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A SPHERICAL FABRICIUS-BJERRE FORMULA WITH APPLICATIONS TO CLOSED SPACE CURVES

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Let $\gamma: C \to S$ be a $\mathscr{C}^3$ immersion of the circle, C, into the 2-sphere, S, of unit radius. We call γ a *closed spherical curve*. Fabricius-Bjerre [1] discovered a formula for a "generic" closed plane curve, c, which involves the number of double points of c, the number of inflection points of c, and the number of double tangents of c. An analogous formula will be obtained for "generic" closed spherical curves which involves all of the above but, moreover, involves the number of pairs of points of $\gamma(C)$ which are antipodal to one another. We will adapt the proof given by Fabricius-Bjerre so that it works for spherical curves. Benjamin Halpern [4] gives an alternate approach to the proof of the formula of Fabricius-Bjerre; presumably this approach could be adapted as well to give a proof of our formula for closed spherical curves.

We will also give some applications of our formula to closed curves in Euclidean 3-space. The results for space curves are obtained by viewing γ as the tangent indicatrix of the given space curve. Particularly noteworthy is Theorem 3 which states that any "generic" non-degenerate closed space curve possesses a pair of parallel tangents or a pair of parallel osculating planes.

1. The formula.

We will first concern ourselves with some definitions. Some restrictions will be imposed on γ in the course of doing this. Let γ' denote the field of positive unit tangent vectors to γ, i.e. those unit tangent vectors pointing in the direction of traverse of γ.

A point $P \in S$ is a *double point of* γ if $\gamma^{-1}(P)$ contains more than one point of C. We will assume that each double point of γ has precisely two preimages in C. Moreover, if $\{x, y\} = \gamma^{-1}(P)$ we require that $\gamma'(x) \neq \pm\gamma'(y)$. For any $P \in S$, let $\bar{P}$ denote its antipode. If $P \in S$, then $\{P, \bar{P}\}$ is called an *antipodal pair of points of* γ if there exists points $x, y \in C$ such that $\gamma(x) = P$ and

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$\gamma(y) = \bar{P}$. We assume that each point of $\{P, \bar{P}\}$ is not a double point. In addition, if $\gamma(x) = P$ and $\gamma(y) = \bar{P}$, we insist that $\gamma'(x) \neq \pm\gamma'(y)$. If $\bar{\gamma}: C \to S$ is defined by $\bar{\gamma}(x) = \overline{\gamma(x)}$, for each $x \in C$, then each point of an antipodal pair of points of γ is a crossing point of γ with $\bar{\gamma}$.

We suppose the reader is familiar with the concept of geodesic curvature of a curve in S; the geodesic curvature of γ will be denoted by k. An *inflection point of* γ is a point at which $k = 0$. We suppose that no inflection point is a double point or a point of an antipodal pair. Also, we insist that at each inflection point k', the derivative of k with respect to arc length, is non-zero.

A *double tangent of* γ is a geodesic, i.e., great circle, l, that is tangent to $\gamma(C)$ at precisely two distinct points. We assume that each point of tangency is not a double point, either point of an antipodal pair of γ, or an inflection point of γ. A double tangent, l, is called an *exterior* double tangent if the curve $\gamma(C)$ lies on the same side of l near each point of tangency, otherwise l is called an *interior* double tangent.

When all the restrictions described immediately above are satisfied for a closed spherical curve γ, we will say that γ is *generic*. We will be concerned with the number of double points, antipodal pairs, etc. of a generic spherical curve. Therefore let:

d = the number of double points of γ,
a = the number of antipodal pairs of γ,
$2i$ = the number of inflection points of γ,
t = the number of exterior double tangents,
s = the number of interior double tangents.

For generic γ it turns out that each of d, a, i, t, and s is finite.

THEOREM 1. *Let $\gamma: C \to S$ be a generic closed spherical curve; then*

$$t - s = d - a + i.$$

PROOF. We will assume that the reader is familiar with the proof of Theorem 1 of [1] and explain how to adjust that proof to give a proof of this theorem.

First, we need something to take the place of the positive half-tangent, p^+, and the negative half-tangent, p^-, used in the proof given by Fabricius-Bjerre. The obvious choice is to use half-geodesics. So suppose $x \in C$; let $\gamma(x) = P$ and $\gamma'(x) = v$, the unit positive tangent vector to $\gamma(C)$ at P. Then let l_x^+, respectively l_x^-, be the geodesic segment of length π emanating from P in the direction v, respectively $-v$. For each $x \in C$, let $N^+(x)$, respectively $N^-(x)$, be the number of points common to $\gamma(C)$ and l_x^+, respectively l_x^-. Then, just as Fabricius-Bjerre, we keep track of $N(x) = N^+(x) - N^-(x)$, or more precisely the changes in $N(x)$ as x traverses C. Note that the changes in $N(x)$ as $\gamma(x)$ passes through a double point, an inflection point, or a point of tangency of a double tangent are just as Fabricius-Bjerre observed in the planar case.

What is new is that there is a change in $N(x)$ as $\gamma(x)$ passes either point of an antipodal pair $\{P, \bar{P}\}$ of γ. Let, in fact, $y \in C$ with $\gamma(y) = \bar{P}$. Then as x passes y note that $\bar{\gamma}(x)$ crosses $\gamma(C)$ at P. Denote the half-geodesics to $\bar{\gamma}$ at $\bar{\gamma}(x)$ by $\bar{l}_x^+$ and $\bar{l}_x^-$. Also let $M^+(x)$, respectively $M^-(x)$, be the number of points of $\bar{l}_x^+$, respectively $\bar{l}_x^-$, in common with $\gamma(C)$, and finally let $M(x) = M^+(x) - M^-(x)$. Since $\bar{\gamma}' = -\gamma'$, it follows that $\bar{l}_x^+ = l_x^-$ and $\bar{l}_x^- = l_x^+$, for all $x \in C$. Hence $N(x) = -M(x)$, for all x. Thus the changes in $N(x)$ are just the opposite of the changes in $M(x)$, but the change in $M(x)$ as x passes y would be the same as the change in $N(x)$ if $\gamma(x)$ had crossed itself at P. Hence the change in $N(x)$ as x passes y is the opposite of the change in $N(x)$ as $\gamma(x)$ passes a double point. Hence, we adjust the formula of Fabricius-Bjerre, $t - s = d + i$, by adding $-a$ to the side of this formula that contains d and obtain $t - s = d - a + i$.

Let H denote an open hemisphere of S. Then the following corollary is obvious.

COROLLARY. *Let $\gamma\colon C \to S$ be a generic closed spherical curve with $\gamma(C) \subset H$. Then $t - s = d + i$.*

REMARK. It is interesting to note that the Corollary follows directly from the formula of Fabricisu-Bjerre. Regard S as a unit sphere in Euclidean 3-space and let E be a plane tangent to S which is parallel to the equator bounding H. Let $\pi\colon H \to E$ denote central projection of H onto E. What is significant about π is that π preserves geodesics, i.e., the half-geodesics in H are mapped by π to straight lines in E. Hence π preserves double tangents of both kinds as well as inflection points of curves. Since π obviously preserves the double points of curves, the formula $t - s = d + i$ for γ "pulls-back" from the formula of Fabricius-Bjerre for $\pi \circ \gamma$.

2. Applications.

Let $\boldsymbol{E}^3$ denote Euclidean 3-space. A $\mathscr{C}^4$ immersion $\alpha\colon C \to \boldsymbol{E}^3$ is called a *closed space curve.* We suppose that α is non-degenerate; saying that α is *non-degenerate* means that α has positive curvature, κ, on C. Let τ denote the torison of α.

We will use a prime to denote differentiation with respect to the arc length of α. Thus α' represents the field of positive unit tangents to α. Define $\gamma: C \to S$ by setting $\gamma(x) = \alpha'(x)$, for all $x \in C$. Then γ is called the *tangent indicatrix* of α. By applying the formula in Theorem 1 to this tangent indicatrix, γ, we obtain a formula for α. It just remains for us to decide what the various properties of γ studied in section 1 mean in terms of α. Of course, we must impose suitable restrictions on α so that γ is generic; in fact, we will say α is *generic* when its tangent indicatrix, γ, is generic. We will leave the details of describing these restrictions on α to the reader. This should be easy after reading the subsequent paragraphs.

Let P be a double point of γ; in fact, suppose $\{x, y\} = \gamma^{-1}(P)$. From the definition of γ, it is immediate that $\alpha'(x) = \alpha'(y)$. Hence each double point of γ corresponds to a pair of points on $\alpha(C)$ whose positive unit tangents are parallel in the same direction; we say these points have *directly parallel tangents.* Likewise, an antipodal pair of points of γ corresponds to a pair of points on $\gamma(C)$ whose positive unit tangents are parallel but oppositely directed. We say these points have *oppositely parallel tangents.*

One may show [3] that the geodesic curvature of γ, k, is related to κ and τ by

$$k = \tau/\kappa.$$

Hence γ has an inflection point at x if and only if $\tau(x) = 0$ but $\tau'(x) \neq 0$; we have taken into account here that γ is generic. A point of $\alpha(x)$ is called a *vertex* of α if $\tau(x) = 0$ and $\tau'(x) \neq 0$. Hence each vertex of α corresponds to an inflection point of γ and conversely.

Now suppose l is a double tangent of $\gamma(C)$. Let $\gamma(x)$ and $\gamma(y)$ be the two points at which l is tangent to $\gamma(C)$, where $x, y \in C$. Since α is non-degenerate we may define its *binormal indicatrix* β: $C \to S$ by

$$\beta = \frac{\gamma \times \gamma'}{\|\gamma \times \gamma'\|}.$$

It follows that $\beta(x) = \pm\beta(y)$; see [3] for details. If we let $\mathcal{O}(z)$ denote the osculating plane to $\alpha(C)$ at $\alpha(z)$, for all $z \in C$, then, of course, this means that $\mathcal{O}(x)$ is parallel to $\mathcal{O}(y)$ since $\beta(z)$ is orthogonal to $\mathcal{O}(z)$, for all $z \in C$. Suppose, in addition, that l is an exterior double tangent. View l as the equator of S and say that $\gamma(C)$ lies in the northern hemisphere near $\gamma(x)$ and $\gamma(y)$. Let N be the north pole. Of course, N may be viewed as a vector in $\boldsymbol{E}^3$ orthogonal to both $\mathcal{O}(x)$ and $\mathcal{O}(y)$. Clearly $(\alpha \cdot N)' = \gamma \cdot N > 0$ for points of C near x or y. Hence α passes through each of $\mathcal{O}(x)$ and $\mathcal{O}(y)$ going in the same (general) direction. If l had been an interior double tangent then α would have passed through each of $\mathcal{O}(x)$ and $\mathcal{O}(y)$ going in the opposite (general) direction. We say the pair of points $\alpha(x)$ and $\alpha(y)$ have *concordant*, respectively *discordant*, parallel osculating planes if $\mathcal{O}(x)$ is parallel to $\mathcal{O}(y)$ and α passes through each of $\mathcal{O}(x)$ and $\mathcal{O}(y)$ going in the same, respectively opposite, direction.

An immediate consequence of Theorem 1 is the following theorem.

THEOREM 2. *Let* $\alpha\colon C \to E^3$ *be a generic non-degenerate closed space curve, then*

$$i = t-s-d+a,$$

where

$2i$ = *the number of vertices of* α,
d = *the number of pairs of directly parallel tangents of* α,
a = *the number of pairs of oppositely parallel tangents of* α,
t = *the number of pairs of concordant parallel osculating planes of* α,
s = *the number of pairs of discordant parallel osculating planes of* α.

Theorem 2 has a number of interesting consequences.

COROLLARY. *Let* $\alpha\colon C \to E^3$ *be a generic non-degenerate closed space curve with positive torsion, then*

$$t-s = d-a,$$

where now:

t = *the number of pairs of directly parallel binormals*,
s = *the number of pairs of oppositely parallel binormals.*

PROOF. Since $\tau > 0$, $i = 0$. Also, since $\tau > 0$, α passes through each of its osculation planes going in the (general) direction of its binormal.

The next theorem is particularly interesting in light of the fact that there exist a closed space curves with no pairs of parallel tangents (see [5]), i.e. $d+a = 0$.

THEOREM 3. *Let* $\alpha\colon C \to E^3$ *be a generic non-degenerate closed space curve. Then* α *must possess a pair of parallel tangents or a pair of parallel osculating planes.*

PROOF. Suppose, to the contrary, that $d = a = t = s = 0$. Then Theorem 2 implies $i = 0$; hence the torsion, τ, does not vanish. But W. Fenchel [2] has shown for closed non-planar space curves with $\kappa > 0$ and $\tau \geqq 0$ that $d \geqq 2$. We have a contradiction.

REMARK. We do not need to assume α is generic in Theorem 3 for this theorem to hold. It is enough to assume that any point at which $\tau = 0$ is a vertex.

COROLLARY. *Let α: $C \to E^3$ be a generic non-degenerate closed space curve with non-vanishing torsion. Then α must possess a pair of parallel principal normals.*

PROOF. Let us assume α has no pair of parallel principal normals. Then one can see that the tangent indicatrix and the binormal indicatrix are what J. White [6] calls SD-generic, since α is generic in the sense of this paper and has no parallel principal normal pairs. By Theorem 3, α must possess a pair of parallel tangents or a pair of parallel binormals. Hence α must posses a pair of parallel principal normals (see [6]). This contradiction proves the corollary.

REFERENCES

1. Fr. Fabricius-Bjerre, *On double tangents of plane closed curves*, Math. Scand. 11 (1962), 113–116.
2. W. Fenchel, *Über Krümmung and Windung geschlossener Raumkurven*, Math. Ann. 101 (1929), 238–252.
3. W. Fenchel, *On the differential geometry of closed space curves*, Bull. Amer. Math. Soc. 57 (1951), 44–54.
4. B. Halpern, *Global theorems for closed plane curves*, Bull. Amer. Math. Soc. 76 (1970), 96–100.
5. B. Segre, *Global differential properties of closed twisted curves*, Rend. Sem. Mat. Fis. Milano 38 (1968), 256–263.
6. J. H. White, *Global properties of immersions into Euclidean spheres*, Indiana Univ. Math. J. 20 (1971), 1187–1194.

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