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A HOLOMORPHIC REPRODUCING KERNEL FOR KOHN-NIRENBERG DOMAINS IN C^2

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Introduction.

Let D be a smooth domain in C^2. Any Leray map $\Psi = (\Psi_1, \Psi_2)$: $\Omega \times \partial\Omega \to C^2$ gives rise to a Cauchy Fantappié formula which reproduces holomorphic functions that are continuous up to the boundary of Ω. In general, it will be impossible to find a Leray map which is holomorphic in the first variable, therefore the Cauchy-Fantappié form will not be holomorphic in this variable either.

For smooth strictly pseudoconvex domains it was proved among others by Henkin [3] that Leray maps and Cauchy-Fantappié forms that are holomorphic in the first variable exist. Range and Siu [5] obtained a kind of Cauchy-Fantappié formula for intersections of smooth strictly pseudoconvex domains.

In this paper we consider the so-called Kohn-Nirenberg domains in C^2:

$$\Omega = \{w \in C^2 : \operatorname{Re} w_2 + P(w_1) < 0\},$$

where P is a real valued homogeneous polynomial in w_1 and $\bar{w}_1$ with $\Delta P > 0$ when $w_1 \neq 0$. To avoid problems stemming from the unboundedness of Ω we will mainly be concerned with $\Omega_R = \Omega \cap \{|w| < R\}$. In general, it is impossible to find a holomorphic Leray map defined in $\Omega \times \partial\Omega$. However, it was shown by the first author [2] that such a map with fairly good properties exists on $\Omega \times \Sigma$, where $\Sigma = \partial\Omega \setminus \{\zeta_1 = 0\}$.

We modify this map slightly and study the related Cauchy-Fantappié formula on Ω_R. Formally this looks exactly like what one would expect in view of the Range-Siu result. Although the kernel we obtain blows up at $\zeta_1 = 0$, we will show that it is integrable over the boundary. It reproduces functions in $A(\Omega_R)$ and maps $C(\partial\Omega_R)$ into $H(\Omega)$.

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1. Preliminaries.

Let $\Omega = \{(w_1, w_2) \in \mathbf{C}^2 ; \operatorname{Re} w_2 + P(w_1) < 0\}$, where P is a homogeneous polynomial of degree $2k$, $\Delta P > 0$ when $w_1 \neq 0$.

Let $\Sigma = \{w \in \partial\Omega, w_1 \neq 0\}$. Let $\Omega_R = \Omega \cap B(0,R)$, $\Sigma_R = \Sigma \cap B(0,R)$.

In this section we give an account of results concerning Ω. All proofs can be found in [1] or [2].

Let $\zeta = (\zeta_1, \zeta_2) \in \Sigma, \theta_1 = \arg \zeta_1$.

LEMMA 1.1. *For every ζ_1 there exists a unique harmonic polynomial of the form* $\operatorname{Re} \alpha w_1^{2k} = P(w_1) + O(|w_1 - \zeta_1|^2)$. *The constant $\alpha = \alpha(\theta_1)$ depends real analytically on θ_1.*

Write $P_1(w_1) = P(\theta_1, w_1) = P(w_1) - \operatorname{Re} \alpha(\theta_1) w_1^{2k}$.

LEMMA 1.2. *There exist $\delta > 0$, $\mathscr{C} > 0$ independent of θ,* such that if

$$|\arg w_1 - \theta_1| \leqq \delta, \text{ then } \frac{1}{\mathscr{C}} |\arg w_1 - \theta_1|^2 |w_1|^{2k} \leqq P_1(w) \leqq \mathscr{C} |\arg w_1 - \theta_1|^2.$$

Introduce

$$F_1(w_1) = F(\zeta_1, w_1) = w_1^{2k}(w_1 - \zeta_1)^2 e^{-i(2k+2)\theta_1}.$$

LEMMA 1.3. *There exist $\delta > 0$, $\mathscr{C} > 0$ such that if $|\arg w_1 - \theta_1| \leqq \delta$, then* $\operatorname{Re} F_1(w_1) \geqq \frac{1}{2} |w_1|^{2k} |w_1 - \zeta_1|^2 - \mathscr{C} |w_1|^{2k+2} (\arg w_1 - \theta_1)^2$.

Let $P_\varepsilon(w_1) = P(w_1) - \varepsilon |w_1|^{2k}$; ε will be chosen very small below, but at least so small that $P_{3\varepsilon}$ is strictly subharmonic if $w_1 \neq 0$. We change coordinates as follows

$$\tilde{w}_1 = w_1, \quad \tilde{w}_2 = \tilde{w}_2(\zeta_1, w, M) = w_2 + \alpha(\theta_1) w_1^{2k} - (\varepsilon/M) F_1(w_1), \quad M \gg 0.$$

Let

$$Q_1(w_1) = Q(\zeta_1, w_1, M) = P_1(w_1) + (\varepsilon/M) \operatorname{Re} F_1(w_1).$$

Then in these coordinates

$$\Omega = \{\operatorname{Re} \tilde{w}_2 + Q_1(\tilde{w}_1) < 0\}.$$

LEMMA 1.4. *For every $R > 0$ there exists $M > 0$ such that if $\tilde{w} \in \Omega$ and $|\zeta_1|$, $|\tilde{w}_1| \leqq R$, then*

$$\tilde{w} \in \Omega_\varepsilon := \{\operatorname{Re} \tilde{w}_2 + P_\varepsilon(\tilde{w}_1) - \operatorname{Re} \alpha(\theta_1) \tilde{w}_1^{2k} < 0\}.$$

By continuity we can find arcs $I_1, \ldots, I_l$ which cover the unit circle, are centered at $e^{i\theta^1}, \ldots, e^{i\theta^l}$, respectively, and are shorter than δ such that if $e^{i\theta_1} \in I_j$, then

$$\Omega_\varepsilon \subset \Omega_j := \{\operatorname{Re}\tilde{w}_2 + P_{2\varepsilon}(\tilde{w}_1) - \operatorname{Re}(\alpha(\theta^j)\tilde{w}_1^{2k}) < 0.$$

Each Ω_j is contained in the even larger pseudoconvex domain

$$\Omega'_j := \{\operatorname{Re}\tilde{w}_2 + P_{3\varepsilon}(\tilde{w}_1) - \operatorname{Re}(\alpha(\theta^j)\tilde{w}_1^{2k}) < 0\}.$$

LEMMA 1.5. *If* $\varepsilon > 0$ *is small enough*, $|\arg\tilde{w}_1 - \theta_1| \leqq \delta$, $|\zeta_1|, |\tilde{w}_1| \leqq R$, *then*

$$Q_1(\tilde{w}_1) \sim (\arg\tilde{w}_1 - \theta_1)^2|\tilde{w}_1|^{2k} + |\tilde{w}_1|^{2k}|\tilde{w}_1 - \zeta_1|^2.$$

LEMMA 1.6. *If* $\varepsilon > 0$ *is small enough*, $|\arg w_1 - \theta_1| = \delta$, *and* $e^{i\theta_1} \in I_j$, *then*

$$P_{3\varepsilon}(\tilde{w}_1) - \operatorname{Re}(\alpha(\theta^j)\tilde{w}_1^{2k}) > 0.$$

We take ε so small that the above requirements are satisfied and such that the sets

$$\{\tilde{w}_1 : P_{3\varepsilon}(\tilde{w}_1) - \operatorname{Re}(\alpha(\theta^j)\tilde{w}_1^{2k}) \leqq 0\}$$

are the closures of their interior for $j = 1, \ldots, l$.

In the $\tilde{}$ coordinates one has $\tilde{\zeta}_1 = \zeta_1$, $\tilde{\zeta}_2 = \zeta_2 + \alpha(\theta_1)\zeta_1^{2k}$. Let $\Phi : \mathbf{C}^2 \to \mathbf{C}^2$ be given by

$$\Phi(\hat{w}_1, \hat{w}_2) = (\hat{w}_1, \hat{w}_2^{2k} + \tilde{\zeta}_2) = (\tilde{w}_1, \tilde{w}_2).$$

Let $\hat{\Omega} = \phi^{-1}(\Omega)$, $\hat{\Omega}_\varepsilon = \phi^{-1}(\Omega_\varepsilon)$, $\hat{\Omega}_j = \phi^{-1}(\Omega_j)$, and $\hat{\Omega}'_j = \phi^{-1}(\Omega'_j)$. Note that $\operatorname{Re}\tilde{\zeta}_2 = 0$. Let $\hat{\zeta} = \phi^{-1}(\zeta)$. One has

$$\hat{\Omega}_j = \{\operatorname{Re}\tilde{w}_2^{2k} + P_{2\varepsilon}(\hat{w}_1) - \operatorname{Re}(\alpha(\theta^j)\hat{w}_1^{2k}) < 0\},$$

$$\hat{\Omega}'_j = \{\operatorname{Re}\hat{w}_2^{2k} + P_{3\varepsilon}(\hat{w}_1) - \operatorname{Re}(\alpha(\theta^j)\hat{w}_1^{2k}) < 0\}.$$

Let $S_0^j \ldots S_{n_j}^j$ be the connected components of $\hat{\Omega}'_j \cap \{\hat{w}_2 = 0\}$. Say S_0^j is the component of $\hat{\zeta}$.

Fix any of the $\hat{\Omega}'_j$. To $\hat{\Omega}'_j$ are associated two open Riemann surfaces $\hat{R}_j \subset\subset R_j$ with the following properties: There is a holomorphic map Π: $R_j \times \mathbf{C} \to \mathbf{C}^2$ of the form $\Pi(p,t) = (\alpha(p)t, \beta(p)t)$ where α, β are holomorphic functions on R_j without common zeros; Π is nonsingular when $t \neq 0$; there is an open set $\check{\Omega}'_j$ in $R_j \times \mathbf{C}$ such that $\Pi|\check{\Omega}'_j \to \hat{\Omega}'_j$ is a biholomorphism. Moreover

$$\check{\Omega}'_j = \bigcup_{p \in \hat{R}_j} \{p\} \times \check{S}^p,$$

where $\check{S}^p$ is a nonempty connected open sector in $\mathbf{C}$. For each complex line $L \subset \mathbf{C}^2$ through 0, $L \cap \hat{\Omega}'_j$ is a union of finitely many disjoint, open, connected sectors $C_1 \ldots C_{n(L)}$ and there exist $q_1 \ldots q_{n(L)}$ in $\hat{R}_j$ such that Π is a linear isomorphism between $q_i \times \check{S}^{q_i}$ and C_i.

Let $p_0 \ldots p_{n_j} \in \hat{R}_j$ be associated to the sectors $S_0^j \ldots S_{n_j}^j$ in $\{\hat{w}_2 = 0\} \cap \hat{\Omega}'_j$.

There are also $2k$ sectors $C_1 \dots C_{2k}$ in $\hat{\Omega}'_j \cap \{\hat{w}_1 = 0\}$ with associated points $q_1 \dots q_{2k} \in \hat{R}_j$.

We fix a holomorphic function $\phi \colon R_j \to \mathbf{C}$, nowhere identically vanishing while ϕ vanishes at least to order $2k+1$ at each of the points $p_1, \dots, p_{n_j}$, $q_1, \dots, q_{2k}$.

2. Construction of the Leray map.

We start with the meromorphic function $1/(\zeta_1 - \hat{w}_1)$ on $\hat{\Omega}'_j$ and pull it back to $R_j \times \mathbf{C}$ to get the meromorphic function $g = 1/(\zeta_1 - \alpha(p)t)$. Fix a small neighborhood V of $\{p_1, \dots, p_n\}$ in R_j. Observe that g is holomorphic as a function of ζ_1, p and t on $S_0^j \times (\check{\Omega}'_j \cap (V \times \mathbf{C}))$ and that there

$$|g| \lesssim \inf\{1/|\zeta_1|, 1/|t|\} \quad (\text{i.e. } |g| \leqq \text{const.}\, \{1/|\zeta_1|, 1/|t|\}).$$

Let $\chi \in C_0^\infty(R_j)$, $\chi \equiv 1$ on a neighborhood $V' \subset\subset V$ of $p_1, \dots, p_n$, $p_0 \notin V$ and $\operatorname{supp} \chi \subset V$. We may assume that $\phi \neq 0$ on $V - \{p_1, \dots, p_n\}$.

We define a $\bar{\partial}$-closed form $\lambda = \lambda_{\zeta_1}$ on $\check{\Omega}'_j$ by

$$\lambda = \begin{cases} \bar{\partial}\chi g/\phi & \text{if } p \in \operatorname{supp} \bar{\partial}\chi \\ 0 & \text{if } p \notin \operatorname{supp} \bar{\partial}\chi. \end{cases}$$

Then $\|\lambda\|_{L^2} \leqq \mathscr{C} |\ln|\zeta_1||^{1/2}$ for a fixed constant $\mathscr{C} > 0$. We now apply Hörmanders theory for solving the $\bar{\partial}$-equation, cf. [4]. Because λ_{ζ_1} is an $L^2-(0,1)$ form for all $\zeta_1 \in S_0^j$, we can use the same L^2-space for solving the $\bar{\partial}$-equation for all $\zeta_1 \in S_0^j$. In particular, choosing the solution in the closure of the range of $\bar{\partial}^*$, we obtain a linear solution operator T that satisfies $\bar{\partial}Tf = f$ and $\|Tf\|_{L^2} \leqq \mathscr{C} \|f\|_{L^2}$ for all $\bar{\partial}$-closed $(0,1)$ forms with coefficients in L^2.

We observe that $T\lambda_{\zeta_1}$ is a holomorphic function of ζ_1 on S_0^j, because T is linear.

Next we define $\Psi_1^1(p, t, \zeta_1) = \chi g - \phi T\lambda$. We have

$$\|\Psi_1^1\|_{L^2} \lesssim |\ln|\zeta_1||^{1/2} + 1. \tag{1}$$

We push ψ_1^1 down to $\hat{\Omega}_j$ to obtain $\psi_1^2(\hat{w}, \zeta_1) = \psi_1^1(\Pi^{-1}(\hat{w}), \zeta_1)$ and return to the $\tilde{}$ coordinates as follows. Let ω be a primitive $2k$-root of unity. Define

$$\psi_1^3(\hat{w}_1, \hat{w}_2, \zeta_1) = \frac{1}{2k} \sum_1^{2k} \psi_1^2(\hat{w}_1, \omega^j \hat{w}_2, \zeta_1).$$

This function is holomorphic in $(\hat{w}_1, \hat{w}_2^{2k}, \zeta_1)$, hence it can be pushed down to Ω_j yielding the holomorphic function

$$\psi_1^4(\tilde{w}_1, \tilde{w}_2, \zeta_1) = \psi_1^3(\tilde{w}_1, (\tilde{w}_2 - \tilde{\zeta}_2)^{1/2k}, \zeta_1).$$

Now $\psi_2^4(\tilde{w}, \zeta)$ is defined implicitly by $1 = (\zeta_1 - \tilde{w}_1)\psi_1^4 + (\tilde{\zeta}_2 - \tilde{w}_2)\psi_2^4$.

Finally this can be written in the original w-coordinates as follows

$$1 = \psi_1^4(\tilde{w},\tilde{\zeta})(\zeta_1 - w_1) + \psi_2^4(\tilde{w},\tilde{\zeta})[\zeta_2 - w_2 + G(\zeta_1, w_1)(\zeta_1 - w_1)],$$

where

$$G(\zeta_1, w_1) = \alpha(\theta_1)(\zeta_1^{2k} - w_1^{2k})/(\zeta_1 - w_1) + (\varepsilon/M)F(\zeta_1, w_1)/(\zeta_1 - w_1)$$

and we define the map ψ^j by

$$\psi_1^j(w,\zeta) = \psi_1^4(\tilde{w},\tilde{\zeta}) + \psi_2^4(\tilde{w},\tilde{\zeta})G(\zeta,\tilde{w})$$

$$\psi_2^j(w,\zeta) = \psi_2^4(\tilde{w},\tilde{\zeta}).$$

The map ψ^j satisfies the requirements for a Leray map, but only for $\zeta \in \Sigma$ with $e^{i\theta_1} \in I_j$.

A global map is now easily defined using a partition of unity. Let $\chi_j \in C_0^\infty(I_j)$, $\chi_j \geqq 0$, $\sum_1^l \chi_j \equiv 1$. Define

$$\psi_i(w,\zeta) = \sum_{j=1}^{l} \chi_j(\zeta_1/|\zeta_1|)\psi_i^j(w,\zeta) \quad (i = 1,2) \quad \text{for } \zeta \in \Sigma. \tag{2}$$

Similarly we can push down each of the functions χ, g, and $\phi T\lambda$. On the $\tilde{\ }$ level this gives functions $\chi^{4,j}$, $g^{4,j}$, and $(\phi T\lambda)^{4,j}$ living on Ω_j. We have

$$\psi_1^4 = \psi_1^{4,j} = \chi^{4,j}g^{4,j} - (\phi T\lambda)^{4,j},$$

where we used that $g^{3,j}$ is independent of $\hat{w}_2$.

3. Estimates concerning the Leray map.

LEMMA 3.1. *There exists a constant $C > 0$ such that for $\zeta \in \Sigma_R$, $\zeta_1/|\zeta_1| \in I_j$, and $\tilde{w} \in \tilde{\Omega}$ the following holds:*

1. $\chi^{4,j}(\tilde{w},\tilde{\zeta}) \equiv 1$ *if* $\tilde{w}_1 \notin S_0^j$ *and* $|\tilde{w}_2 - \tilde{\zeta}_2| < 1/C|\tilde{w}_1|^{2k}$.
2. $\chi^{4,j}(\tilde{w},\tilde{\zeta}) \equiv 0$ *if* $\tilde{w}_1 \in S_0^j$ *or* $|\tilde{w}_2 - \tilde{\zeta}_2| > C|\tilde{w}_1|^{2k}$.
3. $(\phi T\lambda)^{4,j}(\tilde{w},\tilde{\zeta})$ *has a zero at* $\tilde{w}_2 = \tilde{\zeta}_2$.
4. $g^{4,j}(\tilde{w},\tilde{\zeta}) = 1/(\tilde{w}_1 - \tilde{\zeta}_1)$.

PROOF. All these properties are direct consequences of the definition of χ, ϕ, and g and their transformation to $\tilde{\ }$ coordinates.

LEMMA 3.2. *Let K be a compact subset of Ω_R. Then there exists a positive constant κ such that for every $\zeta \in \Sigma_R$, $\zeta_1/|\zeta_1| \in I_j$ the set $\check{K}_3$, the pullback of K to $\tilde{\Omega}'_j$ has distance greater than κ to the boundary of $\tilde{\Omega}'_j$.*

PROOF. The compactum K is for some positive δ contained in $\{\operatorname{Re} w_2 - P(w_1) \leqq -\delta\}$. In $\tilde{}$ coordinates this set corresponds to $\tilde{K}_\zeta := \{\operatorname{Re} \tilde{w}_2 - Q_1(\tilde{w}_1) \leqq -\delta\}$. As the gradient of $\operatorname{Re} \tilde{w}_2(\zeta_1) - Q(\zeta_1, \tilde{w}_1)$ remains uniformly bounded as $\zeta \in \Sigma_R$, it follows that $\operatorname{dist}(\tilde{K}_\zeta, \partial\tilde{\Omega}) \geqq \delta' > 0$. So for $\zeta_1/|\zeta_1| \in I_j$:

$$\operatorname{dist}(\tilde{K}, \partial\Omega'_j) \geqq \delta'$$

by Lemma 1.4 and the observations following it. Pulling back to $\hat{\Omega}'_j$ is done by a translation in the $\operatorname{Im} \zeta_2$ direction, which has no influence on the distance to $\partial\Omega'_j$, followed by taking the inverse image under a proper map which does not depend on ζ. Hence, there is a compactum in $\hat{\Omega}'_j$ which contains the pullbacks of all $\tilde{K}_\zeta$. Finally $\Pi : \check{\Omega}_j \to \hat{\Omega}'_j$ is a biholomorphism and the lemma follows.

LEMMA 3.3. *For every compact* $K \subset\subset \Omega_R$ *there exists a positive constant* $\gamma(K)$ *such that* $|\tilde{\zeta}_1 - \tilde{w}_1| > \gamma(K)$ *on* $\operatorname{supp} \chi^{4,j}$, *and* $|\tilde{\zeta}_2 - \tilde{w}_2| > \gamma(K)$ *on* $\operatorname{supp} 1 - \chi^{4,j}$, *for* $w \in K$ *and* $\zeta \in \Sigma_R$, $\tilde{\zeta}_1/|\zeta_1| \in I_j$.

PROOF. Let $\tilde{K}$ denote K in the $\tilde{}$ coordinates. $\tilde{K}$ depends on ζ, but by Lemma 3.2 and its proof there exists $d > 0$ such that for $\zeta \in \Sigma_R$ distance $(\tilde{K}, \partial\tilde{\Omega}) > d$. Hence

$$|\tilde{w}_1 - \tilde{\zeta}_1| < \tfrac{1}{2}d \Rightarrow |\tilde{w}_2 - \tilde{\zeta}_2| > \tfrac{1}{2}d.$$

By Lemma 3.1, $\chi^{4,j} = 0$ if $|\tilde{w}_1|^{2k} < \frac{1}{2}d/\mathscr{C}$ or if $\tilde{w}_1 \in S_0$. Therefore if $(\tilde{w}, \tilde{\zeta}) \in \operatorname{supp} \chi^{4,j}$, then

$$|\tilde{w}_1 - \tilde{\zeta}_1| > \min\{\tfrac{1}{2}d, d/2\Omega \cdot \min[1, \min_{\substack{\zeta_1/|\zeta_1| \in I \\ \tilde{w}_1 \in \bigcup_{k=1}^{n(j)} S_k^j}} (\arg \zeta_1 - \arg w_1)] := \gamma_1.$$

γ_1 is strictly positive because $I_j \subset S_0^j$ and the sectors S_k^j are separated.

Next there exists $\delta' > 0$ such that $\operatorname{Re} \tilde{w}_2 + Q_1(\tilde{w}_1) \leqq -\delta'$ on $\tilde{K}$ for all $\zeta \in \Sigma_R$. Now $|\tilde{w}_2 - \tilde{\zeta}_2| < \frac{1}{2}\delta'$ implies $|\operatorname{Re} \tilde{w}_2| < \frac{1}{2}\delta'$ because $\operatorname{Re} \tilde{\zeta}_2 = 0$. Hence $Q_1(\tilde{w}_1) < -\frac{1}{2}\delta'$ and by Lemma 1.5, $|\arg \tilde{w}_1 - \arg \zeta_1| > \delta$. By Lemma 1.6 and the remark following it, we conclude that $\tilde{w}_1 \notin S_0^j$. From $Q_1(w_1) < -\frac{1}{2}\delta'$ we also infer that $|w_1| > A(\delta')^{1/2k}$, where A is independent of ζ, $|\zeta| < R$. Application of Lemma 3.1 gives $\chi^{4,j}(\tilde{w}, \tilde{\zeta}) = 1$, if

$$|\tilde{w}_2 - \tilde{\zeta}_2| < \min\{\tfrac{1}{2}\delta', A^{2k}/C\delta'\} =: \gamma_2.$$

Now take $\gamma(K) = \min\{\gamma_1, \gamma_2\}$.

PROPOSITION 3.4. *For every compact* $K \subset\subset \Omega_R$ *the functions* ψ_i, $i = 1, 2$, *defined by* (2) *satisfy*

$$|\psi_i(w,\zeta)| \lesssim |\ln|\zeta_1||^{1/2} + 1, \quad \zeta \in \Sigma_R,\ w \in K.$$

PROOF. It will be enough to show that $|\psi_i^j(w,\zeta)| \lesssim |\ln|\zeta_1||^{1/2} + 1$, if $\zeta_1/|\zeta_1| \in I_j$, $\zeta \in \Sigma_R$, $w \in \bar{K}$. By the definition of ψ_1^j this reduces to proving that the corresponding ψ_1^4, ψ_2^4 are majorized by a constant times $|\ln|\zeta_1||^{1/2} + 1$, because $G(\zeta, \tilde{w})$ remains bounded. Since ψ_1^4 is the pushdown of ψ_1^1 we have

$$\|\psi_1^4(\tilde{w}, \tilde{\zeta})\|_K = \|\psi_1^1(p,t,\zeta)\|_{\tilde{K}_\zeta} \lesssim \|\psi_1^1(p,t,\zeta)\|_{L^2} \lesssim |\ln|\zeta_1||^{1/2} + 1.$$

We used Lemma 3.2 for the first inequality, while the last inequality is just (1).
Next, we deal with

$$\psi_2^4(\tilde{w}, \zeta) = \frac{1 - (\tilde{\zeta}_1 - \tilde{w}_1)\psi_1^4(\tilde{w}, \tilde{\zeta})}{\tilde{\zeta}_2 - \tilde{w}_2}.$$

We have for $\zeta \in \Sigma_R$, $w \in K$, if $|\tilde{\zeta}_2 - \tilde{w}_2| \geqq \gamma(K)/2$

$$|\psi_2^4| \lesssim \frac{2}{\gamma(K)}(|\ln|\zeta_1||^{1/2} + 1),$$

while if $|\tilde{\zeta}_2 - \tilde{w}_2| \leqq \gamma(K)/2$, by Lemmas 3.1, 3.3, and the fact that $(\phi T\lambda)^{4,j}$ is holomorphic if $|\tilde{\zeta}_2 - \tilde{w}_2| \leqq \gamma(K)$

$$|\psi_2^4(w,\zeta)| = \left|\frac{1 - \chi^{4,j}(\tilde{w}, \tilde{\zeta}) + (\tilde{\zeta}_1 - \tilde{w}_1)(\phi T\lambda)^{4,j}}{\tilde{\zeta}_2 - \tilde{w}_2}\right|$$

$$= \left|(\tilde{\zeta}_1 - \tilde{w}_1)\frac{(\phi T\lambda)^{4,j}}{\tilde{\zeta}_2 - \tilde{w}_2}\right| \leqq \|\cdots\|_{L^2} \leqq |\ln|\zeta_1||^{1/2} + 1.$$

4. Reproducing kernels on Ω_R.

Let ψ be the Leray map for Σ as constructed in section 2, let ψ^2 be the Leray map for $\partial B(0,R)$, that is

$$\psi_i^2(w,\zeta) = -\frac{\bar{\zeta}_i}{(w \cdot \zeta - R^2)}, \quad \text{where } w \cdot \zeta = \sum w_i \bar{\zeta}_i,\ (\zeta, w) \in \partial B(0,R) \times B(0,R).$$

We also need the map

$$\psi_i^3(w,\zeta) = \frac{\overline{\zeta_i - w_i}}{\|w - \zeta\|^2},$$

associated with the Bochner Martinelli formula.

We introduce smooth cut-off functions as follows. Let $\alpha \in C^{\infty}(R^+)$, $0 \leqq \alpha \leqq 1$, $\alpha(t) = 1$ for $t \leqq 0$, $\alpha(t) = 0$ for $t \geqq 1$. Put

$$\tau_\varepsilon^1(\zeta) = \alpha\left(\frac{|\zeta_1| - \varepsilon_1}{\varepsilon_2}\right), \qquad \varepsilon = (\varepsilon_1, \varepsilon_2),\ \varepsilon_i > 0,$$

$$\tau_\varepsilon^2(\zeta) = \alpha\left(\frac{||\zeta| - R| - \varepsilon_1}{\varepsilon_2}\right), \quad \varepsilon = (\varepsilon_1, \varepsilon_2),\ \varepsilon_i > 0.$$

We distinguish the following parts in $\partial\Omega_R$

$$F_1 = \overline{\partial\Omega \cap B(0,R)}, \quad F_2 = \overline{\partial B(0,R) \cap \Omega}, \quad F_3 = F_1 \cap F_2.$$

We form for $(w, \zeta) \in \Omega_R \times \partial\Omega_R$

$$\psi_i^{\varepsilon,\eta}(w,\zeta) = \left[\tau_\varepsilon^2(\zeta)\psi_i^2(w,\zeta) + (1 - \tau_\varepsilon^2(\zeta))\psi_i(w,\zeta)\right](1 - \tau_\eta^1(\zeta)) + \tau_\eta^1(\zeta)\psi_i^3(w_1,\zeta)$$

$\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\eta = (\eta_1, \eta_2)$.

Observe that we have for $(w, \zeta) \in \Omega_R \times \partial\Omega_R$

$$1 = \sum_1^2 \psi_i^{\varepsilon,\eta}(w,\zeta)(\zeta_i - w_i)$$

and we can extend $\psi_i^{\varepsilon,\eta}$ smoothly to a neighborhood of $\Omega_R' \times \partial\Omega_R$, such that the above identity remains valid.

PROPOSITION 4.1. *Let $f \in A(\Omega_R)$. Then for $w \in \Omega_R$*

$$4\pi^2 f(w) = \int_{\partial\Omega_R} f(\zeta) K^{\varepsilon,\eta}(w,\zeta),$$

where

$$K^{\varepsilon,\eta}(w,\zeta) = (\psi_1^{\varepsilon,\eta}(w,\zeta)\bar\partial_\zeta \psi_2^{\varepsilon,\eta}(w,\zeta) - \psi_2^{\varepsilon,\eta}(w,\zeta)\bar\partial_\zeta \psi_1^{\varepsilon,\eta}(w,\zeta)) \wedge d\zeta_1 \wedge d\zeta_2.$$

The proof is a copy of the proof for the case of smooth domains: Fix $w \in \Omega_R$, after changing ψ in a small neighborhood of w, we can assume that on this neighborhood $\psi \equiv \psi^3$. Then by using Stokes' Theorem, we see that

$$4\pi^2 f(w) = \int_{\partial\Omega_R^\delta} f(\zeta) K^{\varepsilon,\eta}(w,\zeta),$$

where Ω_R^δ form an increasing family of smooth domains which contain w and exhaust Ω_R when $\delta \to 0$. If we let δ go to 0, we obtain the required formula.

We will let ε and η tend to 0. Then $K^{\varepsilon,\eta}$ will tend to a form K^0 which is holomorphic in w. This already would yield an integral representation, but perhaps only in some principal value sense. We proceed by proving that K^0 is a form with integrable coefficients.

Define

$$K_1(w,\zeta) = (\psi_1\bar\partial_\zeta\psi_2 - \psi_2\bar\partial_\zeta\psi_1) \wedge d\zeta_1 \wedge d\zeta_2$$

$$K_2(w,\zeta) = (\psi_1^2\bar\partial_\zeta\psi_2^2 - \psi_2^2\bar\partial_\zeta\psi_1^2) \wedge d\zeta_1 \wedge d\zeta_2$$

$$K_3(w,\zeta) = (\psi_1\psi_2^2 - \psi_1^2\psi_2) d\zeta_1 \wedge d\zeta_2.$$

THEOREM 4.2. *Let $K \subset\subset \Omega_R$. There exists a constant A such that*

$$\int_{F_i} |K_i(w,\zeta)| < A \quad \textit{for } w \in K \quad (i = 1,2,3).$$

PROOF. The major part is the case $i = 1$. On $\Omega \times \partial\Omega$ we have the following equality

$$\psi_1\bar\partial_\zeta\psi_2 - \psi_2\bar\partial_\zeta\psi_1 = \frac{\bar\partial_\zeta\psi_2(w,\zeta)}{w_1-\zeta_1}$$

(3)
$$= \frac{\bar\partial_\zeta \sum_{j=1}^{l} \chi_j(\zeta_1/|\zeta_1|)\psi_2^j(w,\zeta)}{w_1-\zeta_1}$$

$$= \sum (\bar\partial_\zeta\chi_j)\frac{\psi_2^j(w,\zeta)}{\tilde w_1 - \tilde\zeta_1} + \sum \frac{\chi_j\bar\partial_\zeta\psi_2^j(w,\zeta)}{\tilde w_1 - \tilde\zeta_1}.$$

As χ_j depends only on $\arg\zeta_1$, we have $\|\bar\partial_\zeta\chi_j\| \preccurlyeq 1/|\zeta_1|$. If $|w_1-\zeta_1| > \gamma(K)$ we conclude that (3) is majorized by a constant times

(4)
$$\frac{1}{\gamma(K)}\frac{1}{|\zeta_1|}\cdot \sup_{z\in K} |\psi_2^j(w,\zeta)| + \frac{1}{\gamma(K)} \sup_{w\in K} |\bar\partial_\zeta\psi_2^j(w,\zeta)|.$$

Now

$$\psi_2^j(w,\zeta) = \psi_2^4(\tilde w,\tilde\zeta) = \tilde\psi(\tilde w_1,(\tilde w_2-\tilde\zeta_2),\tilde\zeta_1)$$

which is a holomorphic function of three variables. Hence

$$\frac{\partial\psi_2^j}{\partial\zeta_i} = \frac{\partial\tilde\psi}{\partial\tilde\zeta_1}\frac{\partial\tilde\zeta_1}{\partial\zeta_i} + \frac{\partial\tilde\psi}{\partial(\tilde w_2-\tilde\zeta_2)}\frac{\partial(\tilde w_2-\tilde\zeta_2)}{\partial\zeta_i} + \frac{\partial\tilde\psi}{\partial\tilde w_1}\frac{\partial\tilde w_1}{\partial\zeta_i}.$$

In view of the form of the $\tilde{}$ coordinates we obtain

$$\bar\partial_\zeta\psi_2^j = \frac{\partial\tilde\psi}{\partial(\tilde w_2-\tilde\zeta_2)}\frac{\partial(\tilde w_2-\tilde\zeta_2)}{\partial\bar\zeta_1}d\bar\zeta_1.$$

Fix an open Ω_K such that $K \subset\subset \Omega_K \subset\subset \Omega_R$. Because $\tilde{\psi}$ is holomorphic we have, using Proposition 3.4

$$\sup_{w \in K} \left| \frac{\partial \tilde{\psi}}{\partial(\tilde{w}_2 - \tilde{\zeta}_2)} \right| \leqq \sup_{w \in \Omega_k} |\tilde{\psi}| = \sup_{w \in \Omega_k} |\psi_2^j| \lesssim |\ln|\zeta_1||^{1/2} + 1$$

and also

$$\left| \frac{\partial(\tilde{w}_2 - \tilde{\zeta}_2)}{\partial \tilde{\zeta}_1} \right| \lesssim \frac{|w_1 - \zeta_1|}{|\zeta_1|}.$$

We infer that (4) is bounded by a constant times

$$\frac{|\ln|\zeta_1||^{1/2} + 1}{|\zeta_1|} \quad \text{for } |w_1 - \zeta_1| > \gamma(K).$$

Now for $|w_1 - \zeta_1| < \gamma(K)$ we proceed as follows. We have $|\tilde{w}_2 - \tilde{\zeta}_2| > \gamma(K)$ and by section 2

$$\psi_2^j(w, \zeta) = \psi_2^{4 \cdot j}(\tilde{w}, \tilde{\zeta}) = \frac{1 - \psi_1^{4 \cdot j}(\tilde{w}_1 - \tilde{\zeta}_1)}{(\tilde{w}_2 - \tilde{\zeta}_2)}.$$

Hence

$$\text{(5)} \qquad \sup_{w \in K} \|\textstyle\sum \bar{\partial}_\zeta \chi_j \psi_2^j/(w_1 - \zeta_1)\| \lesssim \frac{1}{\gamma(K)} \frac{1}{|\zeta_1|} \sup_{w \in K} \|\psi_1^{4 \cdot j}(w, \zeta)\|$$

where we used that $\sum_1^l \bar{\partial}_\zeta \chi_j = 0$. Similarly

$$\text{(6)} \quad \sup_{w \in K} \|\textstyle\sum \chi_j \bar{\partial}_\zeta \psi_2^j/(w_1 - \zeta_1)\| \lesssim \sup_{w \in K} \left(\left\| \frac{1}{w_1 - \zeta_1} \bar{\partial}_\zeta \frac{1}{\tilde{w}_2 - \tilde{\zeta}_2} \right\| + \frac{1}{\gamma(K)} \|\textstyle\sum \bar{\partial}_\zeta \psi_1^{4 \cdot j}\| \right).$$

As before, we use that ψ_1^4 is holomorphic as a function of $\tilde{w}_1, \tilde{w}_2 - \tilde{\zeta}_2$ and $\tilde{\zeta}_1$ as well as the estimate for $\partial(\tilde{w}_2 - \tilde{\zeta}_2)/\partial \tilde{\zeta}_1$, to majorize (6) by a constant times

$$\frac{1}{|\zeta_1|} \left(1 + \sup_{w \in \Omega_K} |\psi_1^{4 \cdot j}| \right).$$

Proposition 3.4 combined with (5) the estimate for (6) gives that (3) is bounded by a constant times $(|\ln|\zeta_1||^{1/2} + 1)/|\zeta_1|$ for $w \in K$. On F_1 we can take as local coordinates $\operatorname{Re}\zeta_1$, $\operatorname{Im}\zeta_1$, and $\operatorname{Im}\zeta_2$. The Jacobian determinants remain bounded and we obtain

$$\int_{F_1} |K_1| \leqq C \int_{\substack{|\zeta_1| < R \\ \ln|\zeta_2| < R}} \frac{|\ln|\zeta_1||^{1/2} + 1}{|\zeta_1|} d\operatorname{Re}\zeta_1 d\operatorname{Im}\zeta_1 d\operatorname{Im}\zeta_2 \leqq A$$

for some constant A.

$(i = 2)$: Just note that ψ_i^2 and hence K_2 are smooth and bounded for $w \in K \subset\subset B(0, r)$, $\zeta \in \partial B(0, R)$.

$(i = 3)$: By the boundedness of ψ_i^2 and Proposition 3.4

$$|\psi_1\psi_2^2 - \psi_1^2\psi_2| \lesssim |\ln|\zeta_1||^{1/2} + 1.$$

Also

$$|d\zeta_1 \wedge d\zeta_2| \leqq |dx_1^1 \wedge dx_2^1| + |dx_1^1 \wedge dx_2^2| + |dx_1^2 \wedge dx_2^1| + |dx_1^2 \wedge dx_2^2|,$$

where $\zeta_j = x_j^1 + ix_j^2$, $j = 1, 2$. Using $x_2^1 = P(x_1^1, x_1^2)$ with $\partial P/\partial x_1^1$, $\partial P/\partial x_1^2$ bounded for $|X| < R$ we can estimate

$$\int_{F_j} |K_3(w, \zeta)| \lesssim \int_{-R}^{R} \int_{-R}^{R} (|\ln|t||^{1/2} + 1)\,dt\,ds$$

and the latter integral is bounded.

LEMMA 4.3. *If $\psi_j^i(w, \zeta)$, $i, j = 1, 2$ satisfy*

$$1 = (w_1 - \zeta_1)\psi_1^i(w, \zeta) + (w_2 - \zeta_2)\psi_2^i(w, \zeta),$$

for (w, ζ) in a neighborhood of $\Omega \times \partial\Omega$, then

$$\psi_1^1\bar\partial_\zeta\psi_2^1 - \psi_2^1\bar\partial_\zeta\psi_1^1 = \psi_1^2\bar\partial_\zeta\psi_2^1 - \psi_2^2\bar\partial_\zeta\psi_1^1.$$

PROOF.

$$\begin{aligned}(w_1 - \zeta_1)&[(\psi_1^1 - \psi_1^2)\bar\partial_\zeta\psi_2^1 - (\psi_2^1 - \psi_2^2)\bar\partial_\zeta\psi_1^1]\\ &= (\psi_2^1 - \psi_2^2)[-(w_2 - \zeta_2)\bar\partial_\zeta\psi_2^1 - (w_1 - \zeta_1)\bar\partial_\zeta\psi_1^1]\\ &= -(\psi_2^1 - \psi_2^2)\bar\partial_\zeta[(w_1 - \zeta_1)\psi_1^1 + (w_2 - \zeta_2)\psi_2^1] = 0.\end{aligned}$$

Similarly

$$(w_2 - \zeta_2)[(\psi_1^1 - \psi_1^2)\bar\partial_\zeta\psi_2^1 - (\psi_2^1 - \psi_2^2)\bar\partial_\zeta\psi_1^1] = 0$$

and the Lemma follows.

THEOREM 4.4. *Let f be a continuous function on $\partial\Omega_R$, then*

$$C[f](w) := \sum_{j=1}^{3} \int_{F_j} f(\zeta)K_j(w, \zeta)$$

is a holomorphic function on Ω_R. Moreover, if

$$f \in A(\Omega_R)\ (= C(\bar\Omega_R) \cap H(\Omega_R)),$$

then $C[f] = f$.

PROOF. The first statement follows easily by differentiating under the integral sign, the Cauchy formula and dominated convergence. Next, if $f \in A(\Omega_R)$, $w \in \Omega_R$ we have by Proposition 4.1

$$f(w) = \int_{\partial\Omega_R} f(\zeta) K^{\varepsilon,\eta}(w,\zeta).$$

We put

$$\tilde{\psi}_i = \tilde{\psi}_i^{\varepsilon}(w,\zeta) = \tau_\varepsilon^2(\zeta)\psi_i^2(w,\zeta) + (1-\tau_\varepsilon^2(\zeta))\psi_i(w,\zeta), \quad i = 1,2.$$

Evaluation of $K^{\varepsilon,\eta}$ yields, using Lemma 4.3

$$\begin{aligned}
(6) \qquad K^{\varepsilon,\eta}(w,\zeta) = {} & [(1-\tau_\eta^1)^2(\tilde{\psi}_1\bar{\partial}_\zeta\tilde{\psi}_2 - \tilde{\psi}_2\bar{\partial}_\zeta\tilde{\psi}_1) + \\
& + (\tau_\eta^1)^2(\psi_1^3\bar{\partial}_\zeta\psi_2^3 - \psi_2^3\bar{\partial}_\zeta\psi_1^3) + \\
& + \tau_\eta^1(1-\tau_\eta^1)[\psi_1^3\bar{\partial}_\zeta\tilde{\psi}_2 + \tilde{\psi}_1\bar{\partial}_\zeta\psi_2^3 - \psi_2^3\bar{\partial}_\zeta\tilde{\psi}_1 - \tilde{\psi}_2\bar{\partial}_\zeta\psi_1^3] + \\
& + \tau_\eta^1\bar{\partial}_\zeta(1-\tau_\eta^1)[\psi_1^3\tilde{\psi}_2 - \psi_2^3\tilde{\psi}_1] + \\
& + (1-\tau_\eta^1)\bar{\partial}_\zeta\tau_\eta^1[\psi_2^3\tilde{\psi}_1 - \psi_1^3\tilde{\psi}_2] \wedge d\zeta_1 \wedge d\zeta_2 \\
= {} & [(1-\tau_\eta^1)(\tilde{\psi}_1\bar{\partial}_\zeta\tilde{\psi}_2 - \tilde{\psi}_2\bar{\partial}_\zeta\tilde{\psi}_1) + \tau_\eta^1(\psi_1^3\bar{\partial}_\zeta\psi_2^3 - \psi_2^3\bar{\partial}_\zeta\psi_1^3) + \\
& + \bar{\partial}_\zeta\tau_\eta^1(\psi_2^3\tilde{\psi}_1 - \psi_1^3\tilde{\psi}_2)]d\zeta_1 \wedge d\zeta_2.
\end{aligned}$$

We plug this in (4.1) and let $\eta_2 \to 0$. Then τ_η^1 will tend in measure to the characteristic function of the disc $|\zeta_1| < \eta_1$, while $(\partial\tau'_\eta/\partial\bar{\zeta}_1)d\bar{\zeta}_1 \wedge d\zeta_1$ will tend in measure to arc length on $|\zeta_1| = \eta_1$, compare the proof of a slightly more involved but similar assertion in the sequel. Now we let $\eta_1 \to 0$, then

$$\int_{\partial\Omega_R} f\tau_\eta^1(\psi_1^3\bar{\partial}_\zeta\psi_2^3 - \psi_2^3\bar{\partial}_\zeta\psi_1^3)d\zeta_1 \wedge d\zeta_2$$

will vanish, because the integrand is a continuous function and integration is over $\partial\Omega_R \cap \{|\zeta_1| < \eta_1\}$. Also if $\eta_1 \to 0$

$$\int_{\substack{|\zeta_1| = \eta_1 \\ \zeta \in \partial\Omega_R}} f(\psi_2^3\tilde{\psi}_1 - \psi_1^3\tilde{\psi}_2)d\zeta_2 d\sigma_1 \to 0 \quad (\sigma_1 = \text{arc length on } |\zeta_1| = \eta)$$

because the integrand is bounded by a constant times $|\ln|\zeta_1||^{1/2} + 1$, in view of Proposition 3.4. Therefore

$$f(w) = \int_{\partial\Omega_R} (\tilde{\psi}_1^\varepsilon\bar{\partial}_\zeta\tilde{\psi}_2^\varepsilon - \tilde{\psi}_2^\varepsilon\partial_\zeta\tilde{\psi}_1^\varepsilon)d\zeta_1 \wedge d\zeta_2.$$

We perform the same manipulation to obtain

$$f(w) = \int_{F_1} f(\zeta)(1-\tau_\varepsilon^2(\zeta))(\psi_1(w,\zeta)\bar{\partial}_\zeta\psi_2(w,\zeta)-\psi_2(w,\zeta)\bar{\partial}_\zeta\psi_1(w,\zeta))d\zeta_1 \wedge d\zeta_2 +$$

$$+\int_{\partial\Omega_R} f(\zeta)\tau_\varepsilon^2(\zeta)(\psi_1^2(w,\zeta)\bar{\partial}_\zeta\psi_2^2(w,\zeta)-\psi_2^2(w,\zeta)\bar{\partial}_\zeta\psi_1^2(w,\zeta)) \wedge d\zeta_1 \wedge d\zeta_2 +$$

$$+\int_{F_1} f(\zeta)\bar{\partial}_\zeta(\tau_\varepsilon^2(\zeta))(\psi_1(w,\zeta)\psi_2^2(w,\zeta)-\psi_2(w,\zeta)\psi_1^2(w,\zeta) \wedge d\zeta_1 \wedge d\zeta_2.$$

We used that $\operatorname{supp}(1-\tau_\varepsilon^2) \cap \partial\Omega_R \subset F_1$ for all $\varepsilon_1, \varepsilon_2 > 0$.

If $\varepsilon_1\varepsilon_2 \to 0$, then the first integral tends to $\int_{F_1} f(\zeta)K_1(w,\zeta)$ by dominated convergence, in view of Theorem 4.2. Similarly the second integral tends to $\int_{F_2} f(\zeta)K_2(w,\zeta)$. For the third one, observe that

$$\|\bar{\partial}_\zeta\tau_\varepsilon^2(\zeta)\| \leqq C/\varepsilon_2 \text{ and } \operatorname{supp}\bar{\partial}_3\tau_\varepsilon \subset \partial\Omega \cap \{R-\varepsilon_1-\varepsilon_2 \leqq |\zeta| \leqq R-\varepsilon_1\}.$$

We let first ε_1 go to 0. Again, by dominated convergence

$$\lim_{\varepsilon_1 \to 0} \int_{F_1} f\bar{\partial}_\zeta\tau_\varepsilon^2(\psi_1\psi_2^2-\psi_2\psi_1^2) \wedge d\zeta_1 \wedge d\zeta_2$$

$$= \int_{\partial\Omega} f\bar{\partial}_\zeta\tau_{(0,\varepsilon_2)}^2(\psi_1\psi_2^2-\psi_2\psi_1^2) \wedge d\zeta_1 \wedge d\zeta_2.$$

We claim that

$$\lim_{\varepsilon_2 \to 0} \int_{\partial\Omega} f\bar{\partial}_\zeta\tau_{(0,\varepsilon_2)}(\psi_1\psi_2^2-\psi_2\psi_1^2) \wedge d\zeta_1 \wedge d\zeta_2 = \int_{F_3} f d\zeta_1 \wedge d\zeta_2.$$

This is seen as follows: Put

$$g(w,\zeta) = (\psi_1\psi_2^2-\psi_2\psi_1^2)(w,\zeta),$$

let $\tilde{f}(\zeta)$ be a C_0^∞-function on $\partial\Omega$ such that

$$\sup_{\zeta\in F_1} |f-\tilde{f}| < \delta$$

and let $\tilde{g}(\zeta)$ be a C_0^∞-function on $\partial\Omega$ such that $|\tilde{g}| \leqq g$ on F_1 and $\tilde{g} = g$ on $F_1 \cap \{|\zeta_1| \geqq \delta\}$. Then

$$(7) \qquad \left|\int_{\partial\Omega} fg\bar{\partial}\tau_{0,\varepsilon_2}^2 \wedge d\zeta_1 \wedge d\zeta_2 - \int_{F_3} fgd\zeta_1 \wedge d\zeta_2\right|$$

$$\leqq \left| \int_{\partial\Omega} (fg - \tilde{f}\tilde{g})\bar{\partial}\tau^2_{0,\varepsilon_2} \wedge d\zeta_1 \wedge d\zeta_2 \right| + \left| \int_{\partial\Omega} \tilde{f}\tilde{g}\bar{\partial}\tau^2_{0,\varepsilon_2} d\zeta_1 \wedge d\zeta_2 - \right.$$

$$\left. - \int_{F_3} \tilde{f}\tilde{g} d\zeta_1 \wedge d\zeta_2 \right| + \left| \int_{F_3} (\tilde{f}\tilde{g} - fg) d\zeta_1 \wedge d\zeta_2 \right|.$$

Taking $\operatorname{Re}\zeta_1$, $\operatorname{Im}\zeta_1$, $\operatorname{Im}\zeta$ as coordinates we have, because the involved Jacobian determinants are bounded

$$(8) \quad \left| \int_{\partial\Omega} (fg - \tilde{f}\tilde{g})\bar{\partial}\tau_{0,\varepsilon_2} \wedge d\zeta_1 \wedge d\zeta_2 \right| \leqq C\delta \int_{\substack{\zeta\in\partial\Omega \\ R-\varepsilon^2 < |\zeta| < R}} \left| g/\varepsilon_2 \right| d\operatorname{Re}\zeta_1 d\operatorname{Im}\zeta_1 d\operatorname{Im}\zeta_2 +$$

$$+ C \int_{\substack{\zeta\in\partial\Omega \\ |\zeta_1| < \delta,\, R-\varepsilon_2 < |\zeta_1| < R}} \left| g/\varepsilon_2 \right| d\operatorname{Im}\zeta_1 d\operatorname{Im}\zeta_1 d\operatorname{Im}\zeta_2.$$

Now we integrate first with respect to $\operatorname{Im}\zeta_2$, and observe that for fixed ζ_1, $\operatorname{Im}\zeta_2$ runs over an interval of length $C'\cdot\varepsilon_2$ and that g s bounded by $C''(|\ln|\zeta_1||^{1/2}+1)$, which is integrable. We infer that (8) tends to 0 with δ. For the second term in the righthand side of (7) we have by integrating by parts

$$\int_{\partial\Omega} \tilde{f}\tilde{g}\bar{\partial}_\zeta \tau^2_{0,\varepsilon_2} \wedge d\zeta_1 \wedge d\zeta_2 = \int_{\partial\Omega} \bar{\partial}_\zeta(\tilde{f}\tilde{g})\tau^2_{0,\varepsilon_2} \wedge d\zeta_1 \wedge d\zeta_2$$

which leads to

$$\int_{\partial\Omega\setminus F_1} \bar{\partial}_\zeta \tilde{f}\tilde{g} \wedge d\zeta_1 \wedge d\zeta_2 \quad \text{as} \quad \varepsilon_2 \to 0.$$

By Stokes' Theorem

$$\int_{\partial\Omega\setminus F_1} \bar{\partial}_\zeta(\tilde{f}\tilde{g}) \wedge d\zeta_1 \wedge d\zeta_2 = \int_{\partial\Omega\setminus F_1} d_\zeta(\tilde{f}\tilde{g} d\zeta_1 \wedge d\zeta_2) = \int_{F_3} \tilde{f}\tilde{g} d\zeta_1 \wedge d\zeta_2.$$

Finally the third term in (7) is $O(\delta\log\delta)$ as $\delta \to 0$.

REFERENCES

1. E. Bedford and J. E. Fornæss, *A construction of peak functions on weakly pseudoconvex domains*, Ann. of Math. 107 (1978), 555–568.
2. J. E. Fornæss, *Sup-norm estimates for $\bar{\partial}$ in* $\mathbb{C}^2$, Ann. of Math. 123 (1986), 335–345.
3. G. M. Henkin, *Integral representations of functions holomorphic in strictly pseudoconvex domains and applications to the $\bar{\partial}$-problem*, Mat. Sb. (N.S.) 82 (1970), 300–308; Math. USSR-Sb.11 (1970), 273–281.
4. L. Hörmander, *An Introduction to Complex Analysis in Several Variables* (North-Holland Math. Library 7), North-Holland Publ. Co., Amsterdam - London - New York, 1973.
5. R. M. Range and Y.-T. Siu, *Uniform estimates for the $\bar{\partial}$-equations on domains with piecewise smooth strictly pseudoconvex boundaries*, Math. Ann. 206 (1973), 325–354.

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