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R-TORSION ASSOCIATED TO INFINITE DIMENSIONAL UNITARY REPRESENTATIONS

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Abstract.

Reidemeister torison of $\mathsf{R}[\Gamma]$-complexes associated to certain pairs of ∞-dimensional unitary representations of Γ is dicussed. A qusi-analytical formula for this torison is determined and the resulting extension of the Ray-Singer-Cheeger-Müller Theorem is conjectured.

0. Introduction.

In this note we want to discuss the classical Reidemeister torsion (R-torsion) (see [6], [8]) and an extension related to infinite dimensions. Our hope is to thereby gain information about $\mathsf{R}[\Gamma]$-complexes when Γ is an infinite group, that is not included in the classical theory. For the moment though we are only able to show that the extension we have in mind makes sense and that it is computable in a quasi-analytical way (see [8]). We will begin in section 1 by recalling the classical R-torsion associated to a finite dimensional orthogonal representation of the group Γ. We will also discuss an analytical-combinatorial formulation of this invariant. It was this formula that provided the initial evidence of a link between the combinatorial R-torsion and the analytic torison for smooth compact manifolds [8]. We will discuss in section 3 an extension of the Ray-Singer conjecture (now a theorem (see [1], [7])) to the infinite dimensional setting we have in mind.

In section 2 we will introduce the notion of R-torsion associated to certain pairs, t-pairs, of infinite dimensional unitary representations. We will show that for a stably free, s-based $\mathsf{R}[\Gamma]$-complex one can define an R-torsion associated to a t-pair of representation of Γ. Following this we will, in section 3, verify that this R-torsion is recoverable from the associated laplacians and so obtain an analytical-combinatorial formula for the R-torsion. Finally, as was mentioned above, we will outline how we think the generalization of the Ray-Singer conjecture would be formulated within this

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setting. The proof of this extended conjecture, whether following [1] or [7], would appear to us to involve some very interesting analysis. In particular it seems that perturbation theory, especially scattering theory (see [3], [4]), would necessarily play a distinctive role.

1. Classical R-torsion.

We will follow the treatments and notation of [6], [8]. Let Γ be a finitely group and let

$$(1) \qquad \mathscr{C} = \{C_*, \partial_* : \boldsymbol{c}_*, \boldsymbol{h}_*\} = \{C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 : \boldsymbol{c}_*, \boldsymbol{h}_*\}$$

denote a complex of $\mathsf{R}[\Gamma]$-modules (or more generally modules over any ring). For the moment we will assume that $\mathscr{C}$ is free and provided with preferred bases for the C_* and for its homology modules

$$H_*(\mathscr{C}) = \ker \partial_* / \operatorname{Im} \partial_{*+1}$$

denoted by $\boldsymbol{c}_*$ and $\boldsymbol{h}_*$ respectively. We will also assume that the boundary modules

$$B_* = \operatorname{Im}(\partial_{*+1}) \subset C_*$$

are free. As in [6] these assumptions are not all necessary. After making the appropriate contructions here and in section 2 we will then observe that everything works if only we require each C_* to be stably free and have preferred s-bases. But given $\mathscr{C}$ as above we may more conveniently define the torsion of $\mathscr{C}$, $\tau(\mathscr{C})$, as an element of the reduced Whitehead group of $\mathsf{R}[\Gamma]$, $\tilde{K}_1(\mathsf{R}[\Gamma])$ as follows. Choose anybasis $\boldsymbol{b}_*$ for $B_* \subset C_*$ and select sets of elements $\bar{\boldsymbol{b}}_* \subset C_{*+1}$ such that

$$(2) \qquad \partial_{*+1}(\bar{\boldsymbol{b}}_*) = \boldsymbol{b}_*.$$

Then as was shown in [6] the sets $\boldsymbol{b}_*, \boldsymbol{h}_*, \bar{\boldsymbol{b}}_*$ determine (sufficiently) a basis $(\boldsymbol{b}_*\boldsymbol{h}_*\bar{\boldsymbol{b}}_*)$ of C_* so that the following expression

$$(3) \qquad \tau(\mathscr{C}) = \sum_{j=0}^{n} (-1)^j \cdot [(\boldsymbol{b}_j\boldsymbol{h}_j\bar{\boldsymbol{b}}_j) : \boldsymbol{c}_j]$$

determines a well-defined element of $\tilde{K}_1(\mathsf{R}[\Gamma])$. Recall from [6] that

$$\tilde{K}_1(\mathsf{R}[\Gamma]) = K_1(\mathsf{R}[\Gamma]) / \{0, [(-1)]\}$$

where

$$K_1(\mathsf{R}[\Gamma]) = \mathrm{Gl}(\mathsf{R}[\Gamma])/[\mathrm{Gl}(\mathsf{R}[\Gamma]), \mathrm{Gl}(\mathsf{R}[\Gamma])]$$

$\mathrm{Gl}(\mathsf{R}[\Gamma]) = \lim_n \mathrm{Gl}_n(\mathsf{R}[\Gamma])$, and that for two bases $\boldsymbol{e} = \{e_1, \ldots, e_n\}$ and $\boldsymbol{f} = \{f_1, \ldots, f_n\}$ of a free $\mathsf{R}[\Gamma]$-module,

$$[\boldsymbol{e} : \boldsymbol{f}] = \{\text{change of basis matrix from } \boldsymbol{e} \text{ to } \boldsymbol{f}\} = (a_{ij}),$$

that is

$$e_i = \sum_j a_{ij} f_j, \quad a_{ij} \in \mathsf{R}[\Gamma]. \tag{4}$$

To define R-torsion we need to be in addition provided with an orthogonal representation of Γ

$$\varrho : \Gamma \to \mathrm{O}(m). \tag{5}$$

Extending ϱ linearly to $\mathsf{R}[\Gamma]$ we obtain a ring homomorphism

$$\varrho : \mathsf{R}[\Gamma] \to M_m(\mathsf{R}) = \{m \times m \text{ real matrices}\} = \mathrm{End}(\mathsf{R}^m). \tag{6}$$

So we may also view, via ϱ, R^m as an $\mathsf{R}[\Gamma]$-module and thus consider the tensor product of R^m with the complex $\mathscr{C}$

$$\mathsf{R}^m \otimes_{\mathsf{R}[\Gamma]} \mathscr{C} = \{0 \to \mathsf{R}^m \otimes_{\mathsf{R}[\Gamma]} C_n \xrightarrow{1 \otimes \partial_n} \mathsf{R}^m \otimes_{\mathsf{R}[\Gamma]} C_{n-1} \to \cdots \to 0\} \tag{7}$$

which we will denote simply as $\mathsf{R}^m \otimes_\varrho \mathscr{C}$. *Assume* that $\mathsf{R}^m \otimes_\varrho \mathscr{C}$ is acyclic. Fixing the standard basis $\boldsymbol{e}$ of R^m, the preferred bases of $\mathscr{C}$ determine preferred bases for the vector spaces of $\mathsf{R}^m \otimes_\varrho \mathscr{C}$, $\boldsymbol{e} \otimes \boldsymbol{c}_*$. That is $\mathsf{R}^m \otimes_\varrho \mathscr{C}$ is an acyclic complex of R-vector spaces with preferred bases (the homology being zero also has a preferred basis). So as above we can define

$$\tau(\mathsf{R}^m \otimes_\varrho \mathscr{C}) \in \tilde{K}_1(\mathsf{R}) \underset{\simeq}{\overset{D}{\to}} \mathsf{R}^+ \quad \text{(the positive reals)}$$

where $\tilde{K}_1(\mathsf{R}) \overset{D}{\to} \mathsf{R}^+$ is the homomorphism given by

$$D([a]) = |\det(a)| \quad \text{for } a \in \mathrm{Gl}_n(\mathsf{R}). \tag{8}$$

We define

$$\tau_\varrho(\mathscr{C}) = D(\tau(\mathsf{R}^m \otimes_\varrho \mathscr{C})). \tag{9}$$

$\tau_\varrho(\mathscr{C})$ is then an "alternating product" of determinants of change of basis operators determined by the homomorphisms of $\mathscr{C}$, the preferred bases of the C_*, and the representation ϱ. That is

$$\tau_\varrho(\mathscr{C}) = \prod_{j=0}^{n} |\det[(\boldsymbol{b}_j\bar{\boldsymbol{b}}_j):\boldsymbol{c}'_j]|^{(-1)^j}, \tag{10}$$

where the $\boldsymbol{b}_j, \bar{\boldsymbol{b}}_j$ are chosen as above with respect to the complex $\mathrm{R}^m \otimes_\varrho \mathscr{C}$ and $\boldsymbol{c}'_j$ denotes the preferred basis of $\mathrm{R}^m \otimes_\varrho C_j$. As in [6] we now observe that if we only require that the $\mathrm{R}[\Gamma]$-modules be stably free and each have preferred s-basis then the natural extension of the above construction yields a well-defined torsion $\tau(\mathscr{C}) \in \tilde{K}_1(\mathrm{R}[\Gamma])$ and R-torsion $\tau_\varrho(\mathscr{C}) \in \mathrm{R}^+$. In section 2 we will discuss infinite dimensional representations of Γ which allow an expression similar to (10) to be defined. We hope (knowing of no counter-example) to eventually show that this R-torsion does for infinite groups what the following result of Bass, mentioned in [6], does for finite groups.

THEOREM. *If Γ is a finite group, then the order of $\tau(\mathscr{C}) \in \tilde{K}_1(\mathrm{R}[\Gamma])$ is finite iff $\tau_\varrho(\mathscr{C}) = 1$ for all irreducible orthogonal representations of Γ.*

That is the τ_ϱ determine τ up to torsion. It seems that the point is, for finite Γ, orthogonal representation sufficiently represent Γ's algebraic structure. For instance, the regular representation is orthogonal. If Γ is infinite, this of course can not be so. In fact there are infinite groups which possess no non-trivial orthogonal representations. Thus for infinite Γ one can not expect that the τ_ϱ detect very much of τ.

Now given $\mathscr{C}$ and $\varrho:\Gamma \to \mathrm{O}(m)$ as above, consider $\mathrm{R}^m \otimes_\varrho \mathscr{C}$. The preferred bases of $\mathscr{C}$ determine preferred bases of $\mathrm{R}^m \otimes_\varrho \mathscr{C}$ and so inner products on the vector spaces $\mathrm{R}^m \otimes_\varrho C_j = V_j$ for which these preferred bases are orthonormal. Let $A_j = 1 \otimes \partial_j$ and let A_j^* denote its adjoint with respect to these inner products. Let

$$\Delta_j = A_j^* A_j + A_{j+1} A_{j+1}^* \tag{11}$$

denote the "combinatorial" laplacian of $\mathscr{C}$ at C_j associated to ϱ. Δ_j is self-adjoint and positive. Define

$$\zeta_\varrho^j(s) = \mathrm{Trace}((\Delta_j)^{-s}) \tag{12}$$

$$= \sum_{i=1}^{n_j} \frac{1}{\lambda_i^s}$$

where $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{n_j}$, $n_j = \dim V_j$, are the eigenvalues of Δ_j on V_j. In [8] it is shown that

$$(13) \qquad \log \tau_\varrho(\mathscr{C}) = \frac{1}{2} \sum_{j=0}^{n} (-1)^j \cdot j \cdot \zeta_\varrho^{j\prime}(0).$$

An analogous formula will hold in the infinite dimensional setting and will be discussed in section 3.

2. R-torsion associated to a t-pair.

Let H denote a complex separable Hilbert space. Consider the following:

DEFINITION 1. A pair of unitary representations $\varrho_1, \varrho_2 : \Gamma \to U(H)$ is called a *t-pair* if for all $g \in \Gamma$

$$(14) \qquad \varrho_1(g) - \varrho_2(g) \in \boldsymbol{B}_1(H), \text{ the trace class (see [3], [2]).}$$

We will denote by $\mathscr{L}(H), \mathscr{L}^{-1}(H)$, and $\boldsymbol{B}_1(H)$ the spaces of bounded operators, bounded operators with bounded inverses, and trace class operators on H respectively. Recall (see [3], [2]) that if $T \in \boldsymbol{B}_1(H)$, then for any orthonormal basis $\boldsymbol{e} = \{e_\nu\}$ of H the following sum converges absolutely

$$(15) \qquad \sum_\nu \langle Te_\nu, e_\nu \rangle.$$

It is called the trace of T and denoted $\mathrm{tr}(T)$. Notice that if $\{\varrho_1, \varrho_2\}$ is a t-pair of representations of Γ, then for all $g \in \Gamma$ we have for instance

$$(16) \qquad \varrho_1(g) \cdot \varrho_2(g)^* = \varrho_1(g)\varrho_2(g)^{-1} = \varrho_1(g)\varrho_2(g^{-1}) \in I_H + \boldsymbol{B}_1(H),$$

where I_H = identity operator on H. Notice also that if $\dim H < \infty$ then all operators are of trace class.

We now wish to show, given a complex $\mathscr{C}$ as above and a t-pair $\{\varrho_1, \varrho_2\}$ there is a well-defined torison

$$\tau_{\{\varrho_1, \varrho_2\}}(\mathscr{C}) \in \mathbf{R}^+$$

having the property that if $\dim H < \infty$, then

$$(17) \qquad \tau_{\{\varrho_1, \varrho_2\}}(\mathscr{C}) = \tau_{\varrho_1}(\mathscr{C})/\tau_{\varrho_2}(\mathscr{C}).$$

The problem we have is making sense of expressions like (10) where one would need to take the determinant of operators on Hilbert space. This of course is not always possible. But for t-pairs of representations the needed determinants exist. To begin, let $\mathscr{C} = \{C_*, \partial_* : \boldsymbol{c}_*, \boldsymbol{h}_*\}$ be a free bases $\mathsf{R}[\Gamma]$-complex as with (1). Consider the diagram:

(18)
$$\begin{array}{ccccc}
\cdots \to & H \otimes_{\varrho_1} C_j & \xrightarrow{1 \otimes_{\varrho_1} \partial_j} & H \otimes_{\varrho_1} C_{j-1} & \to \cdots \\
& \phi_1^j \downarrow \simeq & & \simeq \downarrow \phi_1^{j-1} & \\
\cdots \to & H^{n_j} & \underset{A_2^j}{\xrightarrow{A_1^j}} & H^{n_{j-1}} & \to \cdots \\
& \phi_2^j \uparrow \simeq & & \simeq \uparrow \phi_2^{j-1} & \\
\cdots \to & H \otimes_{\varrho_2} C_j & \xrightarrow[1 \otimes_{\varrho_2} \partial_j]{} & H \otimes_{\varrho_2} C_{j-1} & \to \cdots
\end{array}$$

where

$$\phi_\varepsilon^j : H \otimes_{\varrho_\varepsilon} C_j \to H^{n_j} = \underbrace{H \oplus \cdots \oplus H}_{n_j = \dim_{\mathsf{R}[\Gamma]} C_j}, \quad \varepsilon = 1, 2$$

are isomorphisms defined by sending the basis $\boldsymbol{e} \otimes_{\varrho_\varepsilon} \boldsymbol{c}_j$ of $H \otimes_{\varrho_\varepsilon} C_j$ to the basis $\boldsymbol{e} \otimes \boldsymbol{c}_j$ of H^{n_j} (the set $\boldsymbol{c}_j$ is then just a set of n_j symbols when viewed as part of $\boldsymbol{e} \otimes \boldsymbol{c}_j$), and the A_ε^j make their respective squares commute, that is

(19)
$$A_\varepsilon^j(\chi) = \phi_\varepsilon^{j-1} \circ (1 \otimes_{\varrho_\varepsilon} \partial_j) \circ (\phi_\varepsilon^j)^{-1}(\chi).$$

In particular if $h \in H$, $\boldsymbol{c}_j = \{c_{ji}\}_{i=1,\ldots,n_j}$, then

(20)
$$A^j(h \otimes c_{ji}) = \phi_\varepsilon^{j-1}(h \otimes_{\varrho_\varepsilon} \partial_j c_{ji}).$$

So that if $\partial_j c_{ij} = \sum_k a_{ik}^{(j)} c_{j-1,k}$, then

(21)
$$\begin{aligned}
A_\varepsilon^j(h \otimes c_{ij}) &= \phi_\varepsilon^{j-1}\left(h \otimes_{\varrho_\varepsilon} \sum_k a_{ik}^{(j)} c_{j-1,k}\right) \\
&= \phi_\varepsilon^{j-1}\left(\sum_k (a_{ik}^{(j)})_{\varrho_\varepsilon}(h) \otimes_{\varrho_\varepsilon} c_{j-1,k}\right) \\
&= \sum_k (a_{ik}^{(j)})_{\varrho_\varepsilon} h \otimes c_{j-1,k},
\end{aligned}$$

where, for $a = \sum r_g \cdot g \in \mathsf{R}[\Gamma]$,

$$a_{\varrho_\varepsilon} = \sum r_g \cdot \varrho_\varepsilon(g) \in \mathscr{L}(H).$$

Thus if ∂_j is represented by the $n_j \times n_{j-1}$ matrix $(a_{ik}^{(j)})$ with entries in $\mathsf{R}[\Gamma]$, then A_ε^j is represented by the $n_j \times n_{j-1}$ matrix

$$((a_{ik}^{(j)})_{\varrho_\varepsilon}) : H^{n_j} \to H^{n_{j-1}}$$

with entries in $\mathscr{L}(H)$. Notice that if $A = (a_{ij}) \in \mathrm{Gl}_n(\mathsf{R}[\Gamma])$, then for any unitary representation $\varrho : \Gamma \to U(H)$,

$$A_\varrho = ((a_{ij})_\varrho) : H^n \to H^n \in \mathscr{L}^{-1}(H^n)$$

and

$$(A_\varrho)^{-1} = (A^{-1})_\varrho. \tag{22}$$

Definition 2. A pair of operators $\{A_1, A_2\} \subset \mathscr{L}(H)$ is called a *t-pair* of operators if $A_1 - A_2 \in \boldsymbol{B}_1(H)$.

Lemma 1. *Let $A : F \to F$ be an $\mathsf{R}[\Gamma]$-endomorphism of a free based $\mathsf{R}[\Gamma]$-module. If $\{\varrho_1, \varrho_2\}$ is a t-pair of representations then $\{A_{\varrho_1}, A_{\varrho_2}\}$ is a t-pair of operators on H^n, $n = \dim_{\mathsf{R}[\Gamma]} F$.*

Lemma 2. *Let $\boldsymbol{c}_2$ and $\boldsymbol{c}'$ denote two $\mathsf{R}[\Gamma]$-bases of the free $\mathsf{R}[\Gamma]$-module $\mathsf{R}[\Gamma]^n = F$. Let $\{\varrho_1, \varrho_2\}$ be a t-pair of representations of Γ, and together with $\boldsymbol{c}$ let them determine isomorphisms*

$$\phi_\varepsilon : H \otimes_{\varrho_\varepsilon} F \to H^n, \quad \varepsilon = 1, 2$$

as in (18). *Then the change of basis operators*

$$\Theta_\varepsilon = [\phi_\varepsilon(\boldsymbol{e} \otimes_{\varrho_\varepsilon} \boldsymbol{c}') : \boldsymbol{e} \otimes \boldsymbol{c}]$$

form a t-pair of operators on H^n.

Proof of Lemma 1. The isomorphisms ϕ_ε are given by

$$\begin{array}{ccccc} H \otimes_{\varrho_1} F & \xrightarrow{\phi_1} & H^n & \xleftarrow{\phi_2} & H \otimes_{\varrho_2} F \\ h \otimes_{\varrho_1} c_i & \longrightarrow & h \otimes c_i & \longleftarrow & h \otimes_{\varrho_2} c_i \end{array}$$

and $A : F \to F$ induces $A_{\varrho_\varepsilon} = A_\varepsilon : H^n \to H^n$ by

$$A_{\varrho_\varepsilon}(h \otimes c) = \phi_\varepsilon(h \otimes_{\varrho_\varepsilon} Ac).$$

If $A = (a_{ij})$ with respect to $\boldsymbol{c}$, then $A_\varepsilon = ((a_{ij})_{\varrho_\varepsilon}) : H^n \to H^n$ and

$$A_1 - A_2 = ((a_{ij})_{\varrho_1} - (a_{ij})_{\varrho_2}) : H^n \to H^n. \tag{23}$$

But for any $a = \sum r_g \cdot g \in \mathsf{R}[\Gamma]$

$$a_{\varrho_1} - a_{\varrho_2} = \sum r_g \cdot (\varrho_1(g) - \varrho_2(g)) \tag{24}$$

is a finite sum of trace class operators and hence trace class. Therefore

$$A_1 - A_2 \in \boldsymbol{B}_1(H^n), \quad \text{i.e., } \{A_1, A_2\} \text{ is a } t\text{-pair.}$$

Before we prove Lemma 2, notice that if $\{\varrho_1, \varrho_2\}$ is a t-pair and $A \in \mathrm{Gl}_n(\mathsf{R}[\Gamma])$, then

$$A_1 \circ A_2^{-1} \in I_{H^n} + \boldsymbol{B}_1(H^n), \tag{25}$$

and so by [3], [2], $\det(A_1 \circ A_2^{-1})$ is defined.

Proof of Lemma 2. Let $A = (a_{ij}) \in \mathrm{GL}_n(\mathsf{R}[\Gamma])$ denote the change of basis matrix $[\boldsymbol{c}' : \boldsymbol{c}]$, i.e.,

$$c_i' = \sum_j a_{ij} c_j, \quad \text{or briefly } \boldsymbol{c}' = A\boldsymbol{c}. \tag{26}$$

The new bases of H^n, $\phi_\varepsilon(\boldsymbol{e} \otimes_{\varrho_\varepsilon} \boldsymbol{c}')$ can then be written in terms of the first basis $\boldsymbol{e} \otimes \boldsymbol{c}$ by

$$\phi_\varepsilon(\boldsymbol{e} \otimes_{\varrho_\varepsilon} \boldsymbol{c}') = \phi_\varepsilon(\boldsymbol{e} \otimes_{\varrho_\varepsilon} A\boldsymbol{c}) = A_{\varrho_\varepsilon}(\boldsymbol{e} \otimes \boldsymbol{c}).$$

That is

$$[\phi_\varepsilon(\boldsymbol{e} \otimes_{\varrho_\varepsilon} \boldsymbol{c}') : \boldsymbol{e} \otimes \boldsymbol{c}] = A_\varepsilon. \tag{27}$$

The result then follows from Lemma 1.

We may now define an R-torsion of free, based $\mathsf{R}[\Gamma]$ complexes associated to t-pairs. Let $\mathscr{C} = \{C_*, \partial_* : \boldsymbol{c}_*, \boldsymbol{h}_*\}$ and consider the diagram (18). Let $\boldsymbol{b}_*$ be a basis of B_* (assuming for the moment that $B_* \subset C_*$ is free) and let $\bar{\boldsymbol{b}}_*$ be chosen as in (2). Denote the resulting basis $(\boldsymbol{b}_* \boldsymbol{h}_* \bar{\boldsymbol{b}}_*)$ of C_* by $\boldsymbol{c}'_*$. Consider the operators

$$\theta_\varepsilon^j = [\phi_\varepsilon^j(\boldsymbol{e} \otimes_{\varrho_\varepsilon} \boldsymbol{c}_j') : \boldsymbol{e} \otimes \boldsymbol{c}_j], \quad \varepsilon = 1,2, \ j = 0, \ldots, n. \tag{28}$$

As proven in Lemma 2, $\{\theta_1^j, \theta_2^j\}$ is a t-pair of operators on H^{n_j}. We may then define, using the observation (25) and (see [3], [2]),

$$\tau_{\{\varrho_1, \varrho_2\}}(\mathscr{C}) = \prod_{j=0}^{n} |\det(\theta_1^j \cdot (\theta_2^j)^{-1})|^{(-1)^j}. \tag{29}$$

If $\dim H < \infty$, then $\det(\theta_1^j(\theta_2^j)^{-1}) = (\det \theta_1^j)/(\det \theta_2^j)$ and it is easy to see that

$$\tau_{\{\varrho_1, \varrho_2\}}(\mathscr{C}) = \tau_{\varrho_1}(\mathscr{C})/\tau_{\varrho_2}(\mathscr{C}). \tag{30}$$

Now as in [6] we may replace everywhere above the requirements that the R[Γ]-modules be free and based by the conditions that they be stably free and s-based. It is clear that none of the above constructions are affected by such a generalization. In fact all we have done is construct a homomorphism

$$D_{\{\varrho_1, \varrho_2\}} : \tilde{K}_1(\mathrm{R}[\Gamma]) \to \mathrm{R}^+ \qquad [a] \mapsto |\det(a_{\varrho_1} \cdot a_{\varrho_2}^{-1})| \tag{31}$$

given a t-pair of representations $\{\varrho_1, \varrho_2\}$. So we may view our R-torsion as follows:

$$\begin{array}{ccc} \{\text{stably free, } s\text{-based } \mathrm{R}[\Gamma]\text{-complexes}\} & & \\ \downarrow \tau & \searrow^{\tau_{\{\varrho_1, \varrho_2\}}} & \\ \tilde{K}_1(\mathrm{R}[\Gamma]) & \xrightarrow[D\{\varrho_1, \varrho_2\}]{} & \mathrm{R}^+. \end{array} \tag{32}$$

3. An analytical-combinatoial formula for $\tau_{\{\varrho_1, \varrho_2\}}$.

Our aim is to verify Proposition 1 below. It represents the obvious extension of the formula (13) found in [8] to our situation. Let $\mathscr{C} = \{C_*, \partial_* : c_*, h_*\}$ be as above and recall diagram (18) and the operators

$$A_\varepsilon^j = \phi_\varepsilon^{j-1} \cdot (1 \otimes_{\varrho_\varepsilon} \partial_j) \cdot (\phi_\varepsilon^j)^{-1}.$$

The preferred basis c_j of C_j determines the inner products on $H \otimes_{\varrho_\varepsilon} C_j$ and H^{n_j} by making the bases $e \otimes_{\varrho_\varepsilon} c_j$ and $e \otimes c_j$ orthonormal. Let B_ε^j denote the adjoint of A_ε^j, i.e.,

$$H^{n_j} \underset{B_\varepsilon^j}{\overset{A_\varepsilon^j}{\rightleftarrows}} H^{n_{j-1}} \quad \text{with } \langle A_\varepsilon^j(x), y\rangle = \langle x, B_\varepsilon^j(y)\rangle \tag{33}$$

for $x \in H^{n_j}$, $y \in H^{n_{j-1}}$.

We will denote the associated laplacians by

$$\Delta_\varepsilon^j = B_\varepsilon^j \circ A_\varepsilon^j + A_\varepsilon^{j+1} \circ B_\varepsilon^{j+1} \in \mathscr{L}(H^{n_j}). \tag{34}$$

Since we assume always that $H \otimes_{\varrho_\varepsilon} \mathscr{C}$ is acyclic, it then follows that

$$(35) \qquad \Delta_\varepsilon^j \in \mathscr{L}^{-1}(H^{n_j}).$$

LEMMA 3. *If* $\{\varrho_1, \varrho_2\}$ *is a t-pair, then for each* j, $\{\Delta_1^j, \Delta_2^j\}$ *is a t-pair of invertible operators on* H^{n_j}.

Given this for the moment we proceed with the construction of our analytical formula. Let

$$(36) \qquad \zeta^j_{\{\varrho_1,\varrho_2\}}(s) = \operatorname{tr}\{\Delta_1^j)^{-s} - (\Delta_s^j)^{-s}\}$$

which, since $\Delta_\varepsilon^j \in \mathscr{L}^{-1}(H^{n_j})$, represents an entire function of s.

PROPOSITION 1.

$$\log \tau_{\{\varrho_1,\varrho_2\}}(\mathscr{C}) = \frac{1}{2}\sum_j (-1)^j \cdot j \cdot \zeta^{j'}_{\{\varrho_1,\varrho_2\}}(0).$$

As in [8] the proof relates the derivative of $\zeta^*_{\{\varrho_1,\varrho_2\}}$ at $s = 0$ with the determinants of $\Delta_1^*(\Delta_2^*)^{-1}$ and these in turn with the determinants of $\theta_1^*(\theta_2^*)^{-1}$. So that we do not belabor the technicalities we will only prove Proposition 1 in the case that $\mathscr{C}$ is an acylcic complex with $n = 1$. We first prove Lemma 3.

PROOF OF LEMMA 3. We begin by determining how the operators B_ε^j depend on the homomorphisms ∂_j. For simplicity, let $\partial : C \to D$ be an $\mathsf{R}[\Gamma]$-homomorphism between free $\mathsf{R}[\Gamma]$-modules with bases $\boldsymbol{c} = \{c_1, \ldots, c_n\}$ and $\boldsymbol{d} = \{d_1, \ldots, d_m\}$ respectively. Then with respect to $\boldsymbol{c}$ and $\boldsymbol{d}$ let $\partial = (a_{ij})$ and

$$A_\varepsilon = ((a_{ij})_{\varrho_\varepsilon}) : H^n \to H^m \quad \text{as above.}$$

It is then an easy exercise to check that

$$(37) \qquad B_\varepsilon = \text{adjoint of } A_\varepsilon = ((\bar{a}_{ji})_{\varrho_\varepsilon}) : H^m \to H^n$$

where, for $a = \sum r_g \cdot g \in \mathsf{R}[\Gamma]$, $\bar{a} = \sum r_g \cdot g^{-1}$. That is the adjoint of the "operatorization" of an $\mathsf{R}[\Gamma]$-homomorphism is the operatorization of the conjugate transpose of the algebraic map. Because of (37) and (34) we then have

$$(38) \qquad \{B_1^j, B_2^j\} \quad \text{and} \quad \{\Delta_1^j, \Delta_2^j\} \quad \text{are } t\text{-pairs.}$$

In fact, if we let

(39) $$(\mathrm{alg}\Delta^j) = \bar{\partial}_j^t \circ \partial_j + \partial_{j+1} \circ \bar{\partial}_{j+1}^t,$$

where $\bar{\partial}_j^t$ denotes the conjugate transpose of ∂_j, denote the "algebraic laplacian", then

(40) $$\Delta_\varepsilon^j = (\mathrm{alg}\Delta^j)_{\varrho_r}.$$

PROOF OF PROPOSITION 1. From [3],

$$(\Delta_\varepsilon^j)^{-s} = \frac{1}{2\pi i} \oint_\gamma \xi^{-s} \cdot (\xi - \Delta_\varepsilon^j)^{-1} d\xi$$

where γ is a simple closed curve surrounding the (bounded) spectrum of both Δ_1^j and Δ_2^j and avoiding zero (see figure)

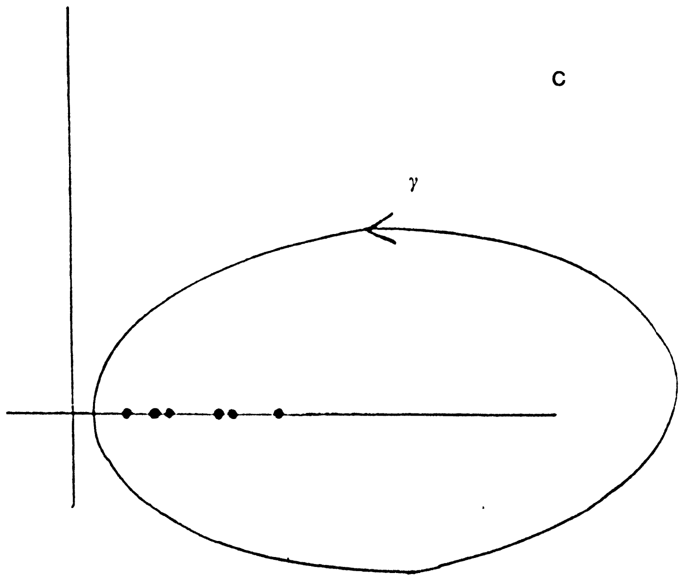

Figure.

But $(\xi - \Delta_1^j)^{-1} - (\xi - \Delta_2^j)^{-1} \in \boldsymbol{B}_1(H^{n_j})$ and so

(41) $$\zeta_{\{\varrho_1, \varrho_2\}}^j(s) = \mathrm{tr}\{(\Delta_1^j)^{-s} - (\Delta_2^j)^{-s}\}$$

$$= \mathrm{tr}\, \frac{1}{2\pi i} \oint_\gamma \xi^{-s}\{(\xi - \Delta_1^j)^{-1} - (\xi - \Delta_2^j)^{-1}\} d\xi$$

$$= \frac{1}{2\pi i} \oint_\gamma \xi^{-s} \mathrm{tr}\{(\xi - \Delta_1^j)^{-1} - (\xi - \Delta_2^j)^{-1}\} d\xi$$

(42)
$$\zeta^{j'}_{\{\varrho_1, \varrho_2\}}(0) = \mathrm{tr}\, \frac{1}{2\pi i} \oint_\gamma -\log \xi \cdot \{(\xi - \Delta_1^j)^{-1} - (\xi - \Delta_2^j)^{-1}\} d\xi$$

$$= -\mathrm{tr} \log(\Delta_1^j (\Delta_2^j)^{-1})$$

$$= -\log \det(\Delta_1^j (\Delta_2^j)^{-1}).$$

For simplicity we now assume that $\mathscr{C}$ is acyclic and $n = 1$. Thus we are considering just an isomorphism between free $\mathsf{R}[\Gamma]$-modules

$$\partial : C_1 \to C_0$$

with preferred bases $\boldsymbol{c}_1, \boldsymbol{c}_0$. Let $\partial = (a_{ij})$ with respect to $c_1, \boldsymbol{c}_0$. Then we can calculate $\tau(\mathscr{C})$ as follows. Since $B_1 = \{0\}$ and $B_0 = C_0$ we may take

(43)
$$\boldsymbol{b}_0 = \boldsymbol{c}_0 \quad \text{and} \quad \bar{\boldsymbol{b}}_1 = \partial^{-1}(\boldsymbol{c}_0).$$

Thus

(44)
$$\tau(\mathscr{C}) = [\boldsymbol{b}_0 : \boldsymbol{c}_0] - [\bar{\boldsymbol{b}}_1 : \boldsymbol{c}_1]$$

$$= [\boldsymbol{c}_0 : \boldsymbol{c}_0] - [\partial^{-1}\boldsymbol{c}_0 : \boldsymbol{c}_1]$$

$$= 1 - [(a_{ij})^{-1}] = 1 + [(a_{ij})] \quad \text{(see [6])}.$$

So if we let $A = (a_{ij})$, then

(45)
$$\tau_{\{\varrho_1, \varrho_2\}}(\mathscr{C}) = |\det(A_{\varrho_1} A_{\varrho_2}^{-1})|.$$

Then to prove Proposition 1 in this case we need to show

$$\log|\det(A_{\varrho_1} A_{\varrho_2}^{-1})| = -\tfrac{1}{2}\zeta^{1'}_{\{\varrho_1, \varrho_2\}}(0)$$

$$= \tfrac{1}{2} \log \det(\Delta_1^1 (\Delta_2^1)^{-1}) \quad \text{by (42)}.$$

Notice though that in this case

(46)
$$\Delta_\varepsilon^1 = A_{\varrho_\varepsilon} \cdot A_{\varrho_\varepsilon}^* \quad (A_{\varrho_\varepsilon}^* = \text{adjoint of } A_{\varrho_\varepsilon}).$$

Hence

$$\begin{aligned}\tfrac{1}{2}\log\det(\Delta_1^1(\Delta_2^1)^{-1}) &= \tfrac{1}{2}\log\det(A_{\varrho_1}A_{\varrho_1}^*(A_{\varrho_2}A_{\varrho_2}^*)^{-1}) \\ &= \tfrac{1}{2}\log\det(A_{\varrho_1}A_{\varrho_2}^{-1}(A_{\varrho_1}A_{\varrho_2}^{-1})^*) \\ &= \log|\det(A_{\varrho_1}A_{\varrho_2}^{-1})|.\end{aligned}$$

Given Proposition 1 we are led to suspect that for the category of compact Riemannian manifolds there lurks an analytic torsion associated to a t-pair of unitary representations of π_1. Being equal to the above R-torsion would by (30) represent a generalized Ray-Singer conjecture (shown to be true by Cheeger [1] and Müller [7]). More precisely, given a smooth compact Riemannian manifold W, possibly with boundary M, and given a t-pair $\{\varrho_1, \varrho_2\}$ of representations of $\pi_1(W)$ we may consider the de Rham complexes of (W, M) with values in the flat Hilbert bundles $\mathscr{H}_\varepsilon$ induced by the representations ϱ_ε, $\varepsilon = 1,2$, $\Lambda^*(W; \mathscr{H}_\varepsilon)$. As all Hilbert bundles over compact spaces are trivial (see [5]) (though not as flat bundles) we may identify

$$\Lambda^*(W; \mathscr{H}_\varepsilon) \underset{\psi_\varepsilon^*}{\simeq} \Lambda^*(W; W \times H).$$

We may also consider the laplacians associated to $\Lambda^*(W; \mathscr{H}_\varepsilon)$ transferred to $\Lambda^*(W; W \times H)$ and denoted there by Δ_ε^*. We would then be led to *conjecture* that

1. $\{\varrho_1, \varrho_2\}$ being a t-pair implies there exist isomorphisms ψ_ε^* so that with respect to the appropriate boundary value problem $\Delta_1^* - \Delta_2^*$ is trace class,

2. $\zeta_{\{\varrho_1, \varrho_2\}}^*(s) = \mathrm{tr}\{(\Delta_1^*)^{-s} - (\Delta_2^*)^{-s}\}$ is defined and analytic for $\mathrm{Re}(s) \gg 0$ and possesses a meromorphic extension to all of $\mathscr{C}$ analytic at $s = 0$ (see [9]), and finally,

3. $\log \tau_{\{\varrho_1, \varrho_2\}}(C^*(\tilde{W}, \tilde{M}; \mathsf{R})) = \dfrac{1}{2} \sum_{j=0}^{\dim W} (-1)^j \cdot j \cdot \zeta_{\{\varrho_1, \varrho_2\}}^{j\prime}(0).$

$\tilde{W}, \tilde{M}$ denotes the universal cover of $\tilde{W}$ and a choice of lift of $\tilde{M}$ (see [8]). $C^*(\tilde{W}, \tilde{M}; \mathsf{R})$ is then an $\mathsf{R}[\Gamma]$-complex with naturally chosen preferred bases (for the cohomology basis one looks to the harmonic forms [8]) to which the above methods may be applied. In contrast to Ray and Singer's work we have as yet no hard evidence that such a conjecture should be true (beyond the Ray-Singer conjecture itself). Our search for examples and/or counter-examples though is leading us down a very interesting and to our knowledge essentially unexplored path. That a proof might, if it follows the spirit of [1] or [7], require the study of a deformation theory or approximation theory for scattering theory (see [3], [4]), seems to us very exciting.

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