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C*-ACTIONS ON GRASSMANN BUNDLES AND THE CYCLE AT INFINITY

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0. Introduction.

This paper describes the Grassmann Graph construction of MacPherson in the analytic category using C*-actions. The details of the algebraic case can be found in [1].

In section 1 we summarize the decomposition theorem of Bailynicki-Birula in the compact Kaehler case, [2], [3]. Section 2 describes a C*-action on Grassmann manifolds and gives the corresponding Bialynicki-Birula decomposition. Examples are given in the next section. In section 4 this C*-action is carried on to Grassmann bundles and Z_∞, the cycle at infinity corresponding to a bundle morphism is defined. It is shown that in the compact Kaehler case Z_∞ is an analytic cycle. The graph construction is finally accomplished in section 5. Examples are given in section 6.

Verdier uses the existence of a closed analytic space S which contains the closure of the graph in transcribing for analytic spaces the results of MacPherson, [9, section 5, proposition], [6]. We show in theorem 1 that in the compact Kaehler case S not only contains but is equal to the closure of the graph.

1. Bialynicki-Birula decomposition.

The references for this section are [2] for the algebraic case and [3] for the complex case. There is also a clear summary in [4, section Ic].

Let M be a compact Kaehler manifold with a C*-action on it. Let this C*-action have nontrivial fixed point set B with components $B_1, \ldots, B_m$. The components of the fixed point set are complex submanifolds of M. For $\lambda \in \mathbf{C}^*$ and $p \in M$ let $\lambda \cdot p$ denote the action of λ on p. The C*-action extends

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to a meromorphic map

$$P^1 \times \{p\} \to M$$

hence $\lim_{\lambda \to 0} \lambda \cdot p$ and $\lim_{\lambda \to \infty} \lambda \cdot p$ exist in M. Clearly these limits are in B. There are two canonical decompositions of M into invariant complex submanifolds. Define

$$M_i^+ = \{p \in M \mid \lim_{\lambda \to 0} \lambda \cdot p \in B_i\}$$

for $i = 1, \ldots, m$. Each M_i^+ is a complex manifold of M and

$$M = \bigcup M_i^+, \quad 1 \leqq i \leqq m.$$

This is called the *plus decomposition* of M. Similarly the *minus decomposition* is defined as

$$M_i^- = \{p \in M \mid \lim_{\lambda \to \infty} \lambda \cdot p \in B_i\}$$

for $i = 1, \ldots, m$. Each M_i^- is a complex submanifold and similarly

$$M = \bigcup M_i^-, \quad 1 \leqq i \leqq m.$$

There are two distinguished components of the fixed point set B, say B_1 and B_m, which are determined by the property that M_1^+ and M_m^- are open and dense in M. B_1 is called the *source* and B_m is called the *sink*.

2. C^*-actions on $G(k, n)$.

In this section we describe a particular C^*-action on $G(k,n)$, the Grassmannian manifold of k-planes in n-space. Fix a coordinate system on C^n. We will use the representation of $G(k,n)$ by matrices. Any point $p \in G(k,n)$ can be represented by a $k \times n$-matrix A of rank k. Two such matrices A and B represent the same point in $G(k,n)$ if there is an invertible $k \times k$-matrix $g \in \mathrm{GL}(k, \mathrm{C})$ such that $gA = B$. For a $k \times n$-matrix A of rank k set $[A]$ = the row space of A.

Given a $k \times n$-matrix $A = (a_{ij})$, $1 \leqq i \leqq k$, $1 \leqq j \leqq n$ define two submatrices

$$A_1 = (a_{ij}), \quad 1 \leqq i, j \leqq k$$

and

$$A_2 = (a_{ij}), \quad 1 \leqq i \leqq k, \; k+1 \leqq j \leqq n.$$

A_1 is a $k \times k$-matrix and A_2 is a $k \times (n-k)$-matrix and $A = (A_1, A_2)$ is a partitioning of A.

Define a C*-action on $G(k,n)$

$$\mathbf{C}^* \times G(k,n) \to G(k,n)$$

by

$$\lambda \cdot [A] = [(A_1, \lambda A_2)].$$

To describe the behaviour of this action define a subset X_{ij} of $G(k,n)$ as the set of all p in $G(k,n)$ which can be represented by a $k \times n$-matrix $A = (A_1, A_2)$ such that rank $A_1 = i$ and rank $A_2 = j$, where $k - \min\{k, n-k\} \leqq i \leqq k$ and $0 \leqq j \leqq \min\{k, n-k\}$. Let $B = (B_1, B_2)$ be another $k \times n$-matrix representing p. Then there is an invertible $k \times k$-matrix g such that $gA = B$

$$gA_1 = B_1 \quad \text{and} \quad gA_2 = B_2.$$

Hence rank $B_1 = \operatorname{rank}(gA_1) = \operatorname{rank} A_1 = i$ and similarly rank $B_2 = j$, and the following definition of X_{ij} is well defined:

$$X_{ij} = \{[A] \in G(k,n) | \operatorname{rank} A_1 = i, \operatorname{rank} A_2 = j\}$$

where $k - \min\{k, n-k\} \leqq i \leqq k$ and $0 \leqq j \leqq \min\{k, n-k\}$. Necessarily we have $i + j \geqq k$; to see this, recall that A represents a point in $G(k,n)$ hence has rank k, and if A_1 has rank i, then A_2 must supply at least the remaining $k-i$ ranks.

To describe the behaviour of the C*-action that is defined above we prove the following lemmas.

Lemma 1. *X_{ik-i} are the fixed point components of the* C*-*action*, $k - \min\{k, n-k\} \leqq i \leqq k$.

Proof. Let $[A] \in X_{ik-i}$, $A = (A_1, A_2)$. We first show that $\lambda \cdot [A] = [A]$. If $i = 0$, then $A_1 = 0$, *and if* $i = k$, then $A_2 = 0$. In both cases $\lambda \cdot [A] = [A]$. Assume $0 < i < k$. Then there exists an invertible $k \times k$-matrix g such that gA is of the form

$$gA = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

where $B_1 \in \mathrm{GL}(i, \mathbf{C})$ and $B_2 \in \mathrm{GL}(k-i, \mathbf{C})$. For $\lambda \in \mathbf{C}^*$ define h_λ to be the diagonal matrix $[1, \ldots, 1, 1/\lambda, \ldots, 1/\lambda]$, where the number of $1/\lambda$'s is $k-i$. We then have the following sequence of equalities:

$$\begin{aligned}\lambda\cdot[A] &= \lambda\cdot[gA]\\ &= \lambda\cdot\left[\begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}\right]\\ &= \left[\begin{pmatrix} B_1 & 0\\ 0 & \lambda B_2\end{pmatrix}\right]\\ &= \left[h_\lambda\begin{pmatrix} B_1 & 0\\ 0 & \lambda B_2\end{pmatrix}\right]\\ &= \left[\begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}\right]\\ &= [A].\end{aligned}$$

Thus we have proven that X_{ik-i} is a subset of the fixed point set. That in fact there are no other fixed points than $\cup X_{ik-i}$, $k-\min\{k,n-k\} \leqq i \leqq k$ follows from the results of the following two lemmas.

LEMMA 2. *If* $[A]\in X_{ij}$, *then* $\lim_{\lambda\to 0}\lambda\cdot[A]\in X_{ik-i}$, *where*

$$k-\min\{k,n-k\} \leqq i \leqq k,\ 0 \leqq j \leqq \min\{k,n-k\} i+j \geqq k.$$

In particular X_{k0} *is the source.*

PROOF. If $i=0$ or $i=k$, then X_{ij} is a component of the fixed point set as in Lemma 1. Assume $0<i<k$. There exists $g\in \mathrm{GL}(k,\mathrm{C})$ such that

$$gA = \left(\begin{array}{c|c} B_1 & \begin{matrix} 0 \\ B_2\end{matrix}\\ \hline 0 & B_3\end{array}\right)$$

where $B_1\in\mathrm{GL}(i,\mathrm{C})$, $B_3\in\mathrm{GL}(k-i,\mathrm{C})$ and B_2 is a $(i+j-k)\times(n-k)$-matrix. Let h_λ be as in Lemma 1. Then

$$h_\lambda \lambda g A = \left(\begin{array}{c|c} B_1 & \begin{matrix} 0 \\ \lambda B_2\end{matrix}\\ \hline 0 & B_3\end{array}\right)$$

and since $\lim_{\lambda\to 0}\lambda B_2 = 0$ we have

$$\lim_{\lambda\to 0} \lambda\cdot[A] = \lim_{\lambda\to 0}[h_\lambda \lambda g A] = \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_3 \end{pmatrix}\right].$$

This last matrix is clearly in X_{ik-i} as claimed.

LEMMA 3. *If* $[A]\in X_{ij}$, *then* $\lim_{\lambda\to\infty}\lambda\cdot[A]\in X_{k-jj}$, *where*

$$k-\min\{k,n-k\}\leqq i\leqq k,\ 0\leqq j\leqq \min\{k,n-k\}.$$

In particular X_{k-mm} *is the sink, where* $m=\min\{k,n-k\}$.

PROOF. If $i=0$ or $i=k$, then X_{ij} is a fixed point component. Assume $0<i<k$. There exists $g\in \mathrm{GL}(k,\mathbf{C})$ such that

$$gA = \left(\begin{array}{c|c} B_1 & 0 \\ \hline B_2 & \\ & B_3 \\ 0 & \end{array}\right)$$

where $B_1\in \mathrm{GL}(k-j,\mathbf{C})$, $B_3\in\mathrm{GL}(j,\mathbf{C})$ and B_2 is a $(i+j-k)\times k$-matrix. Then

$$\lim_{\lambda\to\infty}\lambda\cdot[A] = \lim_{\lambda\to\infty}[\lambda h_\lambda g A] = \lim_{\lambda\to\infty}\left[\begin{pmatrix} B_1 & 0 \\ \lambda^{-1}B_2 & \\ & B_3 \\ 0 & \end{pmatrix}\right] = \left[\begin{pmatrix} B_1 & 0 \\ 0 & B_3\end{pmatrix}\right].$$

This last matrix is in X_{k-jj} as desired.

These last two lemmas show that X_{ik-i} for $k-\min\{n-k\}\leqq i\leqq k$ are the only fixed point components and thus complete the proof of lemma 1.

We can apply these lemmas to examine the behaviour of Schubert cells under the action of $\mathbf{C}^*$ on the Grassmann manifold. We will adopt the terminology of Griffiths and Harris on Schubert cells. For details refer to [5, pp. 195–196].

Let $\{e_1,\dots,e_n\}$ be the standard basis for $\mathbf{C}^n$ and $V_i=\mathrm{span}\{e_1,\dots,e_i\}$. Then $\{V_1,\dots,V_n\}$ defines a flag. For any nonincreasing sequence of nonnegative integers between 0 and $n-k$ define a cell

$$W_a=\{[A]\in G(k,n)\mid \dim(A\cap V_{n-k+i+a_i})=i\}.$$

The sequence of nonincreasing integers $a = (a_1, \ldots, a_k)$ with $0 \leqq a_i \leqq n-k$ is called a Schubert symbol. For $[\Lambda] \in G(k,n)$, let A be a $k \times n$-matrix such that $[A] = [\Lambda]$. If $[A] \in W_a$ for some Schubert symbol $a = (a_1, \ldots, a_k)$, then the rank of the first $k \times (n-k+i-a_i)$ minor is i and the rank of the last $k \times (k-i+a_i)$ minor is $k-i$. The closure of W_a

$$\overline{W_a} = \{[\Lambda] \in G(k,n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) \geqq i\}$$

is called a Schubert variety. If A is a matrix representing $[\Lambda]$ as above, then $[\Lambda]$ is in $\overline{W_a}$ iff the rank of the first $k \times (n-k+i-a_i)$ minor of A is at least i and the rank of the last $k \times (k-i+a_i)$ minor of A is at most $k-i$. It is well-known that $\overline{W_a}$ is an analytic subvariety of $G(k,n)$ and the homology class of $\overline{W_a}$, denoted by σ_a, is independent of the flag used in its definition, [5, p. 196]. σ_a is called the Schubert cycle corresponding to $a = (a_1, \ldots, a_k)$. Regarding the behaviour of Schubert cycles under the $\mathbf{C}^*$-action we give the following corollary to the above lemmas:

Corollary 1. *All Schubert cycles of positive codimension in $G(k, 2k)$ lie in X_{ij}'s where $j < k$. In particular they do not flow to the sink*, i.e. *if $p \in \overline{W_a}$ then $\lim_{\lambda \to \infty} \lambda \cdot p$ is not in the sink.*

Proof. The codimension of $\overline{W_a}$ for $a = (a_1, \ldots, a_k)$ is $\sum a_i$, [5, p. 196]. It suffices to prove the corollary for $a = (1, 0, \ldots, 0)$. For $[\Lambda] \in W_a$ let $A = (A_1, A_2)$ be a matrix representation where A is a $k \times n$-matrix of rank k, and A_1, A_2 are $k \times k$-matrices. The rank of the last $k \times k$ minor of A is of rank at most $k-1$. Hence in particular the rank of A_2 is not k, therefore $[A]$ is not in X_{ik}. Since the only points that flow to the sink belong to the components of the form X_{ik}, $[\Lambda]$ does not flow to the sink. In general if $a = (a_1, \ldots, a_k)$ with $a_1 \geqq 1$ then the last $k \times (k+a_1-1)$ minor has rank at most $k-1$. Since $k+a_1-1 \geqq k$, the rank of A_2 can not be k. Hence $\overline{W_a}$ does not flow to the sink. If $a_1 = 0$, then $a = (0, \ldots, 0)$ and $\overline{W_a}$ does not have positive codimension.

Using the same notation as in the previous corollary we can generalize as follows:

Corollary 2. *Let $\overline{W_a}$, $a = (a_1, \ldots, a_k)$, be a Schubert variety in $G(k,n)$, where $a_1 \geqq n-2k+1$. Then $\overline{W_a}$ does not flow to the sink if $n \geqq 2k$.*

Proof. Let $A = (A_1, A_2)$ be a $k \times n$-matrix with rank k representing a point $[A]$ in $\overline{W_a}$. A_1 is a $k \times (n-k)$-matrix and $[A]$ will flow to the sink if rank A_2 is maximal. Since $n \geqq 2k$ means $n-k \geqq k$, the maximal rank of A_2 is k. The rank of the last $k \times (k+a_1-1)$ minor of A is at most $k-1$. By assumption $k + a_1 - 1 \geqq n - k$, therefore the rank of A_2 cannot be k. Hence $\overline{W_a}$ does not flow to the sink.

3. Examples.

In examples 1 and 2 we assume that the C*-action of the previous section is defined on the spaces $G(2,4)$ and $G(4,9)$.

1) $G(2,4)$. In $G(2,4)$ we have defined the following sets:

$$X_{20}, X_{11}, X_{02}, X_{22}, X_{12}, X_{21}.$$

The first three sets are the fixed point sets. As $\lambda \to 0$ the elements of X_{21} and X_{22} flow to the source X_{20}, and the elements of X_{12} flow to X_{11}. As $\lambda \to \infty$ the elements of X_{22} and X_{12} flow to the sink X_{02}, and the elements of X_{21} flow to X_{11}.

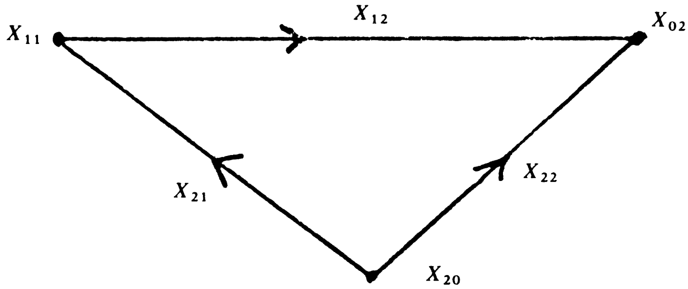

Fig. 1.

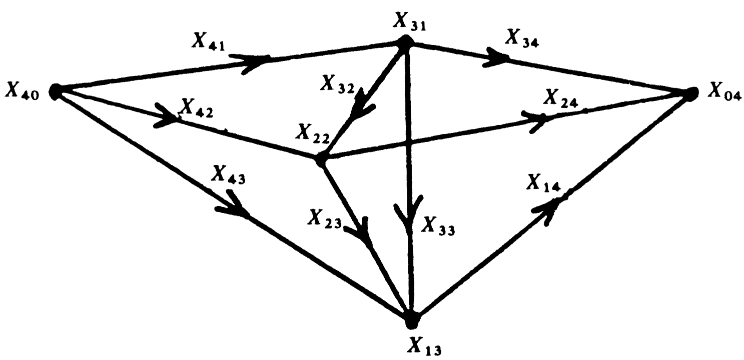

Fig. 2.

See Figure 1 for the direction of these flows for each X_{ij} as $\lambda \to \infty$.

2) $G(4,9)$. For the direction of flow as $\lambda \to \infty$ see Figure 2. From the decomposition of $G(4,9)$ into X_{ij} it can be seen that the points that lie in

$$X_{13} \cup X_{33} \cup X_{31} \cup X_{32} \cup X_{22} \cup X_{23}$$

do not flow to the sink or the source under the action of $\mathbf{C}^*$.

4. C^*-actions on Grassmann bundles.

This section defines in the compact Kaehler case the Grassmann Graph construction of [1, pp. 120–121].

Let E, F be vector bundles of ranks k and n, respectively, on an analytic space M. Let $G(k, E \oplus F) \to M$ denote the Grassmann bundle whose fibre at each $x \in M$ is $G(k, E_x \oplus F_x)$, the Grassmanian of k-planes in $E_x \oplus F_x$. Define a $\mathbf{C}^*$-action on $G(k, E \oplus F)$ as the fibrewise $\mathbf{C}^*$-action. Let

$$\pi_1 : E \oplus F \to E$$

$$\pi_2 : E \oplus F \to F$$

and

$$\pi : G(k, E \oplus F) \to M$$

be the projections. Any $p \in G(k, E \oplus F)$ is represented by a k-plane H in $E_x \oplus F_x$ where $x = \pi(p)$. $\pi_1(H)$ and $\pi_2(H)$ are linear subspaces of E_x and F_x, respectively. The total space $G(k, E \oplus F)$ can be decomposed into $\mathbf{C}^*$-equivariant subbundles

$$X_{ij} = \{[H] \in G(k, E \oplus F) | \dim \pi_1(H) = i, \dim \pi_2(H) = j\}$$

where $k - \min(k,n) \leqq i \leqq k$, $0 \leqq j \leqq \min(k,n)$, and $i + j \geqq k$. It is easy to see that

$$X_{ij} \cong G(i, E) \times G(j. F) \quad \text{if} \quad i + j = k,$$

which are the fixed point sets of the $\mathbf{C}^*$-action. Let

$$\mathrm{Hom}(E, F) \to M$$

be the bundle of morphisms from E to F and let

$$j : \mathrm{Hom}(E, F) \to G(k, E \oplus F)$$

be the natural inclusion defined fibrewise as

$$j_x(\Phi) = \mathrm{graph}(\Phi | E_x) = \{(e, \Phi(e)) \in E_x \oplus F_x\}.$$

Recall that C can be imbedded into P^1 as

$$\mathbf{C} \to P^1$$
$$\lambda \to [1:\lambda],$$

[1, p. 120]. Define a C*-action on $G(k, E \oplus F) \times P^1$

$$\mathbf{C}^* \times G(k, E \oplus F) \times P^1 \to G(k, E \oplus F) \times P^1$$

as

$$(\lambda, p, [\lambda_0 : \lambda_1]) \to (\lambda \cdot p, [\lambda_0 : \lambda\lambda_1])$$

where $\lambda \cdot p$ is the C*-action which is defined above. Also define the C*-action on $M \times \mathbf{C}$,

$$\mathbf{C}^* \times M \times \mathbf{C} \to M \times \mathbf{C}$$

as

$$(\lambda, x, t) \to (x, \lambda t).$$

Every $\Phi \in \operatorname{Hom}(E, F)$ defines an equivariant imbedding $\bar{s}(\Phi)$ of $M \times \mathbf{C}$ into $G(k, E \oplus F) \times P^1$,

$$\bar{s}(\Phi): M \times \mathbf{C} \to G(k, E \oplus F) \times P^1$$

where

$$\bar{s}(\Phi)(x, \lambda) = ([j_x(\lambda\Phi_x)], [1:\lambda]).$$

Let $s(\Phi) = \operatorname{pr}\,(\bar{s}(\Phi))$ where pr is the projection

$$\operatorname{pr}: G(k, E \oplus F) \times P^1 \to G(k, E \oplus F).$$

$s(\Phi)(M, \lambda)$ is the graph of $\lambda\Phi$. *Now define*

$$Z_\infty = \lim_{\lambda \to \infty} s(\Phi)(M, \lambda).$$

THEOREM 1. *If M is a compact Kaehler manifold, then for any $\Phi \in \operatorname{Hom}(E, F)$ the corresponding Z_∞ is an analytic cycle.*

PROOF. Let $\varrho: \mathbf{C}^* \times G(k, E \oplus F) \to G(k, E \oplus F)$ be the C*-action defined above. Consider M as a subspace of $G(k, E \oplus F)$ by the imbedding $s(\Phi)(M, 1)$; i.e. identify M and the graph of Φ. Define a holomorphic map

$$A: M \times \mathbf{C}^* \to G(k, E \oplus F)$$

as

$$A(m, t) = s(\Phi)(m, t),$$

where $m \in M$ and $t \in \mathbf{C}^*$. This map is equivariant with respect to ϱ and the trivial action of C* on $M \times \mathbf{C}^*$, multiplication in the second component; for if $\lambda \in \mathbf{C}^*$ then

$$\begin{aligned} A(m, \lambda \cdot t) &= s(\Phi)(m, \lambda t) \\ &= s(\lambda\Phi)(m, t) \\ &= \lambda \cdot s(\Phi)(m, t) \\ &= \varrho(\lambda, s(\Phi)(m, t)) \\ &= \varrho(\lambda, A(m, t)) \end{aligned}$$

hence equivariance. But Sommese has shown that if $\psi: Y \times \mathbf{C}^* \to X$ is a holomorphic map equivarient with respect to the trivial action of $\mathbf{C}^*$ on $Y \times \mathbf{C}^*$ and the action of $\mathbf{C}^*$ on X with fixed points then ψ extends meromorphically to $Y \times \boldsymbol{P}^1$, [8, p. 111 (Lemma II-B)]. Thus A extends meromorphically to

$$A': M \times \boldsymbol{P}^1 \to G(k, E \oplus F).$$

Let T be the closure of the graph of A in $M \times \boldsymbol{P}^1 \times G(k, E \oplus F)$.

By the definition of a meromorphic map, T is an analytic space. Since

$$M \times \{\infty\} \times Z_\infty = T \cap (M \times \{\infty\} \times G(k, E \oplus F)),$$

being the intersection of two analytic spaces it is analytic. If $\mathrm{pr}: M \times \{\infty\} \times \times Z_\infty \to M$ is the projection, then for any $m \in M$, $\mathrm{pr}^*(m) = \{m\} \times \{\infty\} \times Z_\infty$ is an analytic cycle, from which it follows that Z_∞ is analytic as desired.

Z_∞ is called the cycle at infinity corresponding to the map Φ. Notice that there is an alternate definition of Z_∞, see [1, p. 121];

Let W be the closure of $\bar{s}(\Phi)(M \times \mathbf{C})$ in $G(k, E \oplus F) \times \boldsymbol{P}^1$. Then $Z_\infty \times \{\infty\}$ is the intersection of W and $G(k, E \oplus F) \times \{\infty\}$.

In the algebraic category W is an algebraic variety but in the analytic category the observation that W can be obtained through a $\mathbf{C}^*$-action with fixed points on a compact Kaehler manifold is crucial in concluding that it is analytic.

Clearly $\{Z_\lambda = s(\Phi)(M, \lambda)\}$ defines a family of cycles which are algebraically and hence homologically equivalent.

5. Graph of complexes.

In this section we define the Grassmann Graph construction and the cycle at infinity associated to a complex of vector bundles. This construction was first introduced by MacPherson and used by Baum, Fulton and MacPherson to prove Riemann-Roch theorem for singular algebraic varieties, [1] and [6].

Consider a complex of vector bundles on M,

$$(E.):\ 0 \to E_m \to E_{m-1} \to \cdots \to E_0 \to 0.$$

Denote the maps by γ_i, i.e.

$$\gamma_i:\ E_i \to E_{i-1}$$

where $i = 0, \ldots, m$, $E_{-1} = 0$.

Assume that there is a subvariety S of M such that $(E.)$ is exact on $M-S$.

Let

$$G_i = G(\operatorname{rank} E_i, E_i \oplus E_{i-1}), \quad i = 0, \ldots, m.$$

and let

$$\tau_i \to G_i \text{ the tautological bundle, } i = 1, \ldots, m.$$

Define

$$G = G_0 \times_M \ldots \times_M G_{m'}$$

where $\times_M$ denotes the bundle product on M. On G let τ_i denote the pull back of $\tau_i \to G_i$ by the projection $\mathrm{pr}_i : G \to G_i$ of the ith component, $i = 0, \ldots, m$.

Let

$$\tau = \tau_0 - \tau_1 + \ldots + (-1)^m \tau_m$$

be the virtual tautological bundle on G. Recalling the definition of s from the previous section for any $\lambda \in \mathrm{C}$ define an imbedding

$$s_\lambda^i : M \to G_i$$

as

$$s_\lambda^i(x) = s(\gamma_i)(x, \lambda)$$

where $i = 0, \ldots, m$. Then define for any $\lambda \in \mathrm{C}$ an imbedding

$$s_\lambda : M \to G$$

by

$$s_\lambda(x) = (s_\lambda^0(x), \ldots, s_\lambda^m(x)).$$

Using $s_\lambda(M)$ we define

$$Z_\infty = \lim_{\lambda \to \infty} s_\lambda(M)$$

to be the cycle at infinity corresponding to the complex $(E.)$.

Let $\pi : G \to M$ be the natural projection. Recalling that S is the set off which $(E.)$ is exact we have the following result: (For proofs see [1, p. 121].)

THEOREM (Baum, Fulton, MacPherson). *The cycle Z_∞ has a unique decomposition $Z_\infty = Z_* + M_*$, where*

1) *π maps M_* meromorphically onto M.*
2) *$\pi: M_* - \pi_1(S) \to M - S$ is a biholomorphism.*
3) *π maps Z into S.*
4) *τ restricts on M_* to the zero bundle.*

REMARK. By Theorem 1 of the previous section, Z_∞ is a product of analytic cycles in the product bundle G, hence this theorem can be stated in the analytic category as above. Any cycle can be written as a sum of irreducible cycles. The decomposition of Z_∞ is such a sum. For a proof of (4) see [1, p. 122].

Finally we define two residues on S. Let E be the virtual bundle $\dot{E}_0 - E_1 + \ldots + (-1)^m E_m$ on M. Then $\tau|Z_0$ is isomorphic to E since $Z_0 \cong M$. SinceZ_0 and Z_∞ are rationally equivalent

$$c(E) \cap [M] = c(\tau) \cap Z_0 = c(\tau) \cap Z_\infty$$

where $c(\cdot)$ denotes the Chern class and $\cap$ denotes the cap product. Since Z_∞ decomposes

$$\begin{aligned} c_i(\tau) \cap Z_\infty &= c_i(\tau) \cap (Z_* + M_*) \\ &= c_i(\tau) \cap Z_* + c_i(\tau) \cap M_* \\ &= c_i(\tau) \cap Z_*, \end{aligned}$$

where $i > 0$ and the last equality follows since $\tau|M_* = 0$ by (4) of the above theorem.

Define

$$c_S^i(E.) = \pi_*(c_i(\tau) \cap Z_*) \in H_*(S; \mathrm{C}).$$

Similarly let ch($\cdot$) denote the Chern character, then

$$\begin{aligned} \mathrm{ch}(E) \cap [M] &= \mathrm{ch}(\tau) \cap Z_0 \\ &= \mathrm{ch}(\tau) \cap Z_\infty \\ &= \mathrm{ch}(\tau) \cap Z_* + \mathrm{ch}(\tau) \cap M_* \\ &= \mathrm{ch}(\tau) \cap Z_*. \end{aligned}$$

Similarly define

$$\mathrm{ch}_S(E.) = \pi_*(\mathrm{ch}(\tau) \cap Z_*) \in H_*(S; \mathrm{C}).$$

For basic properties of $\mathrm{ch}_s(E.)$ in the algebraic category see [1, pp. 121–126].

We will use $c_S^i(E)$ for calculating the Baum-Bott residues of singular holomorphic folitations in [7].

6. Examples.

1) Let E, F be vector bundles on M and $\psi \in \mathrm{Hom}(E, F)$. Then the graph $\Gamma(\psi)$ of ψ gives rise to a cycle at infinity Z_∞. Let rank $E = k$, rank $F = n$, and $m = \min\{k, n\}$. For $i = 0, 1, \ldots, m$, let $B_i = X_{k-ii}$, where X_{ij} is as defined in Section 4. $B_0, \ldots, B_m$ are the components of the fixed point set B under the $\mathbf{C}^*$-action on the Grassmann bundle $G(k, E \oplus F)$. To understand the structure of Z_∞ we describe its intersection with B. For this purpose define the following sets

$$\Sigma_i = \{p \in M \mid \mathrm{rank}\, \psi_p \leqq i\}, \quad i = 0, \ldots, r$$

where r is the generic rank of ψ. The behaviour of Z_∞ can now be described as follows:

$$(Z_\infty \cap B_i)_p \neq \phi \quad \text{iff } p \in \Sigma_t \quad \text{and} \quad t \geqq i \geqq r.$$

2) We want to show that the Hironaka Blow-up at a point can be recovered as a Grassmann Graph construction. The problem is local so let M be an open set in $\mathbf{C}^n$. Define two trivial bundles L and F as

$$L = M \times \mathbf{C} \quad \text{and} \quad F = M \times \mathbf{C}^n.$$

Define a morphism $\theta \in \mathrm{Hom}(L, F)$ as:

$$\theta(p, t) = (p, tp) \quad \text{for} \quad p \in \mathbf{C}^n, t \in \mathbf{C}.$$

The cycle at infinity Z_∞ corresponding to θ intersects the sink of $G(1, L \oplus F)$ in M_*, that is $Z_\infty = M_* + Z_*$. M_* is the Hironaka Blow-up of M at the origin. We can see this as follows. Let $p = (p_1, \ldots, p_n) \in M = \mathbf{C}^n$. We also identify $\boldsymbol{P}(L \oplus F)$ with $\boldsymbol{P}^n$. There is a $\mathbf{C}^*$-action

$$\mathbf{C}^* \times M \times \boldsymbol{P}^n \to M \times \boldsymbol{P}^n$$

given as

$$(\lambda, p, [y_0 : y_1 \ldots : y_n]) = (p, [y_0 : \lambda y_1 : \ldots : \lambda y_n]).$$

The graph of θ has the form

$$\Gamma(\theta) = \{(p, [1 : p_1 : \ldots : p_n]) \in M \times \boldsymbol{P}^n\}.$$

The $\mathbf{C}^*$-action moves $\Gamma(\theta)$ as

$$\lambda \cdot \Gamma(\theta) = \{(p, [1 : \lambda p_1 : \ldots : \lambda p_n]) \in M \times \boldsymbol{P}^n\}.$$

Consider the usual imbedding of $\mathbf{C}^*$ in $\boldsymbol{P}^1$ as $\lambda = [1:\lambda] = [\lambda_0:\lambda_1]$, where $\lambda = \lambda_1/\lambda_0$. Since $\lambda \to \infty$ iff $\lambda_0 \to 0$ with $\lambda_1 \neq 0$, we have the following limit

$$\begin{aligned} Z_\infty &= \lim_{\lambda\to\infty} \lambda \cdot \Gamma(\theta) \\ &= \lim_{\lambda_0\to 0} \{(p,[\lambda_0:\lambda_1 p_1:\ldots:\lambda_1 p_n]) \in M \times \boldsymbol{P}^n\} \\ &= \{(p,[0:\lambda_1 p_1:\ldots:\lambda_1 p_n]) \in M \times \boldsymbol{P}^n\}. \end{aligned}$$

Clearly $(p,[\lambda_1 p_1:\ldots:\lambda_1 p_n]) \in M \times \boldsymbol{P}^n$ can be considered as a point $(p,[x_1:\ldots:x_n])$ in $M \times \boldsymbol{P}^{n-1}$ such that

$$p_j x_i = p_i x_j, \quad i \neq j,\ 1 \leqq i, j \leqq n,$$

From here it is easy to see that the intersection of Z_∞ with the sink of the $\mathbf{C}^*$-action is the Hironaka blow-up of M at the origin.

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