MATH. SCAND. 61 (1987), 223-247

FROBENIUS SYMBOLS FOR PARTITIONS AND DEGREES OF SPIN CHARACTERS*

JØRN B. OLSSON

In 1900 Frobenius [4] determined for the first time the irreducible characters of the finite symmetric groups S_n. Although he was aware of the fact that the irreducible characters of S_n may be indexed canonically by the partitions of n, he preferred a "more useful" indexation of the characters by what he called "characteristics". Using these he gave some elegant formulae for special charactervalues. Frobenius also considered β-sets for partitions which arose naturally in his construction of the characters. As is known now, the β-sets are useful in the study of the hook structure of partitions and thus for the practical recursive computation of general character values in S_n. In the first section of this paper we describe a theory of cuts in partition sequences which associates to a given partition λ an infinite sequence of symbols which include all the β-sets for λ as well as Frobenius' "characteristic" for λ. Also we study the relation between an arbitrary Frobenius symbol for λ and the hook structure of λ. The author discovered the Frobenius symbols during a study of the "bar-structure" of 2-regular partitions of n. These partitions index the spin characters of a covering groups $\hat{S}_n$ of S_n. It was first realized by Morris that "bars" play a similar role for spin characters as hooks for ordinary characters of symmetric groups. The results of section 1 may be used to give a perhaps more transparent description of the p-bar quotient of a 2-regular partition (see [12]) which is essential for further work. Theorem (2.3) is particularly important. It leads to an explicit description of the power of an odd prime dividing the degree of a spin character. This is done in section 3. In section 4 we study then the power of 2 dividing the degree of a spin character. Again the results of section 1 are essential. Whereas the distribution of spin characters into p-blocks is known for p odd (see [5], [14]) the same question is open for $p = 2$. We apply the results of section 4 to determine those spin characters, which are contained in 2-blocks of small defect, and finish with a conjecture.

* Partially supported by Deutsche Forschungsgemeinschaft.
Received November 06, 1986.

1. Partition sequences and Frobenius symbols for partitions.

The idea that a partition can be represented as a binary sequence was apparently first used by Comét [2] around 1960 to calculate character values for S_n on a computer. We start by developing this idea further.

We fix the following notation. Let

$$(1) \qquad \lambda = (a_1, a_2, \ldots, a_m), \quad a_1 \geqq a_2 \geqq \ldots \geqq a_m > 0$$

be a partition. If $a_1 + a_2 + \ldots + a_m = n$ we write $\lambda \vdash n$. The *dual* (or *conjugate*) partition of λ will be denoted by

$$(2) \qquad \lambda^0 = (b_1, b_2, \ldots, b_{m'}), \quad b_1 \geqq b_2 \geqq \ldots \geqq b_{m'} > 0$$

(see e.g. [9, p. 2]). The set of *first column hook lengths* of λ is defined as

$$(3) \qquad X_\lambda = \{a_i + (m-i) \mid i = 1, 2, \ldots, m\}$$

so $X_\lambda \subseteqq \mathsf{N}_0\ (= \mathsf{N} \cup \{0\})$. If $X \subseteqq \mathsf{N}_0$ and $r \in \mathsf{N}_0$ we define

$$X^{+r} = \{x + r \mid x \in X\} \cup \{r-1, r-2 \ldots, 1, 0\}$$

(so $X^{+0} = X$). Then the *β-sets for* λ are by definition exactly the sets $X_\lambda^{+r}, r \in \mathsf{N}_0$. From the definition we get immediately:

LEMMA (1.1). *If X is a finite subset of N_0, then there is a unique partition λ having X as a β-set. We then write $P^*(X) = \lambda$.*

A *partition sequence* Λ is a double infinite sequence of zeros and ones, such that if we consider the sequence going from the left to the right we have:

1) All entries to the left of a certain point are zeros.
2) All entries to the right of a certain point are ones.

Thus for example

$$(4) \qquad \ldots 0010011010011 \ldots$$

is a partition sequence, where the dots on the left and the right represent infinite sequences of zeros and ones, respectively. For simplicity we may symbolize infinite sequences of zeros and ones by $\underline{0}$ and $\underline{1}$, so that (4) may be written

$$(5) \qquad \underline{0}1001101001\underline{1}.$$

In a partition sequence Λ, the zeros will be numbered $1, 2, 3, 4, \ldots$ in the order they occur moving from the right to the left in Λ, and similarly the ones will be numbered $1, 2, 3, 4 \ldots$ in the order they occur moving from the left to the right in Λ. We call this the *natural numbering* of the zeros and ones in Λ.

In the above example

$$\begin{array}{lll} & \ldots\ 54\ \ 3\ 21 & \text{Natural numbering of the zeros.} \\ (6) & \underline{0}100110100\underline{1} & \\ & \ \ 1\ \ 23\ 4\ \ \ldots & \text{Natural numbering of the ones.} \end{array}$$

If the number of ones to the left of the ith zero (in the natural numbering of Λ) is a_i, we write

$$P(\Lambda) = (a_1, a_2, \ldots, a_m) \tag{7}$$

if $a_m \neq 0$, $a_{m+1} = 0$, so that $P(\Lambda)$ is in fact a partition.

EXAMPLE. If Λ is the partition sequence in (5) then

$$P(\Lambda) = (4,4,3,1,1) = (4^2, 3, 1^2).$$

This is seen from (6).

It is obvious from the definition that

LEMMA (1.2). *The map P is a bijection between the set of all partition sequences and the set of all partitions of nonnegative integers.*

NOTE. The sequence $\Lambda = \underline{0}\underline{1}$ is mapped onto the empty partition 0 of 0.

If $P(\Lambda) = \lambda$, we call Λ the *partition sequence* of λ. The partition sequence Λ of the partition λ incorporates in a natural way all the β-sets for λ as we shall see. This is one of the reasons why we, contrary to Comét, formally consider infinite sequences.

A *β-numbering* of a partition sequence Λ is obtained by numbering some entry in Λ occurring before the first entry one (or the first entry one itself) as 0 and then the following entries as they occur going from the left to the right by $1, 2, 3, \ldots$. In such a β-numbering the numbers of zero entries form a finite subset of N_0, i.e. a β-set. If $P(\Lambda) = \lambda$, this β-set is in fact X_λ^{+r}, where r is the number of first entry 1 in the β-numbering.

EXAMPLE. Let Λ be as in (4) and (5). A β-numbering of Λ is

$$\begin{array}{cccccccccccccccl} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & \ldots & \beta\text{-numbering} \\ \underline{0} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1\ \underline{1}\,. & \end{array}$$

The corresponding β-set is $\{11, 10, 8, 5, 4, 2, 1, 0\}$ (representing the numbers of the zeros). This equals $\{8, 7, 5, 2, 1\}^{+3}$ and $\{8, 7, 5, 2, 1\}$ is the set of first column hook lengths of $P(\Lambda) = (4^2, 3, 1^2)$.

For completeness we note the following: If Λ is a partition sequence, we may define its *dual* Λ^0 as the partition sequence obtained by reading Λ from the right to the left with zeros and ones interchanged. We then have, of course:

LEMMA (1.3). *In the above notation*

$$P(\Lambda)^0 = P(\Lambda^0).$$

EXAMPLE.

$$\Lambda = \underline{0}10110100\underline{1}, \quad P(\Lambda) = (4, 4, 3, 1)$$
$$\Lambda^0 = \underline{0}11010010\underline{1}, \quad P(\Lambda^0) = (4, 3, 3, 2).$$

Lemma (1.3) is in a certain sense generalized in the theory of cuts described below.

A *β-sequence* is an infinite sequence of zeros and ones, such that if we read the sequence from the left to the right all entries to the right of a certain point are ones. We number the entries in a β-sequence by $0, 1, 2, \ldots$ form the left to the right. Obviously β-sequences correspond bijectively to β-sets. Namely, if χ is a β-sequence we put

(8) $$Q(\chi) = \{i \in \mathrm{N}_0 \mid \text{The } i\text{th entry of } \chi \text{ is zero}\}$$

so that $Q(\chi)$ is a β-set. By adding an infinite number of zeros in front of a β-sequence, it is turned into a corresponding partition sequence.

LEMMA (1.4). *If χ is a β-sequence and Λ the corresponding partition sequence, then $Q(\chi)$ is a β-set for $P(\Lambda)$*, i.e.

$$P^*(Q(\chi)) = P(\Lambda).$$

PROOF. Obviously the numbering of χ described above gives a β-numbering of Λ, so the results follows.

Suppose that the partition λ' is obtained from another partition λ by removing a hook of length l (see [7, §2.3]). If X is a β-set for λ, we obtain a β-set Y for λ' by replacing an element c $(\geq l)$ in X by $c-l$, whereby $c-l \notin X$ (see [7, 2.7.13]). Thus

$$Y = (X \cup \{c-l\}) \setminus \{c\}, \quad |Y| = |X|.$$

Thus the β-sequence of Y is obtained from that of X by exchanging the zero in the cth position and the one in the $(c-l)$th position.

This shows that it is natural to define a *hook* in a partition sequence Λ

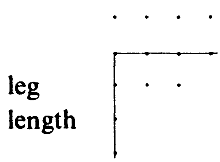
as a pair of entries in Λ, a zero and a one, such that the one (called the *arm*) is to the left of the zero (called the *leg*). The hook is *removed* by exchanging the arm and the leg. The hook is said to be in the ith *row* and the jth *column* if its leg has the number i and its arm has the number j in the natural numbering of the zeros and the ones. By definition there are a_i ones to the left of the ith zero, so there are a_i hooks in the ith row. Thus there is a canonical bijection between the hooks of Λ and the hooks in the (Young diagram of the) partition $\lambda = P(\Lambda)$. Moreover the map P is compatible with the removal of hooks.

The *leg length* b_{ij}^{λ} (*arm length* a_{ij}^{λ}) of the (i, j)-hook in Λ or $\lambda = P(\Lambda)$ is by definition the number of zeros (ones) between the arm and the leg of the hook (excluding these). The *length* l_{ij}^{λ} is defined as $l_{ij}^{\lambda} = a_{ij}^{\lambda}+b_{ij}^{\lambda}+1$. These definitions coincide with usual ones for hooks of partitions (see [7, §2.3]).

EXAMPLE.

$$\Lambda = \underline{0}\ \underbrace{\overset{\text{arm}}{1}\ 0\ 0\ 1\ 1\ 0\ 1\ \overset{\text{leg}}{0}}_{\text{hook}}\ 0\ \underline{1}, \quad \lambda = (4^2, 3, 1^2).$$

The hook considered is the $(2, 1)$-hook in λ as illustrated.



The length is 7, the arm length and the leg length is 3. Removing the hook we get $\Lambda' = \underline{0}\ 1\ 1\ 0\ 1\ 1\ 0\ \underline{1}$ and $\lambda' = P(\Lambda') = (4,2)$.

Let us remark that when hook is removed, its leg length is the difference between the number of the leg in the natural numbering after and before the removal. (In the above example the leg length is $3 = 5-2$.)

A *cut* in a partition sequence Λ is a dividing line between 2 entries, such that Λ is divided into 2 disjoint parts Λ_1 and Λ_2, where Λ_1 (Λ_2) consists of all entries to the right (left) of the dividing line.

EXAMPLE.

$$\Lambda:\ \underbrace{\underline{0}\ 1\ 0\ 0\ 1\ 1}_{\Lambda_2}\ \overset{\text{cut}}{|}\ \underbrace{0\ 1\ 0\ 0\ \underline{1}}_{\Lambda_1}$$

The *coordinates* of the cut are (g, h), where g is the number of zeros in Λ_1

and h is the number of ones in Λ_2. The *position* of the cut is then defined as $h-g$.

EXAMPLE. In the above example the coordinates are $(3, 3)$ and the position is $0 = 3-3$.

Suppose that a cut with the coordinates (g, h) is moved one step to the right, so that an entry is added to Λ_2 and the same entry is deleted in Λ_1. Then the new cut has the coordinates

$$\begin{array}{ll}(g, h+1) & \text{if the entry is one,}\\ (g-1, h) & \text{if the entry is zero.}\end{array}$$

Thus *in any case the position is increased by* 1. This shows:

LEMMA (1.5). *For each $i \in \mathbf{Z}$ there is exactly one cut in Λ with i as its position.*

We fix a partition sequence Λ and let $P(\Lambda) = \lambda$ be as in (1). Moreover we let (g_i, h_i) be the coordinates of the ith cut in Λ (the cut with position i), so that

$$h_i - g_i = i \quad \text{for all } i \in \mathbf{Z}. \tag{9}$$

Each cut in Λ determines 2 β-sequences, namely Λ_1 and Λ_2^0. Here Λ_2^0 is the dual of Λ_2, that is Λ_2 read from the right to the left with zeros and ones interchanged. If the cut is in the ith position, $i \in \mathbf{Z}$, we write

$$F_i(\Lambda) = F_i(\lambda) = (X_i | Y_i) \tag{10}$$

where

$$X_i = Q(\Lambda_1), \quad Y_i = Q(\Lambda_2^0)$$

and call $F_i(\Lambda) = F_i(\lambda)$ the ith *Frobenius symbol* of Λ or λ. By (8), X_i and Y_i are β-sets and by definition we get

$$|X_i| = g_i, \quad |Y_i| = h_i \tag{11}$$

so that $|Y_i| - |X_i| = i$ by (9). In the symbol $(X_i | Y_i)$ we simply write the elements of X_i and Y_i in decreasing order.

EXAMPLE. Consider the cut with position 0 in the previous example. We have

$$X_0 = \{3, 2, 0\}, \; Y_0 = \{3, 1, 0\},$$

so

$$F_0(\Lambda) = (3, 2, 0 | 3, 1, 0).$$

Further examples of Frobenius symbols for our given Λ are

$$F_{-1}(\Lambda) = (4,3,1|3,0), \qquad F_1(\Lambda) = (2,1|5,2,1)$$
$$F_{-5}(\Lambda) = (8,7,5,2,1|\emptyset), \quad F_4(\Lambda) = (\emptyset|8,5,4,2).$$

(Note that $8,7,5,2,1$ and $8,5,4,2$ are the first column and first row hook-lengths of λ.)

From the definition and Lemma (1.3) we get

PROPOSITION (1.6). *If $F_i(\Lambda) = (X|Y)$ then $F_{-i}(\Lambda^0) = (Y|X)$.*

It is also easy to see that the following holds:

PROPOSITION (1.7). *If X and Y are β-sets, there exists a unique partition $\lambda = P(X|Y)$ having $(X|Y)$ as a Frobenius symbol. Then $(X|Y) = F_i(\lambda)$, where $i = |Y| - |X|$.*

PROOF. We simply write the dual β-sequence of Y in front of the β-sequence of X to get a partition sequence with a cut at the place, where the β-sequences meet.

EXAMPLE. $X = \{4,2\}$, $Y = \{3,2,1\}$.

β-sequence for X:	$1\ 1\ 0\ 1\ 0\ \underline{1}$.
Dual β-sequence for Y:	$\underline{0}\ 1\ 1\ 1\ 0$.
Partition sequence:	$\underline{0}\ 1\ 1\ 1\ 0|1\ 1\ 0\ 1\ 0\ \underline{1}$.
Partition $\lambda = (6,5,3)$.	

If again $F_i(\Lambda) = (X_i|Y_i)$, then X_i and Y_i are β-sets for certain partitions, say

$$\lambda_i^c := P^*(X_i), \quad \lambda_i^r := P^*(Y_i).$$

PROPOSITION (1.8). *In the above notation we have*

(i) *The Young diagram of λ_i^c is obtained by removing the first h_i columns from the Young diagram of λ*, i.e.

$$\lambda_i^c = (b_{h_i+1}, b_{h_i+2}, \ldots, b_{m'})^0.$$

(ii) *The Young diagram of $(\lambda_i^r)^0$ is obtained by removing the first g_i rows from the Young diagram of λ*, i.e.

$$\lambda_i^r = (a_{g_i+1}, a_{g_i+2}, \ldots, a_m)^0.$$

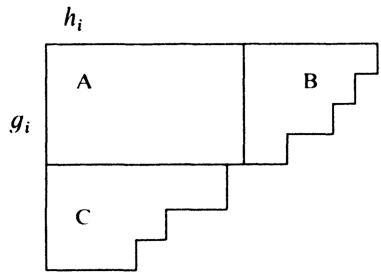
In particular $\lambda_i^c = 0$ if $h_i \geqq m'$ and $\lambda_i^r = 0$ of $g_i \geqq m$.

PROOF. The hooks of λ are recognized as pairs of zeros and ones in Λ as described above. But the partition sequence of λ_i^c is obtained from that of λ

by changing the first h_i ones (in the natural numbering) into zeros. The result (i) follows and (ii) is proved similarly.

We now give a complete description of the hooks of $\Lambda(\lambda)$ in terms of $F_i(\Lambda) = (X_i | Y_i)$, where again $X_i = Q(\Lambda_1)$, $Y_i = Q(\Lambda_2^0)$.

Proposition (1.8) suggests that the Young diagram of λ may be decomposed into 3 parts as follows



Part A is the intersection of the first g_i rows with the first h_i columns. The nodes in A correspond to those hooks of Λ whose, leg is in Λ_1 (to the right of the cut), and whose arm is in Λ_2 (to the left of the cut). These hooks are called *mixed* (relative to $(X_i | Y_i)$). Part B is the Young diagram of λ_i^c and the nodes represent hooks, whose arm and leg are in Λ_1. Similarly part C is the Young diagram of $(\lambda_i^r)^0$ and the nodes represent hooks whose arm and leg are in Λ_2. The hooks in the parts B and C are called *unmixed.*

To compute the mixed hook lengths write

$$X_i = \{c_1^i, c_2^i, \ldots, c_{g_i}^i\}, \quad c_1^i > c_2^i > \ldots > c_{g_i}^i \geqq 0$$

$$Y_i = \{d_1^i, d_2^i, \ldots, d_{h_i}^i\}, \quad d_1^i > d_2^i > \ldots > d_{h_i}^i \geqq 0.$$

As before $a_{kk'}^\lambda$, $b_{kk'}^\lambda$, $l_{kk'}^\lambda$ denotes the arm length, leg length and length of the (k, k')-hook in λ.

LEMMA (1.9).

(i) *For* $1 \leqq k \leqq g_i : c_k^i = a_k - k - i = a_{kk}^\lambda - i.$

(ii) *For* $1 \leqq k' \leqq h_i : d_{k'}^i = b_{k'} - k' + i = b_{k'k'}^\lambda + i.$

PROOF. Since X_i is a β-set for λ_i^c we have that for $1 \leqq k \leqq g_i$, $c_k^i - (g_i - k)$ has to equal the kth part of the partition in λ_i^c, that is $a_k - h_i$ (by (1.8)). Thus, using (9) we have

$$c_k^i = g_i - k + a_k - h_i = a_k - k - i.$$

Trivially $a_k - k = a_{kk}^\lambda$, so (i) is proved and (ii) is proved in a similar way.

(It should perhaps be noted, that in the above notation $a_k - h_i \geqq 0$ for $k = 1, 2, \ldots, g_i$, since by the defintion of the coordinate of a cut, the first h_i ones are to the left of the first g_i zeros. Thus the first g_i parts of λ are at least equal to h_i.)

In particular for $i = 0$ we have

COROLLARY (1.10). *For* $k = 1, 2, \ldots, h_0 = g_0$:

$$c_k^0 = a_k - k, \quad d_k^0 = b_k - k.$$

This corollary shows that the entries of $F_0(\lambda)$ are also the entries in Frobenius' "characteristic" for λ ([4, §4]) (see also [9, p. 3] or [15, p. 49]). So $F_0(\lambda)$ consists of the arm lengths and leg lengths of the diagonal hooks in λ. If $i > 0$, we shift the diagonal by i positions starting then at the $(1, i+1)$-node. If $i < 0$, we start at the $(-i+1, 1)$-node. Going diagonally we shall in any case hit a rim node in the position (g_i, h_i). Then the hook lengths $l_{kk'}^\lambda$ in Part A of the Young diagram are exactly all the the possible sums of an entry in X_i with an entry in Y_i plus 1. Indeed, adding the equations in (1.9) we get:

COROLLARY (1.11). *For* $1 \leqq k \leqq g_i$, $1 \leqq k' \leqq h_i$ *we have*

$$c_k^i + d_{k'}^i = l_{kk'}^\lambda - 1.$$

(Note also that

$$a_{kk'}^\lambda = c_k^i + i, \quad b_{kk'}^\lambda = d_{k'}^i - i, \tag{12}$$

which explains the shifting mentioned above.)

Collecting the information above we have:

PROPOSITION (1.12). *In the above notation*:

(i) *For* $1 \leqq k \leqq g_i$ *the hook lengths in the* kth *row of* λ *are*

$$\{c_k^i + d_{k'}^i + 1 \mid k' = 1, 2, \ldots, h_i\} \cup \{1, 2, \ldots, c_k^i\} \setminus \{c_k^i - c_l^i \mid l > k\}.$$

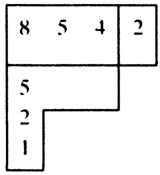
(ii) *For all* $j = 1, 2, \ldots, h_i$

$$\{l_{g_i+1,j}^\lambda, l_{g_i+2,j}^\lambda, \ldots, l_{b_i,j}^\lambda\} = \{1, 2, \ldots, d_j^i\} \setminus \{d_j^i - d_{j'}^i \mid j' > j\}.$$

PROOF. (i) describes the hook lengths in parts A and B using (1.11) and (1.8) (i), and (ii) describe the hook lengths in part C using (1.8) (ii).

EXAMPLE. Let Λ be as in (5),

$$\lambda = (4^2, 3, 1^2), \quad F_1(\Lambda) = (2, 1 \mid 5, 2, 1), \quad g_1 = 2, h_1 = 3.$$



The hook lengths in the first row are

$$\{2+5+1, 2+2+1, 2+1+1\} \cup \{1,2\} \setminus \{2-1\} = \{8,5,4,2\}.$$

(Similarly the hook lengths in the second row may be computed as $\{7,4,3,1\}$). Using (ii) we get for instance

$$\{l^{\lambda}_{31}, l^{\lambda}_{41}, l^{\lambda}_{51}\} = \{1,2,3,4,5\} \setminus \{5-2, 5-1\} = \{5,2,1\}.$$

We may of course also cómpute directly the first column hook lengths of λ, that is X_λ, from the Frobenius symbol:

PROPOSITION (1.13). *In the above notation*

$$X_\lambda = \{c^i_j + d^i_1 + 1 \mid j = 1,2,\ldots,g_i\} \cup \{1,2,\ldots,d^i_1\} \setminus \{d^i_1 - d^i_j \mid j = 2,\ldots,h_i\}.$$

REMARK 1. Mixed hooks (relative to $(X_i|Y_i)$) correspond canonically to pairs of elements (c,d), $c \in X_i$, $d \in Y_i$. The corresponding hook length is $c+d+1$, by (1.11). Removing this hook we get a partition having $(X_i \setminus \{c\} | Y_i \setminus \{d\})$ as Frobenius symbol!

REMARK 2. We may compute $|\lambda| = a_1 + \ldots + a_m$ from $(X_i|Y_i)$. Indeed

$$|\lambda| = \sum_k c^i_k + \sum_{k'} d^i_{k'} + \tfrac{1}{2}(g_i + h_i) - \tfrac{1}{2}(g_i - h_i)^2.$$

This follows easily, since part A contains $g_i \cdot h_i$ nodes, part B constraints $\sum_k c^i_k - \binom{g_i}{2}$ nodes and part C contains $\sum_{k'} d^i_{k'} - \binom{h_i}{2}$ nodes.

For $i = 0$, the formula coincides with Frobenius' [4, formula (7) in §4].

REMARK 3. (1.12) allows us to generalize two further formulas of Frobenius (for the degrees of the irreducible characters in S_n). Let as in [4],

$$\Delta(x_1,\ldots,x_t) = \prod_{1 \le i < j \le t} (x_j - x_i).$$

Let f_λ be the degree of the irreducible character of S_n corresponding to λ. Then

$$f_\lambda = \frac{n!\,\Delta(c^i_1, c^i_2, \ldots, c^i_{g_i})\,\Delta(d^i_1, d^i_2, \ldots, d^i_{h_i})}{c^i_1!\,c^i_2!\ldots c^i_{g_i}!\,d^i_1!\,d^i_2!\ldots d^i_{h_i}! \prod_{k,k'} (c^i_k + d^i_{k'} + 1)}.$$

This is a common generalization of [4, (6) in §3 and (9) in §4].

As a preparation for section 2 we mention the following:

In [12] a mixture of the ordinary notation and the "Frobenius notation" for λ is used. The authors write

$$\lambda \leftrightarrow (a_1, a_2, \ldots, a_i; (c_{i+1}, \ldots, c_{m'}; d_{i+1}, \ldots, d_{m'})),$$

where

$$F_0(a_{i+1}, \ldots, a_m) = (c_{i+1}, \ldots, c_{m'} | d_{i+1}, \ldots, d_{m'}).$$

(Their assumption that $m = m_i$ is generally not correct!) Using (1.9) we get:

COROLLARY (1.14). *For* $i \geqq 0$

$$F_{-i}(\lambda) = (a_1 + (i-1), a_2 + (i-2), \ldots, a_i, c_{i+1}, \ldots, c_{m'} | d_{i+1}, \ldots, d_{m'})$$

(the number m' is really g_{-i}).

This result and a similar result for $F_i(\lambda)$ (which is obvious and left to the reader) shows that Morris' and Yaseen's definition of the p-bar quotient is equivalent to ours, which will be given in section 2.

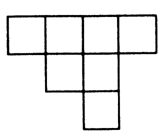
2. On p-bar cores and p-bar quotients.

If λ is a partition of n with all parts different (i.e. a 2-regular partition in the notation of [7]) we write $\lambda \succ n$ and call λ a *bar partition*. We assume that

$$(1) \qquad \lambda = (a_1, \ldots, a_m), \quad a_1 > a_2 > \ldots > a_m > 0$$

is a bar partition of n. Let $S(\lambda)$ be the shifted Young diagram of λ (see e.g. [9, p. 135]). It is obtained from the usual Young diagram of λ by shifting the ith row $(i-1)$-positions to the right. For example, if $\lambda = (4, 2, 1)$ then

$S(\lambda)$:



The jth node in the ith row will be called the (i, j)-node. To each node in $S(\lambda)$, we associate an integer *bar length* as follows: The bar lengths in the ith row are obtained by writing the elements of the following set in decreasing order:

$$(2) \qquad \{1, 2, \ldots, a_i\} \cup \{a_i + a_j | j > i\} \setminus \{a_i - a_j | j > i\}.$$

(So in $\{1, 2, \ldots, a_i\}$ one replaces $a_i - a_j$ by $a_i + a_j$). Thus the first $(m-i)$-bar lengths are $a_i + a_{i+1}, \ldots, a_i + a_m$ (in this order), and then the remaining $a_i - (m-i)$-bar lengths are the *hook lengths* in the ith row of the partition $\lambda^* = P^*(\lambda)$ (considering λ as a β-set!).

In the above example the bar lengths are

$$\begin{array}{cccc} 6 & 5 & 4 & 1 \\ & 3 & 2 & \\ & & 1 & \end{array} .$$

(Note that $\begin{smallmatrix}4 & 1\\ 2 & \\ 1 & \end{smallmatrix}$ is the hook daigram for $(2, 1^2) = P^*(\{4, 2, 1\})$.

We denote the (i, j)-bar length by $\bar{T}_{ij}^{\lambda}$.

To each node (i, j) in $S(\lambda)$ we associate a *bar*, i.e. a subdiagram of $S(\lambda)$ consisting of $\bar{T}_{ij}^{\lambda}$ nodes. If $i+j > m$ the (i, j)-bar consists of the last $\bar{T}_{ij}^{\lambda}$ nodes in the ith row of $S(\lambda)$. Such a bar is called *unmixed*. If $i+j \leqq m$, the (i, j)-bar consists of all the nodes in the ith and all the nodes in the $(i+j)$th row of $S(\lambda)$ (a *mixed* bar). If we remove the nodes of the (i, j)-bar from $S(\lambda)$ and rearrange the rows of the diagram obtained according to size we obtain a new shifted diagram $S(\mu)$ and say, that μ is obtained from λ by *removing the* (i, j)*-bar*. Thus $\mu \succ n - \bar{T}_{ij}^{\lambda}$. The parts of μ are

$$a_1, a_2, \ldots, a_{i-1}, a_i - \bar{T}_{ij}^{\lambda}, a_{i+1}, \ldots, a_m \quad \text{if } i+j > m$$

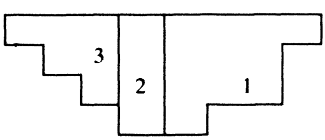
and

$$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{i+j-1}, a_{i+j+1}, \ldots, a_m \quad \text{if } i+j \leqq m.$$

If the (i, j)-bar is removed from λ to get μ we write $\mu = \lambda \setminus \bar{H}_{ij}^{\lambda}$. A bar of length l is called an *l-bar*. The (i, j)-bar is said to be of

$$(3) \qquad \begin{cases} \left.\begin{array}{ll} \textit{Type } 1, & \text{if } i+j \geqq m+2 \\ \textit{Type } 2, & \text{if } i+j = m+1 \end{array}\right\} & \text{unmixed bar.} \\ \begin{array}{ll} \textit{Type } 3, & \text{if } i+j \leqq m \end{array} & \text{mixed bar.} \end{cases}$$

This is illustrated in the following shifted diagram:



mixed, unmixed

We investigate all bars in λ whose lengths are divisible by a fixed positive integer p. As is done in [6] (for β-sets of partitions) and in [12] (for bar partitions) we represent the parts of λ as beads on a *p-abacus*. This abacus has p runners numbered $0, 1, 2, \ldots, p-1$ going from the north to the south. Contrary to [12] the rows will be numbered $0, 1, 2, \ldots$.

For $0 \leqq i \leqq (p-1)$ we define

$$X_i^\lambda := \{a \in \mathbf{N}_0 \mid a_k = ap+i \quad \text{for some } k \in \{1, 2, \ldots, m\}\} \tag{4}$$

so that the X_i^λ are β-sets. The bead configuration of λ on the p-abacus then includes a bead in the jth row of the ith runner, if and only if $j \in X_i^\lambda$.

We define a p-bar core $\lambda_{(\bar{p})}$ (abbreviated *$\bar{p}$-core*) and a p-bar quotient $\lambda^{(\bar{p})}$ (*$\bar{p}$-quotient*) for λ having properties analogous to the p-core and p-quotient for arbitrary partitions ([7, § 2.7]). Especially we want the structure of $\lambda^{(\bar{p})}$ to describe the bars of length divisible by p. In general this is only possible for *p odd*. Consider the following:

EXAMPLE. $\lambda = (3, 1)$. The bar diagram is $\begin{matrix} 4 & 3 & 1 \\ & 1 & \end{matrix}$. So λ contains a 4-bar, but no 2-bar.

This shows that for $p = 2$, there cannot be a $\bar{p}$-quoteint having the property of Theorem (2.3). The same difficulty arises when p is even (then $(3p/2, p/2)$ has a $2p$-bar but no p-bar). This also shows that both statements of the Corollary in [12, p. 31] are false for $p (= q)$ even. Another difficulty arises as well when p is even. Part of it is reflected in [12, Theorem 2 (2)]. In [12], $\bar{p}$-quotients are also defined for p even but they are not suitable for our purposes.

However, a careful examination of the arguments below shows that the difficulties will not arise for λ, when $X_{p/2}^\lambda = \emptyset$. (For a special case of this see section 4.)

We assume from now on that p is odd and put $t = (p-1)/2$. Then we define the $\bar{p}$-quotient

$$\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t),$$

where λ_0 is the bar partition whose parts are the elements in X_0^λ and where for $1 \leqq j \leqq t$

$$\lambda_j = P(X_j^\lambda \mid X_{p-j}^\lambda) \tag{5}$$

in the notation of (1.7). Thus $\lambda_1, \ldots, \lambda_t$ are partitions. The runner 0 determines λ_0 and for $1 \leqq j \leqq t$ the *conjugate runners* j and $p-j$ determines λ_j. (As mentioned in section 1, (1.14) shows that our definition of $\lambda^{(\bar{p})}$ coincides with the one in [12].)

The removal of a p-bar is registered on the abacus as follows (see (3)):

$$(6)\quad \begin{cases} \textit{Type } 1: & \text{Move a bead one position up on the same runner.} \\ \textit{Type } 2: & \text{Remove the bead in the first row on the 0th runner.} \\ \textit{Type } 3: & \text{Remove the 2 beads in the 0th row on the } j\text{th and the} \\ & (p-j)\text{th runner for some } j,\ 1 \leqq j \leqq t. \end{cases}$$

When we remove recursively all p-bars from λ we see (using (6) repeatedly) that the p-abacus configuration is changed until the following configuration is obtained:

(i) There are no beads on the 0th runner.
(ii) For $1 \leqq j \leqq p-1$ the jth runner contains $l_j = \mathrm{Max}(|X_j^\lambda| - |X_{p-j}^\lambda|, 0)$ beads in the l_j first rows.

Thus (see also [12]):

PROPOSITION (2.1). *The $\bar{p}$-core $\lambda_{(\bar{p})}$ of λ obtained by removing all p-bars from λ is uniquely determined by λ and p.*

Putting $f_j = |X_j^\lambda| - |X_{p-j}^\lambda|$ for $1 \leqq j \leqq t$, we see that the t-type $(f_1, \ldots, f_t)$ of integers determines $\lambda_{(\bar{p})}$ completely. We call it the *characteristic* of $\lambda_{(\bar{p})}$. (Not to be confused with Frobenius' concept.)

EXAMPLE. $\lambda = (17, 14, 13, 11, 9, 5, 2)$, $p = 5$

$$\text{5-abacus:}\quad \begin{array}{ccccc} 0 & 1 & 2 & 2 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 17 & 17 & 18 & 19 \end{array}$$

$X_0^\lambda = \{1\}$, $X_1^\lambda = \{2\}$, $X_2^\lambda = \{3, 0\}$, $X_3^\lambda = \{2\}$, $X_4^\lambda = \{2, 1\}$. Thus $\lambda_0 = (1)$, $\lambda_1 = P(2|2, 1) = (4, 2)$, $\lambda_2 = P(3, 0|2) = (3, 1^3)$.

Partition sequence for λ_1: $\underline{0}\ 1\ 1\ 0|1\ 1\ 0\ \underline{1}$.
Partition sequence for λ_2: $\underline{0}\ 1\ 0\ 0|0\ 1\ 1\ 0\ \underline{1}$.
Characteristic: $(-1, 1)$. $\lambda_{(\bar{5})} = (4, 2)$.

Suppose that $(\lambda_0, \lambda_1, \ldots, \lambda_t)$ is given, where λ_0 is a bar partition and $\lambda_1, \ldots, \lambda_t$ are partitions, and suppose that $(f_1, \ldots, f_t) \in \mathbf{Z}^t$. Let X_0 be the set of parts of λ_0 and for $1 \leqq i \leqq t$ let

$$F_{f_i}(\lambda_i) = (X_i | X_{p-i}).$$

Then the bar partition λ having $X_i^\lambda = X_i$ for $i = 0, 1, \ldots, p-1$ has $(\lambda_0, \lambda_1, \ldots, \lambda_t)$ as a $\bar{p}$-quotient and its $\bar{p}$-core has $(f_1, \ldots, f_t)$ as characteristic. (As in section 1, F denotes a Frobenius symbol.) We have proved:

PROPOSITION (2.2). *A bar partition determines and is uniquely determined by its $\bar{p}$-core and its $\bar{p}$-quotient.*

We may generalize (6) as follows:

The removal of an lp-bar, $l \geqq 1$ is registred on the p-abacus as follows:

(7)
- *Type* 1: Move a bead l positions up on the same runner.
- *Type* 2: Remove the bead in the lth row on the 0th runner.
- *Type* 3: (i) Remove the 2 beads in the 0th runner in the l_1th and l_2th row, $1 \leqq l_1 < l_2, l_1 + l_2 = l$.
  (ii) Remove the bead in the l_1th row and the jth runner and the bead in the l_2th row and the $(p-j)$th runner, $1 \leqq j \leqq t$, $l_1 + l_2 = l - 1$.

Defining an *l-bar* in $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$ to be either an l-bar in λ_0 or an l-hook in one of the partitions $\lambda_1, \ldots, y_t$ and the *removal* of an l-bar in $\lambda^{(\bar{p})}$ correspondingly, we have the following important result:

THEOREM (2.3). *There exists a canonical bijection g between the set of bars of λ of length divisible by p and the set of bars in $\lambda^{(\bar{p})}$. Thereby an lp-bar of λ is mapped onto an l-bar of $\lambda^{(\bar{p})}$. Moreover, for the removal of corresponding bars we have*

$$(\lambda \setminus \bar{H})^{(\bar{p})} = \lambda^{(\bar{p})} \setminus g(\bar{H}).$$

PROOF. Suppose that $\bar{H}$ is an lp-bar, $l \geqq 1$. We adapt the notation of (7).

Type 1. The removal of $\bar{H}$ is registred by moving a bead l positions up on (say) the jth runner. In $\lambda^{(\bar{p})}$ this corresponds

if $j = 0$ to the removal of an l-bar of type 1 in λ_0,

if $1 \leqq j \leqq t$ to the removal of an l-hook in part B of the Young diagram of λ_j,

if $t+1 \leqq j \leqq p-1$ to the removal of an l-hook in the part C of the Young diagram of λ_{p-j}.

(The letters B and C refer to the decomposition of a Young diagram described in section 1.)

Type 2. The removal of $\bar{H}$ is registred by removing a bead from the 0th runner. Corresppndingly the l-bar of type 2 is removed in λ_0.

Type 3. (i) The removal of $\bar{H}$ is registred by removing beads representing l_1p and $l_2 p$. Correspondingly a mixed l-bar (of type 3) consisting of the parts l_1, l_2 is removed from λ_0.

(ii) The removal of $\bar{H}$ is registred by removing beads on the jth and p-jth runner ($1 \leqq j \leqq t$). Correspondingly a mixed hook relative to $(X_i^\lambda | X_{p-j}^\lambda)$ is removed from λ_j (in part A of the Young diagram).

The above describes how the map g has to be defined in order to be compatible with the removal of bars and shows its existence. Thus (2.3) is proved.

If $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$ we define the *$\bar{p}$-weight of λ* as

$$w_{\bar{p}}(\lambda) = |\lambda_0| + |\lambda_1| + \ldots + |\lambda_t|.$$

Repeated use of (2.3) (with $l = 1$) shows:

COROLLARY (2.4). *The number of $\bar{p}$-bars to be removed going from λ to $\lambda_{(\bar{p})}$ equals $w_{\bar{p}}(\lambda)$. Thus*

$$|\lambda| = |\lambda_{(\bar{p})}| + p w_{\bar{p}}(\lambda).$$

Moreover exactly $w_{\bar{p}}(\lambda)$ bars in λ have lengths divisible by p.

COROLLARY (2.5). *If λ contains an lp-bar, $l \geqq 2$ then λ contains also a p-bar (p odd).*

NOTE. As a matter of fact an lp-bar may be "decomposed" into l p-bars. This will be discussed in [11].

As we have seen, $w_{\bar{p}}(\lambda)$ p-bars have to be removed going from λ to $\lambda_{(\bar{p})}$. One may ask how many of these p-bars are of type 1, 2 and 3 respectively. This number will depend on $\lambda_{(\bar{p})}$ and $\lambda^{(\bar{p})}$ as follows: (Let $m(\lambda)$ denote the number of parts in the bar partition λ).

COROLLARY (2.6). *Let $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$.*

(i) *The number of p-bars of type 2 being removed going from λ to $\lambda_{(\bar{p})}$ is $m(\lambda_0)$.*
(ii) *The number of p-bars of type 3 being removed going from λ to $\lambda_{(\bar{p})}$ is $\frac{1}{2}(m(\lambda) - m(\lambda_{(\bar{p})}) - m(\lambda_0))$.*

PROOF. (i) is an easy consequence of the proof of (2.3)

(ii) By the removal of a bar of type 2, the number of parts in a bar partition is reduced by 1, and by the removal of a bar of type 3 the number of parts is reduced by 2. So if a is the number of p-bars of type 3 being removed going from λ to $\lambda_{(\bar{p})}$, then by (i)

$$m(\lambda) - m(\lambda_{(\bar{p})}) = m(\lambda_0) + 2a.$$

The result follows.

EXAMPLE. In the previous example, $w_{\bar 5}(\lambda) = 13 = 1+6+6$. Of the 13 5-bars being removed going from λ to $\lambda_{(\bar 5)}$, 1 is of type 2, 2 are of type 3 and the remaining 10 of type 1.

3. The power of an odd prime dividing the degree of a spin character.

In [13] the author introduced the p-core tower of a partition as an array of p-cores which uniquely determined the partition. Using this the power of p dividing the degree of an irreducible character of S_n could be described in a way, which was useful for enumeration (see [8]). The p-core tower has had other uses also. Here we define in an analogous way the $\bar{p}$-core tower of a bar partition and use it to study the degree of a spin character of $\hat{S}_n$ (i.e. an irreducible character of $\hat{S}_n$ which is faithful).

If $\lambda \succ n$, then the degree $g(\lambda)$ of a spin character indexed by λ is (see [10, Theorem 1])

$$g(\lambda) = 2^{[(n-m)/2]} n!/\bar{h}(\lambda) \tag{1}$$

where $\lambda = (a_1, a_2, \ldots, a_m)$, $\bar{h}(\lambda)$ is the product of all the bar lengths of λ (see section 2) and generally $[x]$ denotes the integral part of x. If p is a prime and z an integer we write $v_p(z) = a$ if $p^a | z$ and $p^{a+1} \nmid z$. We denote the multiset of all bar lengths of a bar partition λ by $\bar{\mathscr{H}}(\lambda)$ that is $\bar{\mathscr{H}}(\lambda)$ lists all bar lengths of λ with the multiplicity to which they occur). Similarly $\mathscr{H}(\mu)$ denotes the multiset of the hook lengths of a (general) partition μ. The product of the hook lengths of μ is denoted by $h(\mu)$.

EXAMPLE. (i) $\lambda = (4,3) \succ 7$, $\bar{\mathscr{H}}(\lambda) = \{7,4,3,3,2,2,1\}$, $\bar{h}(\lambda) = 1008$.
(ii) $\lambda = (4,3) \vdash 7$, $\mathscr{H}(\lambda) = \{5,4,3,3,2,1,1\}$, $h(\lambda) = 360$.

From now on we assume that p is an odd prime and that $\lambda \succ n$. Let $\lambda_{(\bar p)}$ and $\lambda^{(\bar p)} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$, $t = p-1/2$, be the $\bar{p}$-core and $\bar{p}$-quotient of λ. Let $w = w_{\bar p}(\lambda)$ be the $\bar{p}$-weight of λ. By (2.4)

$$n = |\lambda_{(\bar p)}| + wp. \tag{2}$$

Moreover by (2.3)

$$v_p(\bar{h}(\lambda)) = w + v_p(\bar{h}(\lambda_0)) + \sum_{i=1}^{t} v_p(h(\lambda_i)). \tag{3}$$

Now we may of course also apply to formula (3) to the bar partition λ_0. The result analogous to (2.3) for hooks shows that if μ is a partition and

$\mu^{(p)} = (\mu_0, \mu_1, \ldots, \mu_{p-1})$ its p-quotient then

$$v_p(h(\mu)) = w' + \sum_{i=0}^{p-1} v(h(\mu_i)) \tag{4}$$

where $w' = \sum_i |\mu_i|$ is the p-weight of μ.

In analogy with [13] we define the *$\bar{p}$-core tower* of $\lambda \succ n$ as follows: It has rows numbered $0, 1, 2, \ldots$, where the ith row contains one $\bar{p}$-core and $(p^i - 1)/2$ p-cores. The zeroth row consists of $\lambda_{(\bar{p})}$. If $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$, the first row consists of $\lambda_{0(\bar{p})}, \lambda_{1(p)}, \ldots, \lambda_{t(p)}$. Now the bar partition λ_0 has a $\bar{p}$-quotient and the partitions $\lambda_1, \ldots, \lambda_t$ have p-quotients. Then the second row consists of the $\bar{p}$-core of the first partition in $\lambda_0^{(\bar{p})}$ and the p-cores of the remaining partitions in $\lambda_0^{(\bar{p})}$, followed by the p-cores of the partitions in $\lambda_1^{(p)}, \ldots, \lambda_t^{(p)}$. Continuing this way we get the $\bar{p}$-core tower.

EXAMPLE. Consider again the example following (2.1). We have $\lambda_{(\bar{5})} = (4, 2)$, $\lambda^{(\bar{5})} = ((1), (4, 2), (3, 1^3))$. The $\bar{5}$-core tower is the following:

$$\begin{array}{ccccccccccccc}
 & & & & & (4,2) & & & & & & & \\
 & (1) & & & & (1) & & & & (3,1^3) & & & \\
(0) & (0) & (0) & (0) & (0) & (0) & (1) & (0) & (0) & (0) & (0) & (0) & (0).
\end{array}$$

The remaining entries are (0).

Using (2.2) and the corresponding fact for partitions and p-cores and p-quotients, we see that we may always recover a bar partition from its $\bar{p}$-core tower. Let $\beta_i(\bar{p}, \lambda)$ be the sum of the cardinalities of the partitions in the ith row of the $\bar{p}$-core tower of λ. Then we get from the definition

$$|\lambda| = \sum_{i \geq 0} \beta_i(\bar{p}, \lambda) p^i. \tag{5}$$

In analogy with the computations in [8] and [13] we get by an inductive argument based in (2), (3), and (4):

PROPOSITION (3.1). *In the above notation*

$$v_p(\bar{h}(\lambda)) = \left(n - \sum_{i \geq 0} \beta_i(\bar{p}, \lambda)\right) / (p-1).$$

If $n = \sum_{i \geq 0} a_i p^i$ is the p-adic decomposition of n then

$$v_p(n!) = \left(n - \sum_{i \geq 0} a_i\right) / (p-1)$$

(see eg. [13]). Using this fact we get:

PROPOSITION (3.2). *In the above notation*

$$v_p(g(\lambda)) = \left(\sum_{i \geqq 0} (\beta_i(\bar{p}, \lambda) - a_i)\right) / (p-1).$$

This proposition makes it possible for a given $a \geqq 0$ to compute the number of bar partitions λ of n with $v_p(g(\lambda)) = a$. The formula is similar to (3.4) in [13]. The results in this section will be applied in [14].

4. The power of 2 dividing a spin character degree.

Let $\lambda = (a_1, a_2, \ldots, a_m) \succ n$. We consider the power of 2 dividing the degree

$$(1) \qquad g(\lambda) = 2^{[(n-m)/2]} n!/\bar{h}(\lambda)$$

(see section 3). We define

$$a(\lambda) := [(n-m)/2] = [(|\lambda| - m(\lambda))/2]$$

(where generally $m(\lambda)$ denotes the number of parts in the partition λ)

$$t_2(\lambda) := v_2(\bar{h}(\lambda))$$

$$t_2^*(\lambda) := t_2(\lambda) - a(\lambda),$$

so $v_2(g(\lambda)) = v_2(n!) - t_2^*(\lambda)$. First we show to reduce the computation of $t_2(\lambda)$ and $t_2^*(\lambda)$ to the case where all parts of λ are odd. In that case a theory of $\bar{4}$-quotients is relevant.

If $\lambda \succ n$ let $\lambda_0(\lambda_e)$ be the partition consisting of all the odd (even) parts of λ.

EXAMPLE. $\lambda = (8, 5, 2, 1)$, $\lambda_0 = (5, 1)$, $\lambda_e = (8, 2)$.

LEMMA (4.1). *In the above notation*

$$t_2(\lambda) = t_2(\lambda_0) + t_2(\lambda_e).$$

PROOF. The power of 2 dividing the product of the integers in the set

$$(\{1, 2, \ldots, a_i\} \cup \{a_i + a_j | j > i\}) \setminus \{a_i - a_j | j > i\}$$

(which are the bar lengths in the ith row of λ) equals the power of 2 dividing the product of the integers in

$$(\{1, 2, \ldots, a_i\} \cup \{a_i + a_j | \, j > i, \; a_i \equiv a_j (\mathrm{mod}\, 2)\} \setminus \{a_i - a_j | \, j > i, \; a_i \equiv a_j (\mathrm{mod}\, 2)\}$$

because when $a_i \not\equiv a_j (\mathrm{mod}\, 2)$, then $v_2(a_i + a_j) = v_2(a_i - a_j) = 0$. From this (4.1) follows easily.

LEMMA (4.2). *If all the parts of λ are even, let λ' be the bar partition obtained* *by dividing all the parts of λ by 2. In that case*

$$t_2(\lambda) = |\lambda'|+t_2(\lambda').$$

PROOF. Let $\lambda = (a_1, \dots, a_m)$, $a_i = 2\alpha_i$. Then the even bar lengths in the ith row of λ are in the set

$$(\{2, 4, \dots, 2\alpha_i\} \cup \{2\alpha_i+2\alpha_j| \, j > i\})\setminus\{2\alpha_i-2\alpha_j| \, j > i\}.$$

Dividing all integers in this set (which is of cardinality α_i) by 2 we get the bar lengths in the th row of λ'.

This is all we need to derive a formula for $t_2(\lambda)$ in terms of partitions with all parts odd. We define for $i \geqq 0$ inductively bar partitions $\lambda^{(i)}$, $\lambda_0^{(i)}$, $\lambda_e^{(i)}$ as follows:

(i) $\lambda^{(0)} = \lambda$,

(ii) $\lambda_0^{(i)}(\lambda_e^{(i)})$ is the partition consisting of the odd (even parts of $\lambda^{(i)}$,

(iii) $\lambda^{(i+1)}$ is obtained by dividing all parts of $\lambda_e^{(i)}$ by 2.

The above lemmas imply

$$|\lambda^{(i)}| = |\lambda_0^{(i)}|+2|\lambda^{(i+1)}| \quad \text{for } i \geqq 0$$

$$t_2(\lambda^{(i)}) = t_2(\lambda_0^{(i)})+|\lambda^{(i+1)}|+t_2(\lambda^{(i+1)}) \quad \text{for } i \geqq 0.$$

Using these equations we get by an easy calculation

$$n = |\lambda| = \sum_{i \geq 0} |\lambda_0^{(i)}|2^i$$

$$t_2(\lambda) = \sum_{i \geq 0} t_2(\lambda_0^{(i)})+ \sum_{i \geq 1} (2^i-1)|\lambda_0^{(i)}|$$

$$= \sum_{i \geq 0} t_2(\lambda_0^{(i)})+ \left(n-\sum_{i \geq 0} |\lambda_0^{(i)}|\right).$$

To simplify notation, we put $\lambda^i = \lambda_0^{(i)}$, $i \geqq 0$ and so each λ^i has only odd parts. In fact, t is a part of λ^i if and only if t is odd and $2^i t$ is a part of λ. We have proved:

PROPOSITION (4.3). *If $\lambda \succ n$ write*

$$\lambda = \lambda^0+2\lambda^1+4\lambda^2+\dots$$

where each λ^i is a bar partition with all parts odd. Then

$$n = \sum_{i \geq 0} |\lambda^i|2^i \tag{2}$$

$$t_2(\lambda) = \sum_{i \geqq 0} t_2(\lambda^i) + (n - \sum|\lambda|). \tag{3}$$

Next we make a similar decomposition for $a(\lambda) = [(n - m(\lambda))/2]$.

LEMMA (4.4. *In the notation of* (4.3)

$$a(\lambda) = \sum_{i \geqq 0} a(\lambda^i) + \left[\frac{1}{2}\left(n - \sum_{i \geqq 0} |\lambda^i|\right)\right]. \tag{4}$$

PROOF. Trivially $m(\lambda) = \sum_{i \geqq 0} m(\lambda^i)$. Moreover $|\lambda^i| - m(\lambda^i)$ is even for all i, since λ^i has only odd parts, so $a(\lambda^i) = (|\lambda^i| - m(\lambda^i))/2$. Using this and (2), then (4) follows by an easy calculation.

PROPOSITION (4.5). *In the notation of* (4.3)

$$\begin{aligned} t_2^*(\lambda) &= \sum_{i \geqq 0} t_2^*(\lambda^i) + \left[\frac{1}{2}\left(n + 1 - \sum_{i \geqq 0} |\lambda^i|\right)\right] \\ &= \sum_{i \geqq 0} t_2^*(\lambda^i) + \left[\frac{1}{2}\left(1 + \sum_{i \geqq 1} |\lambda^i|(2^i - 1)\right)\right]. \end{aligned} \tag{5}$$

PROOF. Use (4.3), (4.4) and the fact that for any integer k we have $k = [k + 1/2] + [k/2]$.

COROLLARY (4.6). *For all* $\lambda \succ n$, $t_2^*(\lambda) \geqq 0$.

PROOF. By (5) it suffices to show $t_2^*(\lambda) \geqq 0$, when λ has all parts odd. In that case, $a(\lambda)$ is just the number of even bar lengths in λ, as is easily seen. (There are $(a_i - 1)/2$ even bar lengths in the ith row of λ.)

We have reduced our problem to the case, where all parts of λ are odd (i.e. $\lambda = \lambda^0$). As we just mentioned, $a(\lambda)$ is then the number of even bar lengths in λ. Therefore, to compute $t_2(\lambda)$ or $t_2^*(\lambda)$ we need only consider bars of a length *divisible by* 4. Let (as in section 2)

$$X_1^\lambda = \{a \in \mathrm{N}_0 \mid a_j = 4a + 1 \quad \text{for some } j \in \{1, 2, \ldots, m\}\}$$

$$X_3^\lambda = \{a \in \mathrm{N}_0 \mid a_j = 4a + 3 \quad \text{for some } j \in \{1, 2, \ldots, m\}\}$$

(so $|X_1^\lambda| + |X_3^\lambda| = m$) and put

$$\mu(\lambda) := P(X_1^\lambda | X_3^\lambda)$$

in the notation of section 1. So $\mu(\lambda)$ is a partition. An argument analogous to the one used in the proof of (2.3) (the bars of type 2 do not occur) shows that we have:

PROPOSITION (4.7). *Let $\lambda \succ n$ have all parts odd. For $l \geqq 1$ there is a canonical bijection g between the set of $4l$-bars in λ and the set of l-hooks in $\mu(\lambda)$. If $\bar{H}$ is a $4l$-bar in λ we have*

$$\mu(\lambda \setminus \bar{H}) = \mu(\lambda) \setminus g(\bar{H}).$$

NOTE. The reason that the difficulties mentioned in section 2 do not occur here is really that λ has no parts which are congruent 2 modulo 4.

If μ is an ordinary partition, let $s_2(\mu) = v_2 h(\mu)$, where $h(\mu)$ is the product of the hook lengths of μ. Then $s_2(\mu)$ may be computed as in [8], [13] (see also section 3). We have

PROPOSITION (4.8). *Let $\lambda \succ n$ have all parts odd. Then*

$$t_2^*(\lambda) = |\mu(\lambda)| + s_2(\mu(\lambda)).$$

PROOF. If $\mu = \mu(\lambda)$, then $a(\lambda)$ bar lengths of λ are divisible by 2 and $|\mu|$ bar lengths are divisible by 4 (using (4.7)). The bars of length divisible by at least 8 are registred as hook of even length in μ (by (4.7)). Thus

$$t_2(\lambda) = a(\lambda) + |\mu| + s_2(\mu)$$

as desired.

This finishes our study of the power of 2 dividing spin character degrees. We apply our result to study 2-blocks of low defect in a covering group $\hat{S}_n$ of S_n.

Let Z be the central subgroup of order 2 in $\hat{S}_n$, so that $S_n \simeq \hat{S}_n/Z$. It is clear that each 2-block B of S_n (considered as a set of irreducible characters) is contained in a unique 2-block $\hat{B}$ of $\hat{S}_n$ and that each $\hat{B}$ will contain both ordinary and spin characters (see [3, V.4]). Obviously we have for the defects of the blocks that $d(\hat{B}) = d(B)+1$. So if $d(B) = 0$, then $\hat{B}$ has a cyclic defect group of order 2. If $d(B) = 1$, then the defect group of $\hat{B}$ is cyclic or elementary abelian of order 4, depending on which of the 2 covering groups of S_n we consider. In any case, the blocks considered will have just one modular character by the general theory (Dade and Brauer). If $d(B) = 0$, then $\hat{B}$ contains one spin character, and if $d(B) = 1$, then $\hat{B}$ contains 2 spin characters. We determine the bar partitions indexing these characters. Since the blocks considered have just one modular characters, the spin characters given in (4.9) and (4.10) below from 2 infinite series of spin characters, which remain irreducible modulo 2. It is not at all obvious from the degree formulas why the ordinary and the spin characters have the same degrees.

Let $\langle\lambda\rangle$ denote a spin character of $\hat{S}_n$ indexed by $\lambda \succ n$. Generally, if $B \leqq \hat{B}$ and $\langle\lambda\rangle \in \hat{B}$, it follows from the definition of the height of a character in a block, that

$$d(B) = t_2^*(\lambda) + \text{height}(\langle\lambda\rangle) \geqq t_2^*(\lambda). \tag{6}$$

Suppose first that $t_2^*(\lambda) = 0$ for some $\lambda \succ n$. By (4.5) we get $t_2^*(\lambda^i) = 0$ for all i and that in fact $\lambda^1 = \lambda^2 = \ldots = 0$, so that $\lambda = \lambda^0$. Then (4.8) forces $|\mu(\lambda)| = 0$, so that λ has no bar of length divisible by 4, by (4.7). Thus either $X_1^\lambda = \phi$ or $X_3^\lambda = \phi$, which shows that λ must have the form

$$\left\{\begin{array}{l} \lambda = (4f+1, 4(f-1)+1, \ldots, 5, 1) \\ \text{or} \\ \lambda = (4f+3, 4(f-1)+3, \ldots, 7, 3). \end{array}\right. \quad \text{for some } f \geqq 0 \tag{7}$$

(Note that also $P^*(X_1^\lambda) = P^*(X_3^\lambda) = 0$.) The partitions in (7) are partitions of triangular numbers. Using (6) we conclude:

Proposition (4.9). *If $d(B) = 0$, that is $B = \{[k, k-1, \ldots, 1]\}$ for some k, then the spin character in the block $\hat{B}$ is $\langle 2k-1, 2k-5, \ldots\rangle$.*

Suppose next that $t_2^*(\lambda) = 1$. By (4.5) we get that either

(i) $\lambda^1 = (1)$, $t_2^*(\lambda^0) = 0$, $\lambda^i = 0$ for $i \geqq 2$ or

(ii) $t_2^*(\lambda^0) = 1$, $\lambda^i = 0$ for $i \geqq 1$.

In case (i), we get from the analysis leading to (7) that λ is obtained by adding a part 2 to a partition in (7), i.e.

$$\left\{\begin{array}{l} \lambda = (4f+1, 4(f-1)+1, \ldots, 5, 2, 1) \\ \text{or} \\ \lambda = (4f+3, 4(f-1)+3, \ldots, 7, 3, 2). \end{array}\right. \quad f \geqq 0 \tag{8}$$

In case (ii), $|\mu| = |\mu(\lambda^0)| = 1$ so that the partition sequence of μ is $\underline{0}\ 1\ 0\ \underline{1}$. Looking at the cuts in this sequence, we get the following possibilities only, which are obtained by removing a single nonmaximal part from the partitions in (7)

$$\left\{\begin{array}{l} \lambda = (4f+1, \ldots, 4(e+1)+1, 4(e-1)+1, \ldots, 5, 1) \\ \text{or} \\ \lambda = (4f+3, \ldots, 4(e+1)+3, 4(e-1)+3, \ldots, 7, 3). \end{array}\right. \quad 0 \leqq e \leqq f-1 \tag{9}$$

Proposition (4.10). *If $d(B) = 1$ (that is $d(\hat{B}) = 2$), then*

$$B = \{[k+2, k-1, k-2, \ldots, 2, 1], [k, k-1, k-2, \ldots, 2, 1^3]\}$$

for some $k \geqq 0$. Then the spin characters in $\hat{B}$ are the 2 characters indexed by $(2k-1, 2k-5, \ldots, 2)$, (which is one of the partitions occurring in (8)).

PROOF. By (6) only spin characters indexed by the partitions in (8) or (9) can occur in $\hat{B}$. There are 2 characters indexed by each partition in (8) and one character indexed by each partition in (9) (see [16]). So if (4.10) is false, the spin characters in $\hat{B}$ must be indexed by one partition of each of the types in (9) (for reasons of cardinality). Thus the spin characters in $\hat{B}$ are $\langle 4f+1, \ldots, 1\rangle$ and $\langle 4g+3, \ldots, 3\rangle$ for suitable $f, g \geqq 0$ (where in each partition a part $4e+1$ or $4e'+3$ is missing). If $4f+1 > 4g+3$, the first character has a nonzero value on a $(4f+1)$-cycle and the other has the value 0 on the same cycle (by the Murnaghan-Nakayama formula for spin characters (see [10], [11])). If $4g+3 > 4f+1$, we have the converse situation. In any case this is impossible: The characters in $\hat{B}$ must have the same value on 2-regular elements, since they are in a block having only one modular character.

Let us consider briefly the other extreme of the principal 2-block. Since a basic spin character $\langle n\rangle$ may be seen as a "spin version" of the trivial character $[n]$, one may ask whether $\langle n\rangle$ is in the principal block of $\hat{S}_n$. Computing the value of the central character of $\langle n\rangle$ on the class $(3, 1^{n-3})$, we get that if $n = 4k+3$, then $\langle n\rangle$ is *not* in the principal block. A general result of Benson [1] shows that these are the only exceptions to the above question.

As mentioned, the spin characters in (4.9) and (4.10) remain irreducible modulo 2, and it is easy to see which is the natural index of the 2-modular they restrict to. But the general question of which spin characters are irreducible modulo 2 and which modular character they then correspond to seems very difficult to answer. It is not even clear what a reasonable conjecture would be. However, there is a conjecture due to R. Knörr and the author, how the spin characters should distribute into 2-blocks, which of course fits the facts mentioned above.

The 2-blocks of S_n (and $\hat{S}_n$) are indexed canonically by the 2-cores of partitions of n. So there is a 2-block of $\hat{S}_n$ for each $k > 0$ such that $n \equiv k(k-1)/2 \bmod 2)$ and $k(k-1)/2 \leqq n$.

CONJECTURE. If $\lambda = (a_1, \ldots, a_m) \succ n$, let $\hat{\lambda} \vdash n$ be defined as

$$\hat{\lambda} = \left(\left[\frac{a_1+1}{2}\right], \left[\frac{a_1}{2}\right], \left[\frac{a_2+1}{2}\right], \left[\frac{a_2}{2}\right], \ldots, \left[\frac{a_m}{2}\right]\right).$$

Then $\langle\lambda\rangle$ is in the 2-block indexed by the 2-core of $\hat{\lambda}$.

EXAMPLE. $\lambda = (7, 2)$, $\hat{\lambda} = (4, 3, 1, 1)$, 2-core of $\hat{\lambda}$: $(2, 1)$.

There is an easy way of computing the 2-core of $\hat{\lambda}$ directly from λ, which may be illuminating for the conjecture: Given λ, disregard all the even parts and consider λ^0. Suppose that f parts of λ^0 are $\equiv 1$ modulo 4 and that g parts of λ^0 are $\equiv 3$ modulo 4. If $a = f - g > 0$, the 2-core of $\hat{\lambda}$ is $(2a-1, 2a-2, \ldots, 1)$. If $b = g - f \geqq 0$, the 2-core of $\hat{\lambda}$ is $(2b, 2b-1, \ldots, 1)$. (The above example corresponds to $g = 1$, $f = 0$, $b = 1$.)

REFERENCES

1. D. Benson, *Some remarks on the decomposition numbers for the symmetric groups*, Proc. Sympos. Pure Math. 47 (1987), 381–394.
2. S. Comét, *Über die Anwendung von Binärmodellen in der Theorie der Charaktere der symmetrischen Gruppen*, Numer. Math. 1 (1959), 90–109.
3. W. Feit, *The Representation Theory of Finite Groups* (North-Holland Math. Library 25), North-Holland Publ. Co., Amsterdam, New York, Oxford, 1982.
4. F. G. Frobenius, *Über die Charactere der symmetrischen Gruppe*, Sitzungsber. Königlich Preussischen Akad. Wiss. (1900), 516–534. (Gesammelte Abhandlungen 3, pp. 148–166. Springer-Verlag, Berlin - Heidelberg - New York, 1968.)
5. J. F. Humphreys, *Blocks of projective representations of the symmetric group*, J. London Math. Soc. 33 (1986), 441–452.
6. G. James, *Some combinatorial results involving Young diagrams*, Math. Proc. Cambridge Philos. Soc. 83 (1978), 1–10.
7. G. James and A. Kerber, *The representation theory of the symmetric group* (Encyclopedia Math. Appl. 16), Addison-Wesley Publ. Co., London, Amsterdam, Sydney, Tokyo, 1981.
8. I. G. Macdonald, On the degree of the irreducible representations of symmetric groups, Bull. London Math. Soc. 3 (1971), 189–192.
9. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.
10. A. O. Morris, *The spin representation of the symmetric group*, Canad. J. Math. 17 (1965), 543–549.
11. A. O. Morris and J. B. Olsson, *On p-quotients for spin characters*, to appear in J. Algebra.
12. A. O. Morris and A. K. Yaseen, *Some combinatorial results involving shifted Young diagrams*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 23–31.
13. J. B. Olsson, *Mckay numbers and heights of characters*, Math. Scand. 38 (1976), 25–42.
14. J. B. Olsson, *On the p-blocks of symmetric and alternating groups and their covering groups*, to appear.
15. G. de B. Robinson, *Representation Theory of the Symmetric group* (Math. Expositions 12), University of Toronto Press, Toronto, 1961.
16. I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. 139 (1911), 155–250. (Gesammelte Abhandlungen 1, pp. 346–441. Springer-Verlag, Berlin - Heidelberg - New York, 1973.)

MATEMATISK INSTITUT
KØBENHAVNS UNIVERSITET
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø
DENMARK