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ON NONSEPARABLE SIMPLEX SPACES

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Abstract.

We construct, for any given cardinality $\kappa \geqq \aleph_0$, two simplices S_1, S_2 satisfying the following:

(i) S_1 and S_2 have dense extreme point sets,

(ii) $A(S_1)$ and $A(S_2)$ are nonisomorphic,

(iii) the density characters of S_1 and S_2 are equal to κ,

(iv) the density characters of $A(S_1)$ and $A(S_2)$ coincide.

1.

It is well-known that there is, up to affine homeomorphisms, one and only one metrizable compact infinite dimensional Choquet simplex S whose extreme point set $\operatorname{ex} S$ is dense in S. This simplex is of some interest e.g. in statistical mechanics. Moreover it has the property that all metrizable compact simplices are affinely homeomorphic to faces of S. The purpose of this note is to show that the condition $\overline{\operatorname{ex} S} = S$ is not a characterization of a unique simplex for any density character except in the metrizable case. We construct, for each density character, even for $\aleph_0$, two examples of simplices S with dense extreme point sets whose corresponding simplex spaces

$$A(S) = \{f : S \to \mathbf{R} \mid f \text{ affine continuous}\}$$

(endowed with the sup-norm) are nonisomorphic. A similar result is proven for unit balls $B(X^*)$ of dual L_1-spaces X^* with the w^*-topology. This is done by extending the results of [7]. We also want to point out that such non-metrizable simplices can be constructed without using deep set theoretical machinery. In fact, the following proofs use only simple topological and geometrical tools.

Recall that the density character $\alpha(X)$ of a topological space X is the infimum of the cardinalities of the dense subsets of X. This definition makes sense since the cardinal numbers are well ordered (see [1]). Theorems 1.1 and 1.4 below are extensions of Theorems 7 and 1 of [7]. We use "simplex" in the sense of compact Choquet simplex.

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1.1. THEOREM. *Let F be a simplex. Then there is a simplex $S \supseteqq F$ satisfying the following*

(1) $\overline{\operatorname{ex} S} = S$,

(2) *F is a face of S,*

(3) *there exists an isometry $T: A(F) \to A(S)$ with $T1_F = 1_S$ and $Tf|_F = f$ for all $f \in A(F)$,*

(4) *$\alpha(S) = \alpha(F)$ and $\alpha(A(F)) = \alpha(A(S))$,*

(5) *S is sequentially compact if F is sequentially compact.*

We postpone the main proofs to the following sections.

1.2. COROLLARY. *If F is metrizable, then S is metrizable.*

PROOF. If F is metrizable, then $A(F)$ is separable. Hence $\alpha(A(F)) = \aleph_0 = \alpha(A(S))$ which means, S is metrizable.

1.3. COROLLARY. *For any infinite cardinal number κ, there are two simplices S_1, S_2 with dense extreme point sets such that $\alpha(S_1) = \alpha(S_2) = \kappa$ and $\alpha(A(S_1)) = \alpha(A(S_2))$, but such that $A(S_1)$ and $A(S_2)$ are not isomorphic. In particular, S_1 and S_2 are not affinely homeomorphic.*

Moreover, if $\kappa \geqq$ the cardinality of the continuum, then there are S_1, S_2 such that in addition $\alpha(A(S_1)) = \alpha(A(S_2)) = \kappa$.

We observe that, for any simplex S, $\alpha(S) \leqq \alpha(A(S)) \leqq 2^{\alpha(S)}$ (see Lemma 3.1.). It seems remarkable that, even if $\kappa = \aleph_0$, simplices S_1, S_2 with the preceding properties exist. In this case we have $\alpha(A(S_1)) = \alpha(A(S_2)) \succ \aleph_0$ because of the fact that metrizable simplices S with dense extreme point sets (e.g. where $\alpha(A(S)) = \aleph_0$) are unique (up to affine homeomorphisms). The analogous results are true for L_1-predual spaces G, where the extreme point sets of the dual unit balls, $\operatorname{ex} B(G^*)$, are w^*-dense in $B(G^*)$. Recall, a biface H of $B(G^*)$ is a subset of the form $H = \operatorname{conv}(F \cup -F)$, where F is a (not necessarily w^*-close) face of $B(G^*)$.

1.4. THEOREM. *Let X be an L_1-predual. Then there is an L_1-predual $G \supset X$ satisfying the following*

(6) $\overline{\operatorname{ex} B(G^*)}^{w^*} = B(G^*)$,

(7) $\alpha(G) = \alpha(X)$ *and* $\alpha(B(G^*)) = \alpha(B(X^*))$,

(8) *there exists a contractive projection $P: G \to X$ such that $P^*B(X^*)$ is a biface of $B(G^*)$. (Here P^* is the adjoint of P.)*

(9) *If $B(X^*)$ is w^*-sequentially compact, so is $B(G^*)$.*

(8) implies that $P^*B(X^*)$ and $B(X^*)$ are w^*-affinely homeomorphic.

1.5. COROLLARY. *For any cardinal number $\kappa \geqq$ the cardinality of the continuum, there are two L_1-predual spaces G_1, G_2 such that $\overline{\operatorname{ex} B(G_i^*)}^{w^*} = B(G_i^*)$, $i = 1,2$, and $\kappa = \alpha(G_1) = \alpha(G_2)$, but G_1 and G_2 are not isomorphic to each other.*

If $\kappa = \aleph_0$, the situation is different. According to the introductory remarks, in this case all such spaces G are isometrically isomorphic to each other. Nevertheless, it is possible to construct two nonisomorphic spaces G_1, G_2 such that $\alpha(G_1) = \alpha(G_2)$ and $\alpha(B(G_1^*)) = \alpha(B(G_2)^*) = \aleph_0$ (see section 4).

2.

Section 2 is devoted to the proof of Theorem 1.1. Let X be a Banach space. We need a well-known result concerning density characters. To make the paper self-contained, we include a proof.

2.1. LEMMA. *If $V \subset X^*$ is a bounded subset endowed with the w^*-topology, then*

$$\alpha(V) \leqq \alpha(X).$$

PROOF. Let Γ be a dense subset of X of cardinality $\alpha(X)$. Let Δ be the set of all pairs $(\{x_1,\ldots,x_n\}, r)$, where $x_1,\ldots,x_n \in \Gamma$, $n = 1,2,\ldots$, and r is a positive rational number. Hence the cardinality of Δ is equal to $\alpha(X)$. If

$$\delta = (\{x_1,\ldots,x_n\}, r) \in \Delta,$$

put $E_\delta = \operatorname{span}\{x_1,\ldots,x_n\}$. Fix a finite subset $\Lambda_\delta \subset V$ such that for each $x^* \in V$, there is $y^* \in \Lambda_\delta$ with $\|(x^* - y^*)_{|E_\delta}\| \leqq r$. Then $\bigcup_{\delta\in\Delta}\Lambda_\delta$ is a w^*-dense subset of V whose cardinality is $\leqq \alpha(X)$.

For any Γ let $l_\infty(\Gamma)$ be the Banach space of all bounded real valued functions and $c_0(\Gamma)$ the subspace of all functions vanishing at infinity. Clearly, the unit ball $B(c_0(\Gamma)^*)$ of $c_0(\Gamma)^*$ is w^*-sequentially compact. This follows from the fact that each $\mu \in c_0(\Gamma)^*$ is a measure on Γ which is supported by only a countable number of elements in Γ. (We identify in the following a regular bounded Borel measure μ on a locally compact Hausdorff space K with the corresponding functional on

$$C_0(K) = \{f: K \to \mathsf{R} \mid f \text{ continuous, } f \text{ vanishes at } \infty\}.)$$

Now let $X = A(F)$ be a simplex space. Denote by e the function of F which is one everywhere. Let $S(X)$ be the state space of X with respect to e, that is

$$S(X) = \{x^* \in X^* \mid \|x^*\| = 1 = x^*(e)\}.$$

Then $F \cong S(X)$ (affinely homeomorphic). Here $S(X)$ is endowed with the w^*-topology.

Let $\Gamma \subset S(X)$ be w^*-dense, such that the cardinality of Γ is $\alpha(S(X))$. Consider $(X \oplus l_\infty(\Gamma))_{(\infty)}$, where

$$\|(x,f)\| = \max(\|x\|, \|f\|), \quad x \in X, f \in l_\infty(\Gamma).$$

We identify X with the subspace of $(X \oplus l_\infty(\Gamma))_{(\infty)}$ consisting of all $(x, \hat{x})$, $x \in X$. Here $\hat{x} \in l_\infty(\Gamma)$ is defined by $\hat{x}(\mu) = \mu(x)$, $\mu \in \Gamma$. This identification is clearly an isometry. Put

$$Y = Y(X,\Gamma) = \operatorname{span}(\{(x,\hat{x}) \mid x \in X\} \cup \{(0,f) \mid f \in c_0(\Gamma)\}). \tag{10}$$

Y is a subspace of $(X \oplus l_\infty(\Gamma))_{(\infty)}$. Let $S(Y)$ be the state space of $Y(X,\Gamma)$ with respect to $(e, \hat{e})$.

2.2. Lemma. *$Y(X,\Gamma)$ is a simplex space. The underlying simplex is affinely homeomorphic to $S(Y)$.*

Proof. By assumption on e, we have $\hat{e} \equiv 1$. $(e, \hat{e})$ is an extreme point of the unit ball of $Y(X,\Gamma)$). Hence it suffices to show that $Y(X,\Gamma)^*$ is an L_1-space (see [3]). To this end, let B_i, $i = 1,\dots,4$, be balls in $Y(X,\Gamma)$ with centers $(x_i, \hat{x}_0 + f_i)$ and radii r_i such that $B_i \cap B_j \neq \emptyset$ for all i and j. We claim that $\bigcap_{i=1}^4 B_i \neq \emptyset$ which proves that $Y(X,\Gamma)^* \cong L_1$ (see [4]). Fix $\varepsilon > 0$. We obtain by our assumptions on B_i and the definition of the norm in $Y(X,\Gamma)$ that

$$\|x_i - x_j\| \leqq r_i + r_j \quad \text{and} \quad \|\hat{x}_i + f_i - (\hat{x}_j + f_j)\| \leqq r_i + r_j$$

for all i and j. Since X is an L_1-predual, there is $x \in X$ with $\|x - x_i\| \leqq r_i$, $i = 1,\dots,4$, (see [4]). Since $f_i \in c_0(\Gamma)$, there is a finite subset $\Omega \subset \Gamma$ such that $|f_i(\mu)| \leqq \varepsilon$ for $\mu \in \Gamma \setminus \Omega$, $i = 1,\dots,4$. We have

$$\|\hat{x}_i + f_i - \hat{x} - (\hat{x}_j + f_j - \hat{x})\| \leqq r_i + r_j$$

for all i and j. Since $l_\infty(\Omega)$ is a (finite dimensional) L_1-predual, there is $f \in c_0(\Gamma)$ supported by Ω, such that

$$|(\hat{x}_i + f_i - \hat{x} - f)(\omega)| \leqq r_i$$

for all i and $\omega \in \Omega$. By assumption on Ω we obtain

$$\|\hat{x}_i + f_i - \hat{x} - f\| \leqq r_i + \varepsilon$$

for all i. This means $\bigcap_{i=1}^4 \tilde{B}_i \neq \emptyset$, if $\tilde{B}_i$ are the balls with centers

$(x_i, \hat{x}_i + f_i)$ and radii $r_i + \varepsilon$. Since $\varepsilon > 0$ was arbitrary this implies $\bigcap_{i=1}^4 B_i \neq \emptyset$ (see [4]).

Let $P_X: Y(X, \Gamma) \to X$ be defined by

$$(11) \qquad P_X(x, f) = x, \quad (x, f) \in Y(X, \Gamma).$$

Clearly, P_X is a contractive projection onto X. (Recall, we identified x with $(x, \hat{x})$, $x \in X$.)

Let $\hat{S}(X) = \{(\mu, 0) \in Y(X, \Gamma)^* \mid \mu \in S(X)\}$ and

$$(12) \qquad M(\Gamma) = \{(0, \mu) \in Y(X, \Gamma)^* \mid \mu \ \text{ a probability measure on } \Gamma\}.$$

Note that any probability measure μ on Γ is of the form $\mu = \sum_{i=1}^{\infty} \lambda_i \hat{\mu}_i$, where $0 \leqq \lambda_i$, $\sum_{i=1}^{\infty} \lambda_i = 1$, $\mu_i \in \Gamma$ and $\hat{\mu}_i(f) = f(\mu_i)$, $f \in l_\infty(\Gamma)$.

If

$$S(X \oplus l_\infty(\Gamma)) = \{(\mu, \nu) \in (X \oplus l_\infty(\Gamma))^* \mid 0 \leqq \mu(e), \nu(\hat{e}) \leqq 1, \mu(e) + \nu(\hat{e}) = 1, \|\mu\| + \|\nu\| = 1\},$$

then by Hahn-Banach $S(X \oplus l_\infty(\Gamma))_{|Y} = S(Y)$. Hence in view of (10) every element in $S(Y)$ is of the form (μ, ν), where $\mu \in [0, 1] \cdot S(X)$, $\nu \in (\hat{X} + c_0(\Gamma))^*$, $0 \leqq \nu(\hat{e}) \leqq 1$, $\|\mu\| + \|\nu\| = 1$. ($\hat{X} = \{\hat{x} \in l_\infty(\Gamma) \mid x \in X\}$.)

2.3. Lemma.

(a) $S(Y) = w^*\,\text{clos.conv}(\hat{S}(X) \cup M(\Gamma))$.

(b) $P_X^*(S(X)) = \hat{S}(X)$ *is a face of* $S(Y)$.

Proof. $\hat{S}(X) \cup M(\Gamma)$ is a subset of $S(Y)$ and separates the points in $Y(X, \Gamma)$. So (a) follows from the Hahn-Banach separation theorem. If $(\mu_i, \nu_i) \in S(Y)$, $i = 1,2$, and $0 < \lambda < 1$ such that

$$\lambda(\mu_1, \nu_1) + (1 - \lambda)(\mu_2, \nu_2) \in P_X^*(S(X)),$$

then there is $\mu \in S(X)$ with $(\mu, 0) = (\lambda\mu_1 + (1-\lambda)\mu_2, \lambda\nu_1 + (1-\lambda)\nu_2)$. Hence $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ which implies $\mu_1, \mu_2 \in S(X)$. Since $\|\mu_i\| + \|\nu_i\| = 1$, $i = 1,2$, we obtain $\nu_1 = \nu_2 = 0$. Thus

$$(\mu_i, \nu_i) = (\mu_i, 0) \in \hat{S}(X) = P_X^*(S(X)).$$

2.4. Corollary. $\alpha(Y(X, \Gamma)) = \alpha(X)$ *and* $\alpha(S(Y)) = \alpha(S(X))$. *If* $S(X)$ *is* w^**-sequentially compact, then* $S(Y)$ *is* w^**-sequentially compact. Moreover*

$$(13) \qquad \text{ex}\, S(Y)_{|X} \text{ is } w^*\text{-dense in } S(X).$$

PROOF. At first we observe that the elements $(0, \mu)$, where

$$\mu = \sum_{i=1}^{\infty} \lambda_i \hat{\mu}_i, \quad 0 \leqq \lambda_i,$$

λ_i rational numbers, $\Sigma\lambda_i = 1$, $\mu_i \in \Gamma$, are w^*-dense in $M(\Gamma)$. Hence

$$\alpha(M(\Gamma)) \prec \aleph_0 \cdot \aleph_0 \cdot (\text{cardinality of } \Gamma) = \alpha(S(X)).$$

In view of Lemma 2.3 (a), this implies $\alpha(S(Y)) \lessapprox \alpha(S(X))$. On the other hand, the restriction map $S(Y) \to S(X)$ is surjective and continuous which yields $\alpha(S(X)) \lessapprox \alpha(S(Y))$. We conclude $\alpha(S(X)) = \alpha(S(Y))$. By (10) and (11) we have $P_X Y = X$, $(\mathrm{id} - P_X)Y = c_0(\Gamma)$. Moreover,

$$\alpha(c_0(\Gamma)) = \text{cardinality of } \Gamma = \alpha(S(X)) \lessapprox \alpha(X)$$

(in view of Lemma 2.1). It follows that $\alpha(Y) = \alpha(X)$. Furthermore,

$$S(Y) \subset P_X^* S(X) + (\mathrm{id} - P_X)^* B(c_0(\Gamma)^*).$$

Since $B(c_0(\Gamma)^*)$ is w^*-sequentially compact, $S(Y)$ is w^*-sequentially compact provided $S(X)$ is w^*-sequentially compact. Finally, (13) is a consequence of the fact that $(0, \hat{\mu})$ is an extreme point of $M(\Gamma)$, if $\mu \in \Gamma$ and $\hat{\mu}(\hat{x}) = \mu(x)$, $x \in X$.

2.5. PROOF OF THEOREM 1.1. We use induction to define

$$X \subset Y(X, \Gamma_1) \subset Y(Y(X, \Gamma_1), \Gamma_2) \subset Y(Y(Y(X, \Gamma_1), \Gamma_2), \Gamma_3) \subset \dots$$

as follows: Put $Y_0 = X$. If we have Y_{n-1} already, let $\Gamma_n \subset S(Y_{n-1})$ be w^*-dense such that cardinality of $\Gamma_n = \alpha(S(Y_{n-1}))$. Put $Y_n = Y(Y_{n-1}, \Gamma_n)$. Put

$$P_1 = P_X \colon Y(X, \Gamma_1) = Y_1 \to X,$$

$$P_n = P_{n-1} \circ P_{Y_{n-1}} \colon Y_n \to X, \quad n = 1, 2, \dots .$$

Here $P_{Y_{n-1}}$ is the corresponding projection from $Y(Y_{n-1}, \Gamma_n) = Y_n$ onto U_{n-1} ((11) with Y_{n-1} instead of X).

Put $Z = \overline{\cup Y_n}$ (completion). Note that by our construction the function e corresponding to one in the simplex space X is the one function in $Y(X, \Gamma_1)$ once we have identified X with a subspace of $Y(X, \Gamma_1)$. The same applies to all other simplex spaces Y_n involved. So, $B(Z)$ has an extreme point (namely the one function e in $X \subset Z$). On the other hand, Z is an L_1-predual space since it is the union of an increasing chain of L_1-preduals (see [4]). Hence Z is a simplex space. Therefore the state space $S(Z)$ of Z with respect to e is a simplex and satisfies $S(Z)_{|Y_n} = S(Y_n)$, $n = 0, 1, 2, \dots$. (13), applied to Y_n instead of X, implies that $\operatorname{ex} S(Z)$ is w^*-dense in $S(Z)$. The P_n define a global contractive projection $P: Z \to X$ since $P_{n|Y_{n-1}} = P_{n-1}$ for all n. Lemma 2.3. (b) applied to all Y_n instead of X yields that $P_n^* S(X)$ is a face of $S(Y_n)$ for all n, hence $P^*S(X)$ is a face of $S(Z)$. Corollary 2.4 yields that $\alpha(Y_n) = \alpha(Y_{n-1})$, $\alpha(S(Y_n)) = \alpha(S(Y_{n-1}))$, $n = 1, 2, \ldots$. Hence $\alpha(Z) = \alpha(X)$, $\alpha(S(Z)) = \alpha(S(X))$. Furthermore, if $S(X)$ is w^*-sequentially compact, then by Corollary 2.4, all $S(Y_n)$ are w^*-sequentially compact. Then a routine diagonalization argument yields that $S(Z)$ is w^*-sequentially compact. This concludes the proof of Theorem 1.1.

3.

Here we prove Corollary 1.3. To round out the discussion we include the following:

3.1. Lemma. *If S is an infinite dimensional simplex then $\alpha(S) \leqq \alpha(A(S)) \leqq 2^{\alpha(S)}$.*

Proof. $\alpha(S) \leqq \alpha(A(S))$ follows from Lemma 2.1. On the other hand, if Γ is a dense subset of S with cardinality $\alpha(S)$, then $A(S)$ is isometrically isomorphic to a subspace of $l_\infty(\Gamma)$ (restriction to Γ). We have $\alpha(l_\infty(\Gamma)) = 2^{\alpha(S)}$ (consider the linear span of the functions in $l_\infty(\Gamma)$ with modulus one which is a dense set with minimal cardinality). Since $l_\infty(\Gamma)$ is a metric space we obtain $\alpha(A(S)) \leqq \alpha(l_\infty(\Gamma))$.

3.2. Lemma. *There is a separable simplex E which is sequentially compact such that $\alpha(A(E)) = \alpha(l_\infty)$.*

Proof. This follows from a well-known example in general topology: There is a separable compact Hausdorff space E which is first countable, hence sequentially compact, but which is nonmetrizable (see [2, p. 164 "Helly space"]). It turns out that E is a simplex:

Let $E = \{f : [0,1] \to [0,1] \mid f \text{ nondecreasing}, f(1) = 1\}$. E as a subset of $[0,1]^{[0,1]}$ is compact (topology of pointwise convergence). With the usual pointwise addition of functions, E is convex; we have

$$\operatorname{ex} E = \{f \in E \mid f([0,1]) \subset \{0,1\}\}.$$

For $x \in [0,1]$, let δ_x be the function on E with $\delta_x(f) = f(x)$. δ_x is continuous and affine. Moreover, $\delta_1 \equiv 1$. The δ_x separate the points of E, hence the closed linear span of the δ_x is equal to the space of all continuous affine functions, $A(E)$, on E. To see that $A(E)$ is a simplex space it suffices to prove that for each $0 \leqq x_1 < x_2 < \ldots < x_n \leqq 1$, the δ_{x_i} span a subspace isometrically isomorphic to l_∞^n (see [4]). The latter fact follows from

$$\left\|\sum_{i=1}^{n} \lambda_i \delta_{x_i}\right\| = \sup_{f \in E}\left|\sum_{i=1}^{n} \lambda_i f(x_i)\right|$$
$$= \sup_{f \in \operatorname{ex} E}\left|\sum_{i=1}^{n} \lambda_i f(x_i)\right| = \sup_{k \leqq n}\left|\sum_{i=k}^{n} \lambda_i\right|$$

for all scalars λ_i. Hence $\delta_{x_1}, \ldots, \delta_{x_n}$ is equal to the summing basis of l^n_∞. We obtain, $A(E)$ is an L_1-predual. Since $1 \in A(E)$, $A(E)$ is a simplex space and E is a simplex.

E is separable, since every $f \in E$ is the uniform limit of a sequence of step functions. This also shows that f has only a countable number of points of discontinuity in $[0, 1]$. The latter fact implies that E is first countable. On the other hand $\|\delta_x - \delta_y\| = 1$ whenever $x \neq y$. Hence

$$\alpha(A(E)) \geqq \text{cardinality of } \{\delta_x \mid x \in [0, 1]\} = 2^{\aleph_0}.$$

Since E is separable, $A(E)$ is a subspace of l_∞. Hence

$$2^{\aleph_0} \leqq \alpha(A(E)) \leqq \alpha(l_\infty) = 2^{\aleph_0}.$$

3.3. Lemma. *Let κ be any cardinal number $\geqq \aleph_0$. Then there are simplices F_1, F_2 such that $\alpha(F_1) = \alpha(F_2) = \kappa$, $\alpha(A(F_1)) = \alpha(A(F_2))$, F_1 is sequentially compact, F_2 is not sequentially compact. Moreover, if $\kappa \geqq 2^{\aleph_0}$ then F_1, F_2 can be arranged such that in addition $\alpha(A(F_1)) = \alpha(A(F_2)) = \kappa$.*

Proof. Let E_1 be the simplex of Lemma 3.2 and let $E_2 = \operatorname{Prob}(\beta\mathrm{N})$ (regarded as subset of l^*_∞ with the w^*-topology). Then $\alpha(E_1) = \alpha(E_2) = \aleph_0$, E_1 is sequentially compact, E_2 is not sequentially compact. Moreover

$$\alpha(A(E_1)) = \alpha(A(E_2)) = \alpha(l_\infty) = 2^{\aleph_0}.$$

Let Γ be a set of cardinality κ and let $c(\Gamma)$ be the subspace of $l_\infty(\Gamma)$ spanned by $c_0(\Gamma)$ and 1_Γ. Clearly, $\alpha(c(\Gamma)) = \kappa$ and $c(\Gamma)$ is a simplex space whose underlying simplex is $\operatorname{Prob}(\Gamma \cup \{\infty\})$. Here $\Gamma \cup \{\infty\}$ is the Alexandrov compactification of Γ with the discrete topology. Put

$$F_1 = \operatorname{conv}(E_1 \times \{0\} \cup \{0\} \times \operatorname{Prob}(\Gamma \cup \{\infty\}))$$

and

$$F_2 = \operatorname{conv}(E_2 \times \{0\} \cup \{0\} \times \operatorname{Prob}(\Gamma \cup \{\infty\})).$$

Hence $\alpha(F_1) = \alpha(F_2) = \kappa$, $\alpha(A(F_1)) = \alpha(A(F_2))$, since $A(F_i) = A(E_i) \oplus c(\Gamma)$, $i = 1, 2$. F_1 is sequentially compact, F_2 is not sequentially compact. Finally, if $\kappa \geqq 2^{\aleph_0}$, then $\alpha(A(F_i)) = 2^{\aleph_0} \cdot \kappa = \kappa$.

3.2. Proof of Corollary 1.3. Take the simplex spaces $A(F_1)$, $A(F_2)$ of

Lemma 3.3 and apply Theorem 1.1. We obtain simplices S_1, S_2 with dense extreme point sets, such that

$$\kappa = \alpha(S_1) = \alpha(S_2) = \alpha(F_1) = \alpha(F_2),\ \alpha(A(S_1)) = \alpha(A(S_2)) = \alpha(A(F_1)) = \alpha(A(F_2))$$

(by (1), (4)). If $\kappa \geqq 2^{\aleph_0}$, then $\kappa = A(F_i) = A(S_i)$, $i = .1, 2$. S_1 is sequentially compact (by (5)), but S_2 is not sequentially compact since $F_2 \subset S_2$ and F_2 is not sequentially compact. If $T: A(S_1) \to A(S_2)$ were a surjective isomorphism, then $T^*S(A(S_2))$ (state space of $A(S_2)$) would be a subset of $\|T^*\|B(A(S_1)^*)$. Since

$$B(A(S_1)^*) = \text{conv}(S(A(S_1)) \cup -S(A(S_1)),$$

the set $\|T^*\|B(A(S_2))$ and therefore S_2 would be sequentially compact, a contradiction. This means, $A(S_1)$ and $A(S_2)$ are not isomorphic, in particular, S_1 and S_2 are not affinely homeomorphic.

4.

Here we prove Theorem 1.4.

Let X be an L_1-predual space. We proceed in complete analogy to section 2. Take a w^*-dense subset $\Gamma \subset B(X^*)$ whose cardinality is $\alpha(B(X^*))$. Define $Y = Y(X, \Gamma)$ as in (10), $P_X: Y \to X$ as in (11), and $M(\Gamma)$ as in (12).

4.1. Lemma. *Y is an L_1-predual space.*

$$B(Y^*) = w^*\text{clos.conv}(\{(X^*, 0) | x^* \in B(X^*)\} \cup M(\Gamma)).$$

$$\alpha(B(Y^*)) = \alpha(B(X^*)), \quad \alpha(Y) = \alpha(X).$$

$\text{ex}\, B(Y^*)_{|X}$ is w^*-dense in $B(X^*)$. $B(Y^*)$ is w^*-sequentially compact if $B(X^*)$ is w^*-sequentially compact. $P_X^* B(X^*)$ is a biface of $B(Y^*)$.

Proof. The proofs of the first assertions are identical with the proofs of 2.2, 2.3, 2.4. Only the last assertion of Lemma 4.1 is somewhat different: Since $X^* \cong L_1$, there is a (not necessarily w^*-closed) face F of $B(X^*)$ with $B(X^*) = \text{conv}(F \cup -F)$. Now exactly as in the proof of 2.3 (b), one sees that $P_X^*(F)$ is a face of $B(Y^*)$. Hence

$$P_X^* B(X^*) = \text{conv}(P_X^*(F) \cup -P_X^*(F))$$

is a biface of $B(Y^*)$.

4.2. Proof of Theorem 1.4. Repeat the argument of 2.5 with Lemma 4.1 instead of 2.2, 2.3, 2.4.

4.3. Proof of Corollary 1.5. Use Lemma 3.3 to find L_1-predual spaces

X_1, X_2 with the same density character, so that $B(X_1^*)$ is w^*-sequentially compact, $B(X_2^*)$ is not w^*-sequentially compact. (Take $X_i = A(F_i)$, F_i as in Lemma 3.3, $i = 1, 2$, and use $B(X_i^*) = \operatorname{conv}(\hat{F}_i \cup -\hat{F}_i)$, where $\hat{F}_i$ is the state space of F_i.) Then apply Theorem 1.4 to find L_1-predual spaces $G_i \supset X_i$ such that $\alpha(G_i) = \alpha(X_i)$, $\alpha(B(G_i^*)) = \alpha(B(X_i^*))$,

$$\overline{\operatorname{ex} B(G_i^*)}^{w^*} = B(G_i^*), \quad i = 1, 2,$$

$B(G_i^*)$ is w^*-sequentially compact, $B(G_2^*)$ is not w^*-sequentially compact. Then G_1 and G_2 cannot be isomorphic. If we start with $X_1 = A(E)$, E as in Lemma 3.2 and $X_2 = l_\infty$, we obtain spaces G_1, G_2 such that in addition $\alpha(B(G_1^*)) = \alpha(B(G_2^*)) = \aleph_0$.

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