Q) = dt_1^N \ldots t_r^N.$$

If $X_i \subset L_i$ with $L_i \cap L_1 \ldots \check{L}_i \ldots L_r = \phi$ we must prove $p = q = 1$. If L is a joining $(r-1)$-plane, then L meets each L_i in one point, giving $p = 1$. Through each general point in $L_1 \ldots L_r$, there passes exactly one $(r-1)$-plane meeting all the L_i's, giving $q = 1$. More precisely, one can prove, see [2], that

$$\mathbf{P}_\Pi\left(\bigoplus_{i=1}^{r} p_i^* O_{X_i}(1)\right) \text{ is the blow-up of } X_1 \ldots X_r \text{ with center } \bigcup_{i=1}^{r} X_i.$$

We will now study closer the case $r = 2$. Let

$$s^k(X_1 \cap X_2, X_1 \times X_2) \in A_k(X_1 \cap X_2)$$

be the kth Segre class of $X_1 \cap X_2$ in $X_1 \times X_2$, see [7, 4.2] for the definition, and let

$$\sigma^k(X_1, X_2) = \deg s^k(X_1 \cap X_2, X_1 \times X_2),$$

where the degree is taken by the inclusion of $X_1 \cap X_2$ in $\Delta \cong \boldsymbol{P}$. For $\max(n_1, n_2) \leqq m \leqq n_1 + n_2$ define

$$\varphi_m = d_1 d_2 - \sum_{i=\alpha}^{m} \binom{m+1}{m-i} \sigma^{n_1+n_2-i}(X_1, X_2)$$

where $\alpha = \alpha(X_1, X_2) = n_1 + n_2 - \dim(X_1 \cap X_2)$.

THEOREM 2.4. *We have* $\varphi_m \geqq 0$ *and* $\varphi_m = 0$, *iff* $\dim X_1 X_2 \leqq m$. *If* $\dim X_1 X_2 = n_1 + n_2 + 1$, *then* $p \cdot q \cdot \deg X_1 X_2 = \varphi_{n_1+n_2}$.

PROOF. We must find the relation between $\pi_*(\gamma[B].s_m(\gamma^*Q))$ and φ_m. To do this it is enough to replace $X \times X$ by $X_1 \times X_2$ in any proof of the double point formula for projections, see [11], [12] or the theory of embedding-obstructions [10], [13]. We will stay close to the proof in [12].

On $\boldsymbol{F} \times_G \boldsymbol{F}$ we have the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I \otimes L_1 & \longrightarrow & L_1 & \longrightarrow & \pi^*F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \gamma^*Q & \longrightarrow & L_1 \oplus L_2 & \longrightarrow & \pi^*F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_2 & = & L_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $L_i = \pi^* p_i^* O_{\boldsymbol{P}}(1)$ and $I = F^0(\pi^*F)$ the ideal of the exceptional divisor. By [12, V, 29] we have the formula

$$s(\gamma^*Q) = s(L_1)s(L_2) \sum_{i=0}^{2N} s(L_1)^i c_1(I)^i.$$

If we let i denote the inclusion $X_1 \cap X_2 \subset \boldsymbol{P} \times \boldsymbol{P}$ then [12, II, 43] gives

$$i_* s^k(X_1 \cap X_2, X_1 \times X_2) = -\pi_*(c_1(I)^{n_1+n_2-k}.[B]), \quad k = 0, \dots, n_1 + n_2 - 1.$$

Using the projection formula, we get

$$\pi_*([B].s(\gamma^*Q)) = d_1 d_2 t_1^{N-n_1} t_2^{N-n_2}(1 + t_1 + \dots + t_1^N)(1 + t_2 + \dots + t_2^N) -$$
$$- \sum_{i=\alpha}^{n_1+n_2} i_* s^{n_1+n_2-i}(X_1 \cap X_2, X_1 \times X_2)(1 + t_1 + \dots + t_1^N)^{i+1}(1 + t_2 + \dots + t_2^N)$$

where $t_i = c_1(p^* O_{\boldsymbol{P}}(1))$, $i = 1, 2$.

Let M^l denote the sum of all monomials of degree l in $A(\boldsymbol{P}\times\boldsymbol{P}) = Z[t_1,t_2]/(t_1^{N+1},t_2^{N+1})$. Then

$$i_*s^k = \Delta_*(\sigma^k\,.\,\Delta^*t_1^{N-k}) = \sigma^k M^N t_1^{N-k} = \sigma^k M^{2N-k},$$

where $s^k = s^k(X_1\cap X_2, X_1\times X_2)$ and $\sigma^k = \sigma^k(X_1,X_2)$. Collecting terms of degree $2N-n_1-n_2+m$ we obtain

$$\pi_*([B]\,.\,s_m(\gamma^*Q)) = \varphi_m M^{2N-n_1-n_2+m}.$$

Finally we look at the trivial cases, when $X_1\times X_2\cap U_2=\emptyset$ that is $X_1\times X_2\subset\Delta$. Then $X_1=X_2$ is a point P, or one of the X_i's is empty. In the first case one can easily check that the theorem holds, in the second case the theorem becomes "empty".

We will now give an application of this formula.

Suppose $X_1\cap X_2$ has irreducible components $W_1,\ldots,W_s$ of dimension m and possibly other components of less dimension. Then

$$s^m(X_1\cap X_2, X_1\times X_2) = \sum_{i=1}^{s} e_i[W_i]$$

where e_i is the multiplicity of W_i on $X_1\times X_2$, see [7, 4.3].

Corollary 2.5. *We have the following inequalities*

(*) $\dim X_1X_2 \geqq \dim X_1+\dim X_2-\dim(X_1\cap X_2)$,

(**) $\deg X_1\cdot\deg X_2 \geqq \sum_{i=1}^{s} e_i\deg W_i$.

If $X_1\cap X_2\neq\emptyset$ then (*) is an equality if and only if (**) is an equality.

Proof. If $\dim(X_1\cap X_2)<\min(n_1,n_2)$, then $\varphi_{\alpha-1}$ is defined. Since $\varphi_{\alpha-1}=d_1d_2\neq 0$, (*) holds.

If $\dim(X_1\cap X_2)=\min(n_1,n_2)$, then $\dim X_1X_2\geqq\max(n_1,n_2)=\alpha$ and (*) holds in this case too.

If $X_1\cap X_2\neq\emptyset$ then φ_α is defined. Since

$$\varphi_\alpha = d_1d_2-\sigma^m(X_1,X_2) \quad\text{and}\quad \sigma^m(X_1,X_2)=\sum_{i=1}^{s} e_i\deg W_i,$$

(**) holds. If $X_1\cap X_2=\emptyset$, then (**) is trivial since the sum equals zero.

Suppose $X_1\cap X_2\neq\emptyset$. Then equality holds in (*), iff $\varphi_\alpha=0$ or equivalently, (**) is an equality.

REMARK. The inequality (**) is a special case of Fulton's refined Bezout theorem, more precisely, it follows directly from Theorem 12.3 and Example 12.3.7 in [7].

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